Graceful Colorings of Graphs

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Abstract

A graceful labeling of a graph G of order n and size m is a one-to-one function $f : V(G) \to \{0, 1, \ldots, m\}$ that induces a one-to-one function $f' : E(G) \to \{1, 2, \ldots, m\}$ defined by $f'(uv) = |f(u) - f(v)|$. A graph that admits a graceful labeling is a graceful graph. A proper coloring $c : V(G) \to \{1, 2, \ldots, k\}$ is called a graceful k-coloring if the induced edge coloring c' defined by $c'(uv) = |c(u) - c(v)|$ is proper. The minimum positive integer k for which G has a graceful k-coloring is its graceful chromatic number $\chi_g(G)$. The graceful chromatic numbers of cycles, wheels and caterpillars are determined. An upper bound for the graceful chromatic number of trees is determined in terms of its maximum degree.

Key Words: grace labeling, grace coloring, graceful chromatic numbers.
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1 Introduction

In 1967, Rosa [5] introduced a vertex labeling of a graph that he called a *β-valuation*. In 1972, Golomb [4] referred to this labeling as a graceful labeling - terminology that has become standard. Let G be a graph of order n and size m. A *graceful labeling* of G is a one-to-one function $f : V(G) \to \{0, 1, \ldots, m\}$ that, in turn, assigns to each edge uv of G the label $f'(uv) = |f(u) - f(v)|$ such that no two edges of G are labeled the same. Therefore, if f is a graceful labeling of G, then the set of edge labels is $\{1, 2, \ldots, m\}$. A graph possessing a graceful labeling is a *graceful graph*. A major problem in this area is that of determining which graphs are graceful. One of the best known conjectures dealing with graceful graphs involves trees and is due to Kotzig and Ringel (see [3]).

The Graceful Tree Conjecture *Every nontrivial tree is graceful.*

The *gracefulness* grac(G) of a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the smallest positive integer k for which it is possible to label the vertices of G with distinct elements of the set $\{0, 1, 2, \ldots, k\}$ in such a way that distinct edges receive distinct labels. The gracefulness of every such graph is defined, for if we label v_i by 2^{i-1} for $1 \leq i \leq n$, then a vertex labeling with this property exists. Thus, if G is a graph of order n and size m, then $m \leq \text{grac}(G) \leq 2^{n-1}$. If grac$(G) = m$, then G is graceful. The gracefulness of a graph G can be considered as a measure of how close G is to being graceful – the closer the gracefulness is to m, the closer the graph is to being graceful. The exact values of grac(K_n) were determined for $1 \leq n \leq 10$ in [4]. For example, grac$(K_4) = 6$, grac$(K_5) = 11$ and grac$(K_6) = 17$. The exact value of grac(K_n) is not known in general, however. On the other hand, Erdős showed that grac$(K_n) \sim n^2$ (see [4]).

Graceful labelings have also been looked at in terms of colorings. A *rainbow vertex coloring* of a graph G of size m is an assignment f of distinct colors to the vertices of G. If the colors are chosen from the set $\{0, 1, \ldots, m\}$, resulting in each edge uv of G being colored $f'(uv) = |f(u) - f(v)|$ such that the colors assigned to the edges of G are also distinct, then this *rainbow* vertex coloring results in a *rainbow* edge coloring $f' : E(G) \to \{1, 2, \ldots, m\}$. So, such a rainbow vertex coloring is a graceful labeling of G (also see [7]).

The colorings of graphs that have received the most attention, however, are *proper vertex colorings* and *proper edge colorings*. In such a coloring of a graph G, every two adjacent vertices or every two adjacent edges are assigned distinct colors. The minimum number of colors needed in a proper vertex coloring of G is its *chromatic number*, denoted by $\chi(G)$, while the minimum number of colors needed in a proper edge coloring of G is its *chromatic index*, denoted by $\chi'(G)$.

Inspired by graceful labelings, we now consider a natural and new type of vertex colorings that induce edge colorings, both of which are proper rather than rainbow. We refer to the book [2] for graph theory notation and terminology not described in this paper.

2 Graceful Chromatic Numbers of Graphs

It is useful to describe notation for certain intervals of integers. For positive integers a, b with $a \leq b$, let $[a, b] = \{a, a+1, \ldots, b\}$ and $[b] = [1, b]$. A *graceful k-coloring* of a nonempty graph G is a proper vertex coloring $c : V(G) \to [k]$, where $k \geq 2$, that induces a proper edge coloring $c' : E(G) \to [k-1]$ defined by $c'(uv) = |c(u) - c(v)|$. A vertex coloring c of a graph G is a *graceful coloring* if c is a graceful k-coloring for some $k \in \mathbb{N}$. The minimum k for which G has a graceful k-coloring is called the *graceful chromatic number* of G, denoted by $\chi_g(G)$. Note that in a graceful labeling of a nonempty graph of size m, the colors are chosen from the set $\{0, 1, \ldots, m\}$ and so the color 0 could be used; while in a graceful coloring, each color is a positive integer. There are immediate lower and upper bounds for the graceful chromatic number of a graph.

Observation 2.1 *If G is a nonempty graph of order n, then $\chi_g(G)$ exists and*

$$\chi(G) \le \chi_g(G) \le \mathrm{grac}(G) \le 2^{n-1}.$$

Figure 1 shows two graceful graphs K_4 and C_4 together with a graceful coloring for each of these two graphs. In fact, $\chi_g(K_4) = 5 < \mathrm{grac}(K_4) = 6$ and $\chi_g(C_4) = \mathrm{grac}(C_4) = 4$.

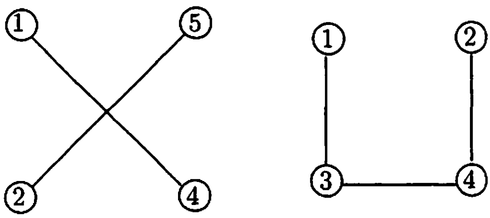

Figure 1: Graceful colorings of K_4 and C_4

We make some additional useful observations. For a graceful k-coloring c of a graph G, the *complementary coloring* $\overline{c} : V(G) \to [k]$ of G is a k-coloring defined by $\overline{c}(v) = k+1-c(v)$ for each vertex v of G. If $xy \in E(G)$, then the color $\overline{c}'(xy)$ of xy induced by $\overline{c}$ is

$$\begin{aligned}\overline{c}'(xy) &= |\overline{c}(x) - \overline{c}(y)| = |[(k+1) - c(x)] - [(k+1) - c(y)]| \\ &= |c(x) - c(y)| = c'(xy).\end{aligned}$$

This results in the following observation.

Observation 2.2 *The complementary coloring of a graceful coloring of a graph is also graceful.*

If c is a graceful k-coloring of a graph G, then the restriction of c to a subgraph H of G is also a graceful coloring. Thus, we have the following observation.

Observation 2.3 *If H is a subgraph of a graph G, then*

$$\chi_g(H) \le \chi_g(G).$$

If G is a disconnected graph having p components $G_1, G_2, \ldots, G_p$ for some integer $p \geq 2$, then $\chi_g(G) = \max\{\chi_g(G_i) : 1 \leq i \leq p\}$. Thus, it suffices to consider only nontrivial connected graphs. If c is a graceful coloring of a nontrivial connected graph G and $v \in V(G)$, then c must assign distinct colors to the vertices in the closed neighborhood $N[v]$ of v. Thus, if $u, w \in V(G)$ such that $u \neq w$ and $d(u, w) \leq 2$, then $c(u) \neq c(w)$. Furthermore, if (x, y, z) is an $x - z$ path in G, where $c(x) > c(z)$, say, then $c(x) - c(y) \neq c(y) - c(z)$ and so $c(y) \neq \frac{c(x)+c(z)}{2}$. We state these useful observations next.

Observation 2.4 *Let $c : V(G) \rightarrow [k]$, $k \geq 2$, be a coloring of a nontrivial connected graph G. Then c is a graceful coloring of G if and only if (i) for each vertex v of G. the vertices in the closed neighborhood $N[v]$ of v are assigned distinct colors by c and (ii) for each path (x, y, z) of order 3 in G, $c(y) \neq \frac{c(x)+c(z)}{2}$.*

As a consequence of condition (i) in Observation 2.4, it follows that if G is a nontrivial connected graph, then

$$\chi_g(G) \geq \Delta(G) + 1. \tag{1}$$

As an illustration, we determine $\chi_g(Q_3)$. Figure 2 shows a graceful 5-coloring of Q_3 and so $\chi_g(Q_3) \leq 5$. By (1), $\chi_g(Q_3) \geq 4$. Therefore, either $\chi_g(Q_3) = 4$ or $\chi_g(Q_3) = 5$. We show that $\chi_g(Q_3) \neq 4$. Assume, to the contrary, that Q_3 has a graceful 4-coloring using colors from the set [4]. By Observation 2.4, the four vertices in a 4-cycle in Q_3 must be colored differently. Thus, some vertex v of Q_3 is colored 3. However then, the three neighbors of v must be colored $1, 2, 4$, which implies that two incident edges of v are colored 1. This is impossible. Hence, $\chi_g(Q_3) = 5$.

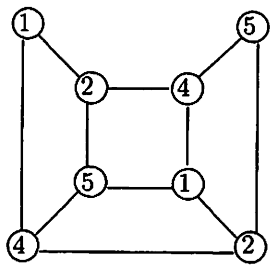

Figure 2: A graceful 5-coloring of Q_3

This example also illustrates the following observation.

Observation 2.5 *If G is an r-regular graph where $r \geq 2$, then*

$$\chi_g(G) \geq r + 2.$$

Since $\chi_g(K_{1,n-1}) = n = \Delta(K_{1,n-1}) + 1$, the bound in (1) is attained for all stars and consequently, the bound is sharp. By Brooks' theorem [1], $\chi(G) \leq \Delta(G) + 1$ for every graph G and, when G is connected, $\chi(G) = \Delta(G) + 1$ if and only if G is a complete graph or an odd cycle. Furthermore, by Vizing's theorem [6], $\chi'(G) \leq \Delta(G) + 1$ for every nonempty graph G. Thus, $\chi_g(G) \geq \max\{\chi(G), \chi'(G)\}$. These observations together with Observation 2.5 yield the following.

Proposition 2.6 *If G is a nontrivial connected graph of order at least 3, then*

$$\chi_g(G) \geq \max\{\chi(G), \chi'(G)\} + 1.$$

The *diameter* $\text{diam}(G)$ of a connected graph G is the largest distance between any two vertices of G. The following result is also a consequence of Observation 2.4.

Corollary 2.7 *If G is a connected graph of order $n \geq 3$ with diameter at most 2, then $\chi_g(G) \geq n$.*

While the star $K_{1,n-1}$, $n \geq 3$, is a graph of order n and diameter 2 having graceful chromatic number n, there are infinite classes of connected graphs having diameter 2 whose graceful chromatic number is its order.

Proposition 2.8 *If G is a complete bipartite graph of order $n \geq 3$, then*

$$\chi_g(G) = n.$$

Proof. Let $G = K_{s,t}$ be a complete bipartite graph of order $n = s+t$ with partite sets U and W, where $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_t\}$. Since the diameter of G is 2, it follows by Corollary 2.7 that $\chi_g(G) \geq n$. Next, consider a proper coloring $c : V(G) \to [n]$ defined by $c(u_i) = i$ for $1 \leq i \leq s$ and $c(w_j) = s + j$ for $1 \leq j \leq t$. Thus, $c'(u_iw_j) = |s + (j - i)|$ for $1 \leq i \leq s$ and $1 \leq j \leq t$. If i is fixed and $1 \leq j_1 \neq j_2 \leq t$, then $|s + (j_1 - i)| \neq |s + (j_2 - i)|$ and similarly, if j is fixed and $1 \leq i_1 \neq i_2 \leq s$, then $|s + (j - i_1)| \neq |s + (j - i_2)|$. Hence, c' is a proper edge coloring and c is a graceful n-coloring. Therefore, $\chi_g(G) = n$.

In fact, there are also infinite classes of connected graphs G of order n such that $\text{diam}(G) = 2$ and $\chi_g(G) > n$.

Proposition 2.9 *If G is a nontrivial connected graph of order n such that $\delta(G) > n/2$, then $\chi_g(G) > n$.*

Proof. Since $\delta(G) > n/2$, it follows that $\text{diam}(G) \leq 2$. Assume, to the contrary, that there is a graceful n-coloring c of G. By Observation 2.4, all vertices are assigned distinct colors by c and so there is a vertex v of G such that $c(v) = \lceil \frac{n}{2} \rceil$. Let $S = [1, \lceil \frac{n}{2} \rceil - 1]$ and $T = [\lceil \frac{n}{2} \rceil + 1, n]$, where then $|S| \leq |T| = n - \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor$. By Observation 2.4, at most one element in each set $\{\lceil \frac{n}{2} \rceil - i, \lceil \frac{n}{2} \rceil + i\}$ $(1 \leq i \leq \lceil \frac{n}{2} \rceil - 1)$ can be used to color the vertices in $N(v)$. Hence, there are at most $\lfloor \frac{n}{2} \rfloor$ colors that are available for the vertices in $N(v)$. Since $\deg v > n/2 \geq \lfloor \frac{n}{2} \rfloor$, this is impossible. Therefore, $\chi_g(G) > n$. ∎

3 Graceful Colorings of Cycles and Wheels

In this section, we determine the graceful chromatic numbers of some well-known graphs, namely cycles, paths and wheels. In order to determine the graceful chromatic number of a cycle, we first introduce some useful notation. Let $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ be a cycle of order $n \geq 3$ where $e_i = v_i v_{i+1}$ for $i = 1, 2, \ldots, n$. For a vertex coloring c of C_n, let

$$s_c = (c(v_1), c(v_2), \ldots, c(v_n)).$$

Similarly, for an edge coloring c' of C_n, let

$$s_{c'} = (c'(e_1), c'(e_2), \ldots, c'(e_n)).$$

Proposition 3.1 *For each integer $n \geq 4$,*

$$\chi_g(C_n) = \begin{cases} 4 & \text{if } n \neq 5 \\ 5 & \text{if } n = 5. \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ be a cycle of order $n \geq 4$ where $e_i = v_i v_{i+1}$ for $i = 1, 2, \ldots, n$. First, suppose that $n = 5$. Since $\text{diam}(C_5) = 2$, it follows by Corollary 2.7 that $\chi_g(C_5) \geq 5$. Define a vertex coloring c such that $s_c = (1, 5, 3, 4, 2)$. Then the induced edge coloring c' satisfies $s_{c'} = (4, 2, 1, 2, 1)$. Thus c is a graceful 5-coloring and so $\chi_g(C_n) = 5$.

Next, suppose that $n \neq 5$. First, we show that $\chi_g(C_n) \geq 4$. Assume, to the contrary, that there is a graceful 3-coloring c of C_n, say $c(v_1) = 1$. Since c is a graceful coloring, $\{c(v_2), c(v_n)\} = \{2, 3\}$, say $c(v_2) = 2$ and $c(v_n) = 3$. However then, $c(v_3) = 3$ and so $c'(v_1 v_2) = c'(v_2 v_3) = 1$, which is impossible. Hence, $\chi_g(C_n) \geq 4$. It remains to define a graceful 4-coloring c of C_n.

- $n \equiv 0 \pmod 4$. For $n = 4$, let $s_c = (1,2,4,3)$. Then $s_{c'} = (1,2,1,2)$. For $n \geq 8$, let $s_c = (1,2,4,3,\ldots,1,2,4,3)$. Then $s_{c'} = (1,2,\ldots,1,2)$.
- $n \equiv 1 \pmod 4$. For $n = 9$, let $s_c = (1,2,4,1,2,4,1,2,4)$.
  So $s_{c'} = (1,2,3,1,2,3,1,2,3)$.
  For $n \geq 13$, let $s_c = (1,2,4,3,\ldots,1,2,4,3,1,2,4,1,2,4,1,2,4)$.
  Then $s_{c'} = (1,2,1,2,\ldots,1,2,1,2,3,1,2,3,1,2,3)$.
- $n \equiv 2 \pmod 4$. For $n = 6$, let $s_c = (1,2,4,1,2,4)$.
  Then $s_{c'} = (1,2,3,1,2,3)$.
  For $n \geq 10$, let $s_c = (1,2,4,3,\ldots,1,2,4,3,1,2,4,1,2,4)$.
  Then $s_{c'} = (1,2,1,2,\ldots,1,2,1,2,3,1,2,3)$.
- $n \equiv 3 \pmod 4$. In this case, $n \geq 7$.
  Let $s_c = (1,2,4,3,\ldots,1,2,4,3,1,2,4)$.
  Then $s_{c'} = (1,2,1,2,\ldots,1,2,1,2,3)$.

In each case, there is a graceful 4-coloring of C_n. Therefore, $\chi_g(C_n) = 4$ when $n \neq 5$. ■

It is easy to see that $\chi_g(P_4) = 3$. For $n \geq 5$, the following is a consequence of Proposition 3.1.

Proposition 3.2 *For each integer $n \geq 5$, $\chi_g(P_n) = 4$.*

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$ where $n \geq 5$. For $n = 5$, a graceful 4-coloring c^* of P_5 is defined by

$$(c^*(v_1), c^*(v_2), c^*(v_3), c^*(v_4), c^*(v_5)) = (1,2,4,1,2)$$

and so $\chi_g(P_5) \leq 4$. For $n \geq 6$, since P_n is a subgraph of C_n, it follows by Observation 2.3 and Proposition 3.1 that $\chi_g(P_n) \leq 4$. We show that $\chi_g(P_n) \neq 3$. Suppose that there is a graceful 3-coloring c of P_n. Necessarily, $c(v_3) \neq 2$ and so we may assume that $c(v_3) = 1$. Thus, $\{c(v_2), c(v_4)\} = \{2,3\}$, say $c(v_2) = 2$. However then, $c(v_1) = 3$ and so $c'(v_1v_2) = c'(v_2v_3) = 1$, which is impossible. Therefore, $\chi_g(P_n) = 4$.

We now turn our attention to wheels W_n of order $n \geq 6$, constructed by joining a new vertex to every vertex of an $(n-1)$-cycle.

Theorem 3.3 *If W_n is the wheel of order $n \geq 6$, then $\chi_g(W_n) = n$.*

Proof. Let $G = W_n$, where $C_{n-1} = (v_1, v_2, \ldots, v_{n-1}, v_1)$ and whose central vertex is v_0. By Corollary 2.7, $\chi_g(G) \geq n$. Thus, it suffices to show that G has a graceful n-coloring. Figure 3 shows a graceful n-coloring of W_n for $n = 6, 7, 8$, where the central vertex is colored 1 and the graceful n-coloring of W_n for $n = 7, 8$ is obtained from the graceful $(n-1)$-coloring of W_{n-1} by inserting a new vertex into the cycle C_{n-2} of W_{n-1}, joining this vertex to the central vertex and then assigning the color n to this vertex.

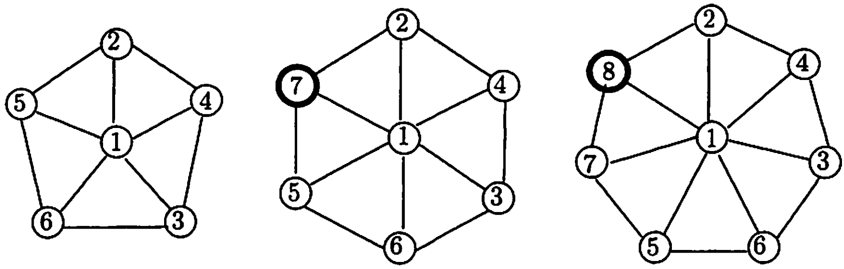

Figure 3: Graceful colorings of W_6, W_7, W_8

Next, we show that for a given graceful $(n-1)$-coloring of W_{n-1} for some integer $n \geq 7$, in which the central vertex is colored 1, there is an edge xy on the $(n-2)$-cycle C_{n-2} of W_{n-1} such that (1) a new vertex v can be inserted into the edge xy and joined to the central vertex v_0 of W_{n-1} to produce W_n and (2) the color n can be assigned to v to produce a graceful n-coloring of the resulting graph W_n. Now, let there be given a graceful $(n-1)$-coloring c of W_{n-1} for some integer $n \geq 7$, in which the central vertex is colored 1. It suffices to show that there exists an edge xy on C_{n-2} such that $c(x)$ and $c(y)$ satisfy the following two conditions:

(i) $c(x) \neq \frac{n+1}{2}$ and $c(y) \neq \frac{n+1}{2}$.

(ii) If (x', x, y, y') is a path on C_{n-2}, then $c(x) \neq \frac{c(x')+n}{2}$ and $c(y) \neq \frac{c(y')+n}{2}$.

Let $C_{n-2} = (v_1, v_2, \ldots, v_{n-2}, v_1)$. Since the diameter of W_{n-1} is 2, all vertices of W_{n-1} are assigned different colors by c. Hence, if $c(v_{i+1}) = \frac{c(v_{i+2})+n}{2}$ for some i, then $c(v_j) \neq \frac{c(v_{i+2})+n}{2}$ for all $j \neq i+1$ (where the subscripts are expressed as integers modulo $n-2$). We consider two cases.

Case 1. *n is odd.* Suppose that $c(v_{i+1}) = \frac{c(v_{i+2})+n}{2}$ for some i, in which case the edge $v_i v_{i+1}$ fails condition (ii). Since $n = 2c(v_{i+1}) - c(v_{i+2})$ is odd, it follows that $c(v_{i+2})$ is odd. Because there are $\frac{n-3}{2}$ vertices of C_{n-2} that are assigned odd colors by c (as the central vertex is colored 1), at most $\frac{n-3}{2}$

edges on C_{n-2} fail condition (ii). Hence, there are at least $(n-2) - \frac{n-3}{2} = \frac{n-1}{2} \geq 3$ edges on C_{n-2} that satisfy condition (ii). Among these edges that edges satisfy condition (ii), at most two of them fail condition (i). Thus, there is at least one edge xy on C_{n-2} such that $c(x)$ and $c(y)$ satisfy both (i) and (ii).

Case 2. *n is even.* Suppose that $c(v_{i+1}) = \frac{c(v_{i+2})+n}{2}$ for some i. Since $n = 2c(v_{i+1}) - c(v_{i+2})$ is even, it follows that $c(v_{i+2})$ is even. Because there are $\frac{n-2}{2}$ vertices on C_{n-2} that are assigned even colors by c, at most $\frac{n-2}{2}$ edges fail condition (ii). Hence, there are at least $(n-2) - \frac{n-2}{2} = \frac{n-2}{2} \geq 4$ edges that satisfy condition (ii). Since $(n+1)/2$ is not an integer, all of these edges satisfy condition (i) Therefore, there is at least one edge xy such that $c(x)$ and $c(y)$ satisfy both (i) and (ii).

4 Graceful Colorings of Regular Complete Multipartite Graphs

For the regular complete bipartite graph $K_{p,p}$, it follows by Proposition 2.8 that $\chi_g(K_{p,p}) = 2p$. Since $\delta(K_{p,p}) = p = n/2$, the result stated in Proposition 2.9 is best possible. This suggests considering other regular complete multipartite graphs. For integers p and k where $p \geq 2$ and $k \geq 3$, let $K_{k(p)}$ be the regular complete k-partite graph, each of whose partite sets consists of p vertices. Thus, the order of $K_{k(p)}$ is $n = kp$ and the degree of regularity is $r = \frac{n(k-1)}{k} = (k-1)p$. The following result gives an upper bound for the graceful chromatic number of $K_{k(p)}$. Because the verification of this bound is quite lengthy, we omit the proof.

Theorem 4.1 *For integers p and k where $p \geq 2$ and $k \geq 3$,*

$$\chi_g(K_{k(p)}) \leq \begin{cases} \left(2^{\frac{k+2}{2}} - 2\right)p - 2^{\frac{k-2}{2}} + 1 & \text{if } k \text{ is even} \\ \left(2^{\frac{k+3}{2}} - 3\right)p - 2^{\frac{k-1}{2}} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

The upper bound for $\chi_g(K_{k(p)})$ presented in Theorem 4.1 is almost certainly not sharp. While $\chi_g(K_{p,p,p}) \leq 5p-1$ for $p \geq 2$ according to Theorem 4.1, the following result gives an improved upper bound in this case. First, we introduce some useful notation. For a vertex coloring c of a graph G and a set X of vertices of G, let $c(X) = \{c(x) : x \in X\}$ be the set of colors of the vertices of X.

Theorem 4.2 *For each integer $p \geq 2$,*

$$\chi_g(K_{p,p,p}) \leq \begin{cases} 4p-1 & \text{if } p \text{ is even} \\ 4p & \text{if } p \text{ is odd.} \end{cases}$$

Proof. Let $G = K_{p,p,p}$ with partite sets V_1, V_2, V_3, where $|V_i| = p$ for $1 \le i \le 3$. First, suppose that p is even. Define a proper coloring $c : V(G) \to [4p-1]$ of G such that $c(V_1) = [p]$, $c(V_2) = [p+1, 2p - \frac{p}{2}] \cup [2p + \frac{p}{2}, 3p-1]$ and $c(V_3) = [3p, 4p-1]$. To show that c is a graceful coloring of G, it suffices to show that if (x, z, y) is a path of order 3 in G, then

$$\frac{c(x)+c(y)}{2} \neq c(z). \tag{2}$$

Let $x \in V_i, y \in V_j, z \in V_t$, where $1 \le i, j, t \le 3$ and $t \ne i, j$. We may assume that $j \le i$ and $c(y) \le c(x)$. If $t < j$, then $c(z) < c(y) \le \frac{c(x)+c(y)}{2}$. If $t > i$, then $\frac{c(x)+c(y)}{2} \le c(x) < c(z)$. Hence, we may assume that $j < t < i$ and so $j = 1, t = 2$ and $i = 3$. Observe that

$$\begin{aligned} \frac{c(x)+c(y)}{2} &\ge \frac{3p+1}{2} = 2p - \frac{p-1}{2} > 2p - \frac{p}{2} \\ \frac{c(x)+c(y)}{2} &\le \frac{p+4p-1}{2} = 2p + \frac{p-1}{2} < 2p + \frac{p}{2}. \end{aligned}$$

Thus, (2) holds.

Next, suppose that p is odd. A proper coloring $c : V(G) \to [4p]$ of G is defined by $c(V_1) = [p]$, $c(V_2) = [p+1, 2p - \lceil \frac{p}{2} \rceil] \cup [2p + \lceil \frac{p}{2} \rceil, 3p]$ and $c(V_3) = [3p+1, 4p]$. Let (x, z, y) be a path of order 3 in G. Suppose that $x \in V_i, y \in V_j, z \in V_t$, where $1 \le i, j, t \le 3$ and $t \ne i, j$. By an argument similar to the one used in Case 1, we may assume that $j = 1, t = 2$ and $i = 3$. Observe that

$$\begin{aligned} \frac{c(x)+c(y)}{2} &\ge \frac{(3p+1)+1}{2} > \frac{3p+1}{2} = 2p - \frac{p-1}{2} > 2p - \left\lceil \frac{p}{2} \right\rceil \\ \frac{c(x)+c(y)}{2} &\le \frac{p+4p}{2} < 2p + \frac{p+1}{2} = 2p + \left\lceil \frac{p}{2} \right\rceil. \end{aligned}$$

Thus, (2) holds. ∎

Indeed, there is a reason to believe that the upper bound for $\chi_g(K_{p,p,p})$ presented in Theorem 4.2 is the actual value of $\chi_g(K_{p,p,p})$ for every integer $p \ge 2$.

Conjecture 4.3 *For each integer $p \ge 2$,*

$$\chi_g(K_{p,p,p}) = \begin{cases} 4p-1 & \textit{if } p \textit{ is even} \\ 4p & \textit{if } p \textit{ is odd.} \end{cases}$$

Conjecture 4.3 has been verified when $2 \le p \le 6$. As an illustration, we verify this for $p = 3$.

Proposition 4.4 $\chi_g(K_{3,3,3}) = 12$.

Proof. By Theorem 4.2, $\chi_g(K_{3,3,3}) \leq 12$. Hence, it remains to show that there is no graceful 11-coloring of $G = K_{3,3,3}$. Let V_1, V_2, V_3 be the partite sets of G. Assume, to the contrary, that G has a graceful coloring $c : V(G) \to [11]$. Since $\text{diam}(G) = 2$, no two vertices of G are assigned the same color. First, we claim that the color 6 cannot be used; for otherwise, say $6 \in c(V_1)$. Then at least one color in each of the five sets $\{i, 12-i\}$ $(1 \leq i \leq 5)$ is either not used by c or is in $c(V_1)$. Since $|c(V_1)| = 3$ and exactly two colors in $[11]$ are not used by c, this is impossible. Thus, 6 is not used and so exactly nine of the ten colors in $[11] - \{6\}$ are used by c. We consider two cases.

Case 1. $5, 7 \in c(V(G))$. If $5, 7 \in c(V_i)$ for some $i = 1, 2, 3$, say $i = 1$, then one color in each of the four sets $\{1, 9\}$, $\{2, 8\}$, $\{3, 11\}$, $\{4, 10\}$ is either in $c(V_1)$ or is not used by c. Since $|c(V_1)| = 3$ and exactly one color in $[11] - \{6\}$ is not used c, this is impossible. Thus, we may assume that $5 \in c(V_1)$ and $7 \in c(V_2)$. Then the color 3 is either not used or is in $c(V_1)$ and the color 9 is either not used or is in $c(V_2)$. We may assume that $3 \in c(V_1)$ and so the color 4 is either not used or is in $c(V_1)$.

Subcase 1.1. $9 \in c(V_2)$. Then the color 8 is either not used or is in $c(V_2)$. We saw that the color 4 is either not used or is in $c(V_1)$. By symmetry, we may assume that $4 \in c(V_1)$. Then each of 10 and 11 is either not used or in $c(V_2)$. Therefore, each of the three colors 8, 10, 11 is either not used or is in $c(V_2)$. Since (i) $7, 9 \in c(V_2)$, (ii) at most one of 8, 10, 11 belongs to $c(V_2)$ and (iii) at most one of 8, 10, 11 is not used by c, at least one of 8, 10, 11 is in $c(V_3)$, a contradiction.

Subcase 1.2. 9 *is not used.* Then the colors used by c are 1, 2, 3, 4, 5, 7, 8, 10, 11. Since $3, 4, 5 \in c(V_1)$, it follows that $c(V_1) = \{3, 4, 5\}$. Because $2, 8 \notin c(V_1)$, the vertex colored 5 is incident with two edges colored 3, a contradiction.

Case 2. *Exactly one of* 5 *and* 7 *is used by* c, *say* 5. Then the colors used by c are $1, 2, 3, 4, 5, 8, 9, 10, 11$. We may assume that $5 \in c(V_1)$. Thus, at least one color in $\{2, 8\}$ and at least one color in $\{1, 9\}$ belongs to $c(V_1)$. Assume that $2 \in c(V_1)$. Thus, exactly one color in $\{1, 3\}$ belongs to $c(V_1)$. Since at least one color in $\{1, 9\}$ belongs to $c(V_1)$, it follows that $1 \in c(V_1)$ and so $c(V_1) = \{1, 2, 5\}$. However then, $3 \in c(V_2 \cup V_3)$ and the vertex colored 3 is incident with two edges colored 2, a contradiction. Thus, $2 \notin c(V_1)$ and so $8 \in c(V_1)$.

Next, suppose that $1 \in c(V_1)$. Thus, $c(V_1) = \{1, 5, 8\}$. However then, $3 \in c(V_2 \cup V_3)$ and the vertex colored 3 is incident with two edges colored 2, a contradiction. Thus, $1 \notin c(V_1)$ and so $9 \in c(V_1)$. Hence, $c(V_1) = \{5, 8, 9\}$.

We may assume that $1 \in c(V_2)$. Since $5 \in c(V_1)$, it follows that $3 \in c(V_2)$ and so $2 \in c(V_2)$. Thus, $c(V_2) = \{1, 2, 3\}$ and $c(V_3) = \{4, 10, 11\}$. However then, the vertex colored 4 is incident with two edges colored 1, producing a contradiction. ∎

The proof of Proposition 4.4 shows not only that $\chi_g(K_{3,3,3}) = 12$ but that there is a vertex coloring $c : V(G) \to [11]$ of $G = K_{3,3,3}$ that is a proper vertex coloring, namely $c(V_1) = \{5, 8, 9\}$, $c(V_2) = \{1, 2, 3\}$ and $c(V_3) = \{4, 10, 11\}$, whose induced edge coloring c' results only in one pair of adjacent edges having the same color.

5 Graceful Colorings of Caterpillars

We now determine the graceful chromatic numbers of some well-known trees. A *caterpillar* is a tree T of order 3 or more, the removal of whose leaves produces a path (called the *spine* of T). Thus, every path, every star (of order at least 3) and every double star (a tree of diameter 3) is a caterpillar.

Theorem 5.1 *Let T be a caterpillar with maximum degree $\Delta \geq 2$. If T has a vertex of degree Δ that is adjacent to two vertices of degree Δ in T, then $\chi_g(T) = \Delta + 2$.*

Proof. Since the theorem holds when $\Delta = 2$ by Proposition 3.2, we may assume that $\Delta \geq 3$. First, we show that $\chi_g(T) \geq \Delta + 2$. Assume, to the contrary, that $\chi_g(T) \leq \Delta + 1$. It then follows by (1) that $\chi_g(T) = \Delta + 1$ and so T has a graceful coloring c using colors from $[\Delta + 1]$. Let $v \in V(G)$ with $\deg v = \Delta$. Suppose that $c(v) = a$. If $1 < a < \Delta + 1$, then there are two neighbors u and w of v such that $c(u) = a + 1$ and $c(w) = a - 1$. However then, $c'(uv) = c'(wv) = 1$, which is impossible. Since $c(N[v]) = [\Delta + 1]$, there is $u \in N(v)$ such that $c(u) = a + \Delta$ or $c(u) = a - \Delta$. Because $a \leq \Delta + 1$, either $a = 1$ or $a = \Delta + 1$. Hence, every vertex of degree Δ is colored 1 or $\Delta + 1$. However, T has a vertex of degree Δ that is adjacent to two vertices of degree Δ in T and so c is not proper, which is a contradiction. Therefore, $\chi_g(T) \geq \Delta + 2$.

To verify that $\chi_g(T) \leq \Delta + 2$, it suffices to show that there is a graceful coloring c of T using colors in $[\Delta + 2]$. First, we consider a caterpillar T^* with $\Delta(T^*) = \Delta$ such that each non-end-vertex of T^* has degree Δ and the spine of T^* is a path $(v_1, v_2, \ldots, v_{3k+1})$ of order $3k + 1$. Thus $\deg_{T^*} v_i = \Delta$ for each i with $1 \leq i \leq 3k + 1$. We show that $\chi_g(T^*) \leq \Delta + 2$. Define a proper coloring $c : V(T^*) \to [\Delta + 2]$ of T^* as follows. First, let

$$c(v_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod{3} \\ 2 & \text{if } i \equiv 2 \pmod{3} \\ \Delta + 1 & \text{if } i \equiv 0 \pmod{3}. \end{cases}$$

Let $e_i = v_iv_{i+1}$ for $1 \le i \le 3k$. The induced edge coloring c' satisfies

$$(c'(e_1), c'(e_2), \ldots, c'(e_n)) = (1, \Delta - 1, \Delta, 1, \Delta - 1, \Delta, \ldots, 1, \Delta - 1, \Delta).$$

For each integer i with $1 \le i \le 3k + 1$, let $L(v_i)$ be the set of leaves that are adjacent to v_i.

- ⋆ Let $c(L(v_1)) = [3, \Delta + 1]$ and let $c(L(v_{3k+1})) = [2, \Delta]$.
- ⋆ If $i \equiv 1 \pmod 3$ and $i \ne 1, 3k + 1$, let $c(L(v_i)) = [3, \Delta]$.
- ⋆ If $i \equiv 2 \pmod 3$, let $c(L(v_i)) = [4, \Delta] \cup \{\Delta + 2\}$.
- ⋆ If $i \equiv 0 \pmod 3$, let $c(L(v_i)) = [3, \Delta]$.

This is illustrated in Figure 4 for $k = 1$. For each vertex v_i $(1 \le i \le 3k+1)$, let E_{v_i} be the set of edges incident with v_i. Then $c'(E_{v_i}) = [\Delta]$. Hence, c' is proper and so c is a graceful $(\Delta + 2)$-coloring of T^*. Therefore, $\chi_g(T^*) = \Delta + 2$.

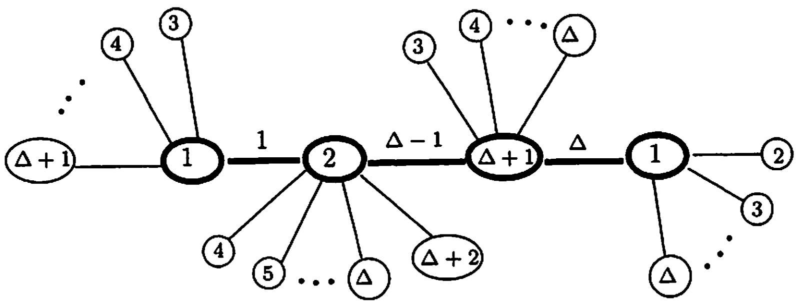

Figure 4: Illustrating the coloring c for $k = 1$

Next, let T be a caterpillar with maximum degree $\Delta \ge 3$ such that some vertex of degree Δ in T is adjacent to at least two vertices of degree Δ in T. Then there is a caterpillar T^* with $\Delta(T^*) = \Delta$ having the structure as described above such that T is a subtree of T^*. By Observation 2.3, $\chi_g(T) \le \chi_g(T^*) = \Delta + 2$. Therefore, $\chi_g(T) = \Delta + 2$.

By Theorem 5.1, if T is a caterpillar with maximum degree 3 containing a vertex of degree 3 adjacent to two vertices of degree 3 in T, then $\chi_g(T) = 5$. If T has no such vertex, then we show that $\chi_g(T) = 4$.

Theorem 5.2 *Let T be a caterpillar with maximum degree 3. If every vertex of degree 3 in T is adjacent to at most one vertex of degree 3 in T, then $\chi_g(T) = 4$.*

Proof. Let $P = (v_1, v_2, \ldots, v_t)$ be the spine of T. Thus, every vertex of P has degree 2 or 3 in T and every vertex of T not on P is an end-vertex of T. Since the result holds for a star by Proposition 2.8, we may assume that $t \geq 2$ and at least one vertex of P has degree 3 in T. If P has only vertices of degree 3 in T, then $t = 2$ and both v_1 and v_2 have degree 3 in T. Color one of these vertices 1 and the other 4. We may assume that P contains a vertex of degree 3 in T immediately followed by a vertex of degree 2 in T (otherwise, we may let $P = (v_t, v_{t-1}, \ldots, v_1)$).

Let v_a be the first vertex of degree 3 on P immediately followed by a vertex of degree 2 in T. Assign v_a the color 1 or 4. If v_{a-1} also has degree 3 in T, then assign this vertex a color such that v_{a-1}, v_a are colored $1, 4$ or $4, 1$. If no vertex of P following v_a has degree 3, then color the vertices $v_{a+1}, v_{a+2}, \ldots, v_t$ with $2, 4, 3, 1, 2, 4, 3, 1, \ldots$ if v_a is colored 1; while $v_{a+1}, v_{a+2}, \ldots, v_t$ are colored with $3, 1, 2, 4, 3, 1, 2, 4, \ldots$ if v_a is colored 4. If all vertices prior to v_a on P have degree 2 in T, then color $v_{a-1}, v_{a-2}, \ldots, v_1$ with $3, 4, 2, 1, 3, 4, 2, 1, \ldots$ if v_a is colored 1; or $2, 1, 3, 4, 2, 1, 3, 4, \ldots$ if v_a is colored 4. If v_{a-1} has degree 3 and is colored 1, then $v_{a-2}, v_{a-3}, \ldots, v_1$ are colored $3, 4, 2, 1, 3, 4, 2, 1, \ldots$; while if v_{a-1} has degree 3 and is colored 4, then $v_{a-2}, v_{a-3}, \ldots, v_1$ are colored $2, 1, 3, 4, 2, 1, 3, 4, \ldots$.

Thus, we may now assume that there are one or more vertices following v_a that have degree 3 in T. Let v_b be the first vertex following v_a that has degree 3 in T. We now consider two cases.

Case 1. a and b are of the same parity. Then $v_{a+1}, v_{a+2}, \ldots, v_b$ are colored $2, 4, 3, 1, 2, 4, 3, 1, \ldots$ if v_a is colored 1; while $v_{a+1}, v_{a+2}, \ldots, v_b$ are colored $3, 1, 2, 4, 3, 1, 2, 4, \ldots$ if v_a is colored 4.

Case 2. a and b are of opposite parity. If v_{a-1} also has degree 3, then $v_{a+1}, v_{a+2}, \ldots, v_b$ are colored with $3, 2, 4, 3, 1, 2, 4, 3, 1, \ldots$ if v_a is colored 1; while $v_{a+1}, v_{a+2}, \ldots, v_b$ are colored $2, 3, 1, 2, 4, 3, 1, 2, 4, \ldots$ if v_a is colored 4. If $a = 1$ or v_{a-1} has degree 2 in T, then $v_{a+1}, v_{a+2}, \ldots, v_b$ are colored with $4, 3, 1, 2, 4, 3, 1, 2, 4, \ldots$ if v_a is colored 1; while $v_{a+1}, v_{a+2}, \ldots, v_b$ are colored $1, 2, 4, 3, 1, 2, 4, 3, 1, \ldots$ if v_a is colored 4.

If there are vertices of degree 2 and 3 in T following v_b on P, then relabel v_b as v_a if v_{b+1} has degree 2 in T or relabel v_{b+1} as v_a if v_{b+1} has degree 3 in T. We then proceed as above.

Since each vertex of degree 3 is colored 1 or 4, there is a color available for each end-vertex of T that results in both a proper vertex coloring of T and a proper induced edge coloring of T. Thus, $\chi_g(T) = 4$.

Next, we show that a theorem analogous to Theorem 5.2 holds when $\Delta \geq 4$.

Theorem 5.3 *Let T be a caterpillar with maximum degree $\Delta \geq 4$. If no vertex of degree Δ in T is adjacent to two vertices of degree Δ in T, then*

$$\chi_g(T) = \Delta + 1.$$

Proof. Since $\chi_g(T) \geq \Delta + 1$, it remains to show that $\chi_g(T) \leq \Delta + 1$. Let T be a caterpillar with maximum degree $\Delta \geq 4$ in which

(i) no vertex of degree Δ in T is adjacent to two vertices of degree Δ in T.

Adding leaves to the tree T if necessarily, we may further assume that

(ii) each vertex on the spine of T has degree Δ and $\Delta - 1$ in T and that no vertices of degree $\Delta - 1$ is adjacent to two vertices of degree $\Delta - 1$ in T.

We establish the following stronger statement:

> If T is a caterpillar with maximum degree $\Delta \geq 4$ that satisfies (i) and (ii), then T has a graceful $(\Delta + 1)$-coloring c such that a vertex v on the spine of T is colored 1 or $\Delta + 1$ by c if and only if $\deg_T v = \Delta$.

We proceed by induction on the order ℓ of the spine of a tree. If $\ell = 1$, then T is a star and the statement is true by Proposition 2.8. If $\ell = 2$, then T is a double star. Let v_1 and v_2 be the two central vertices (non-end-vertices) of T. First, suppose that $\deg_T v_1 = \deg_T v_2 = \Delta$. Assign the color 1 to v_1, the color $\Delta + 1$ to v_2, assign the colors in $[2, \Delta]$ to the vertices in $N(v_1) - \{v_2\}$ and the vertices in $N(v_2) - \{v_1\}$. Next, suppose that exactly one of v_1 and v_2 has degree Δ, say v_1. Assign the color 1 to v_1, the color 2 to v_2, the colors in $[3, \Delta + 1]$ to the vertices in $N(v_1) - \{v_2\}$ and the colors in $[4, \Delta + 1]$ to the vertices in $N(v_1) - \{v_2\}$. In each case, T has a graceful $(\Delta + 1)$-coloring with the desired property. This establishes the base step.

Assume that if T' is a tree of maximum degree $\Delta' \geq 4$, the length of whose spine is $\ell - 1$ for some $\ell \geq 3$ such that T' satisfies (i) and (ii), then T' has a graceful $(\Delta' + 1)$-coloring such that a vertex on the spine of T' is colored 1 or $\Delta' + 1$ if and only if $\deg_{T'} v = \Delta'$. Let T be a tree of maximum degree $\Delta \geq 4$ the length of whose spine is ℓ such that T satisfies (i) and (ii). Let $P = (v_1, v_2, \ldots, v_\ell)$ be the spine of T. We may assume that there is $i \in \{2, 3, \ldots, \ell\}$ such that $\deg_T v_i = \Delta$; for otherwise, v_1 is the only vertex of degree Δ in T and let $P = (v_\ell, v_{\ell-1}, \ldots, v_1)$. Let T' be the caterpillar obtained from T by removing all leaves adjacent to v_1. Then T' is a caterpillar of maximum degree $\Delta \geq 4$, whose spine is $P' = (v_2, v_3, \ldots, v_\ell)$ of length $\ell - 1$. Since $\deg_{T'} v_i = \deg_T v_i$ for $2 \leq i \leq \ell$, it follows that T' satisfies (i) and (ii). By the induction hypothesis, T' has a graceful $(\Delta+1)$-coloring c such that a vertex v on P' is colored 1 or $\Delta+1$ by c if and only if $\deg_{T'} v = \Delta$.

Next, we show that that T has a graceful $(\Delta+1)$-coloring c_T such that

$$\text{a vertex } v \text{ on } P \text{ is colored } 1 \text{ or } \Delta+1 \text{ by } c_T \text{ if and only if } \deg_T v = \Delta. \tag{3}$$

We consider four cases, according to the degrees of v_1 and v_2.

Case 1. $\deg_T v_1 = \deg_T v_2 = \Delta$. Let $N_T(v_1) = \{v_2, x_1, x_2, \ldots, x_{\Delta-1}\}$. Then $\deg_T v_3 = \Delta - 1$. By the induction hypothesis, $c(v_2) \in \{1, \Delta+1\}$ and $c(v_3) \notin \{1, \Delta+1\}$. By Observation 2.2, we may assume $c(v_2) = \Delta+1$. Thus, one of the leaves adjacent to v_2 is colored 1, say v_1. Define a vertex coloring c_T of T by $c_T(v) = c(v)$ if $v \in V(T')$ and $c_T(x_i) = i+1$ for $x_i \in N(v_1)$. Then c_T is a graceful $(\Delta+1)$-coloring of T that satisfies (3).

Case 2. $\deg_T v_1 = \Delta$ *and* $\deg_T v_2 = \Delta-1$. Let $N_T(v_1)=\{v_2, x_1, x_2, \ldots, x_{\Delta-1}\}$. By the induction hypothesis, $c(v_2) \notin \{1, \Delta+1\}$. Furthermore, we may assume that $c(v_3) \neq 1$ by Observation 2.2. If v_2 is adjacent to a leaf that is colored 1, then we may assume that this leaf is v_1; while if v_2 is not adjacent to a leaf colored 1, then $c(v_2) = 2$ and there exists a leaf adjacent to v_2 colored 3. In this case, we may assume that this leaf is v_1 and change the color of v_1 to 1 such that the resulting coloring is still graceful. Define a vertex coloring c_T of T by $c_T(v) = c(v)$ if $v \in V(T')$ and $c_T(x_i) = i+2$ for $x_i \in N_T(v_1)$. Then c_T is a graceful $(\Delta+1)$-coloring of T satisfying (3).

Case 3. $\deg_T v_1 = \Delta - 1$ *and* $\deg_T v_2 = \Delta$. Let $N_T(v_1)= \{v_2, x_1, x_2, \ldots, x_{\Delta-2}\}$. Then $c(v_2) \in \{1, \Delta+1\}$. By Observation 2.2, we may assume $c(v_3) \neq 2$. Hence, v_2 is adjacent to a leaf adjacent that is colored 2, say v_1. Define a vertex coloring c_T of T by $c_T(v) = c(v)$ if $v \in V(T')$, and $c_T(x_i) = i+2$ for each $x_i \in N_T(v_1) - \{x_1\}$. If $c(v_2) = 1$, then let $c_T(x_1) = \Delta+1$; while if $c(v_2) = \Delta+1$, then let $c_T(x_1) = 3$. Then c_T is a graceful $(\Delta+1)$-coloring of T satisfying (3).

Case 4. $\deg_T v_1 = \deg_T v_2 = \Delta-1$. Let $N_T(v_1) = \{v_2, x_1, x_2, \ldots, x_{\Delta-2}\}$. Since no vertex of degree $\Delta-1$ is adjacent to two vertices of degree $\Delta-1$, it follows that $\deg_T v_3 = \Delta$. Thus, $c(v_3) \in \{1, \Delta+1\}$. By Observation 2.2, we may assume $c(v_2) \neq 2$. Then v_2 is adjacent to a leaf colored 2, say v_1. Define a vertex coloring c_T of T by $c_T(v) = c(v)$ if $v \in V(T')$, $c_T(x_i) = i+2$ for $1 \leq i \leq \Delta-3$ and $c_T(x_{\Delta-2}) = \Delta+1$. Then c_T is a graceful $(\Delta+1)$-coloring of T satisfying (3).

By the Principle of Mathematical Induction, if T is a caterpillar with maximum degree $\Delta \geq 4$ that satisfies (i) and (ii), then T has a graceful $(\Delta+1)$-coloring c such that a vertex v on the spine of T is colored 1 or $\Delta+1$ by c if and only if $\deg_T v = \Delta$. Since every caterpillar with maximum degree Δ and satisfying (i) is a subtree of a caterpillar with maximum degree Δ and satisfying (i) and (ii), the result follows by Observation 2.3.

By Theorems 5.1–5.3, the following result provides the graceful chromatic number of every caterpillar.

Theorem 5.4 *If T is a caterpillar with maximum degree $\Delta \geq 2$, then*

$$\Delta + 1 \leq \chi_g(T) \leq \Delta + 2.$$

Furthermore, $\chi_g(T) = \Delta + 2$ if and only if T has a vertex of degree Δ that is adjacent to two vertices of degree Δ in T.

6 An Upper Bound for the Graceful Chromatic Number of a Tree

We have seen examples of trees T for which $\chi_g(T) = \Delta(T) + 1$ and trees T for which $\chi_g(T) = \Delta(T) + 2$. This brings up the question of whether there exists a tree T such that $\chi_g(T) - \Delta(T) > 2$. To answer this question, we consider the tree T_0 with $\Delta(T_0) = 4$ shown in Figure 5. First, we claim that there is no graceful 6-coloring of T_0. Suppose that there is such a coloring $c : V(T_0) \to [6]$. The vertices in $N[u]$ are then colored with five colors from the set $[6]$. If $c(u) = 3$, then no two vertices in $N(u)$ can be colored both 2 and 4 or both 1 or 5 by Observation 2.4. Similarly, it is impossible that $c(u) = 4$. Thus, $c(u) \in \{1, 2, 5, 6\}$. The same can be said of v, w, x and y. This implies that two vertices of $N[u]$ are colored same, which is impossible. Since the 7-coloring of T_0 shown in Figure 5 is a graceful coloring, it follows that $\chi_g(T_0) = 7 = \Delta(T_0) + 3$.

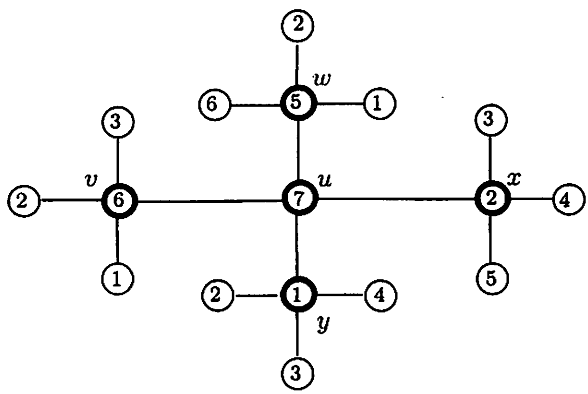

Figure 5: A tree T_0 with $\chi_g(T_0) = \Delta(T_0) + 3$

For the tree T_0 in Figure 5, observe that $\chi_g(T_0) = 7 = \left\lceil \frac{5\Delta(T_0)}{3} \right\rceil$. Indeed, for every tree T with maximum degree Δ, the graceful chromatic number of T can never exceed $\lceil \frac{5\Delta}{3} \rceil$, as we now show.

Theorem 6.1 *If T is a nontrivial tree with maximum degree Δ, then*

$$\chi_g(T) \leq \left\lceil \frac{5\Delta}{3} \right\rceil.$$

Proof. Let $S_1 = [\lceil \frac{2\Delta}{3} \rceil]$, $S_2 = [\Delta + 1, \lceil \frac{5\Delta}{3} \rceil]$ and $S = S_1 \cup S_2$. In order to show that T has a graceful coloring using the colors in S, we first verify the following claim.

> **Claim.** For each $a \in S$, there are at least Δ distinct elements $a_1, a_2, \ldots, a_\Delta \in S - \{a\}$ such that all of the Δ integers $|a - a_1|, |a - a_2|, \ldots, |a - a_\Delta|$ are distinct.

We consider three cases, according to the values of Δ modulo 3.

Case 1. $\Delta \equiv 0 \pmod 3$. Let $\Delta = 3k$ for some positive integer k. Then $\lceil \frac{2\Delta}{3} \rceil = 2k$ and so $S_1 = [2k]$ and $S_2 = [3k + 1, 5k]$. Let $a \in S$. By Observation 2.2, we may assume that $a \in S_1$. For each $i = 1, 2, \ldots, 2k$, let $a_i = 3k + i$. Then all of $|a - a_1|, |a - a_2|, \ldots, |a - a_{2k}|$ are distinct and $|a - a_i| = 3k + i - a \geq 3k + i - 2k = k + i \geq k + 1$ for $1 \leq i \leq 2k$. If $a \leq k$, then choose $a_{2k+j} = a + j$ for $1 \leq j \leq k$; while if $a \geq k + 1$, then choose $a_{2k+j} = a - j$ for $1 \leq j \leq k$. Then all of $|a - a_{2k+1}|, |a - a_{2k+2}|, \ldots, |a - a_{3k}|$ are distinct and $|a - a_{2k+j}| = j \leq k$. Since $|a - a_i| \geq k + 1$ for $1 \leq i \leq 2k$ and $|a - a_i| \leq k$ for $2k + 1 \leq i \leq 3k$, it follows that $|a - a_1|, |a - a_2|, \ldots, |a - a_{3k}|$ are distinct.

Case 2. $\Delta \equiv 1 \pmod 3$. Let $\Delta = 3k + 1$ for some nonnegative integer k. Then $\lceil \frac{2\Delta}{3} \rceil = 2k + 1$ and so $S_1 = [2k + 1]$ and $S_2 = [3k + 2, 5k + 2]$. Let $a \in S$. As observed in Case 1, we may assume that $a \in S_1$. For each $i = 1, 2 \ldots, 2k + 1$, let $a_i = 3k + 1 + i$. Then all of $|a - a_1|, |a - a_2|, \ldots, |a - a_{2k+1}|$ are distinct and $|a - a_i| = 3k + 1 + i - a \geq 3k + 1 + i - (2k + 1) = k + i \geq k + 1$ for $1 \leq i \leq 2k + 1$. If $a \leq k$, then choose $a_{2k+1+j} = a + j$ for $1 \leq j \leq k$; while if $a \geq k + 1$, then choose $a_{2k+1+j} = a - j$ for $1 \leq j \leq k$. Then all of $|a - a_{2k+2}|, |a - a_{2k+3}|, \ldots, |a - a_{3k+1}|$ are distinct and $|a - a_{2k+1+j}| = j \leq k$. Since $|a - a_i| \geq k + 1$ for $1 \leq i \leq 2k + 1$ and $|a - a_i| \leq k$ for $2k + 2 \leq i \leq 3k + 1$, it follows that $|a - a_1|, |a - a_2|, \ldots, |a - a_{3k+1}|$ are distinct.

Case 3. $\Delta \equiv 2 \pmod 3$. Let $\Delta = 3k + 2$ for some nonnegative integer k. Then $\lceil \frac{2\Delta}{3} \rceil = 2k + 2$ and so $S_1 = [2k + 2]$ and $S_2 = [3k + 2, 5k + 4]$. The argument is similar to the one in Case 2.

Therefore, the claim holds. It remains to construct a graceful coloring c of T using the colors in S. Let $v \in V(T)$ such that $\deg v = \Delta$ and let $V_i =$

$\{w \in V(T) : d(v,w) = i\}$ for $0 \le i \le e(v)$, where $e(v)$ is the eccentricity of v. Thus, $V_0 = \{v\}$ and $V_1 = N(v)$. Let $c(v) = a$ for some $a \in S$ and let $a_1, a_2, \ldots, a_\Delta \in S - \{a\}$ for which $|a - a_1|, |a - a_2|, \ldots, |a - a_\Delta|$ are distinct. Color the vertices of V_1 such that $\{c(w) : w \in V_1\} = \{a_1, a_2, \ldots, a_\Delta\}$. Thus each vertex in $V_0 \cup V_1$ has been assigned a color from S such that all vertices and edges of the tree $T_1 = T[V_0 \cup V_1]$ are properly colored. Suppose then, for some integer i where $1 \le i < e(v)$, that the colors of vertices in the tree $T_i = T\left[\cup_{j=0}^{i} V_j\right]$ have been assigned colors from S such that all vertices and edges of T_i are properly colored. Next, we define the colors of vertices in V_{i+1}. Let $w \in V_i$ that is not an end-vertex of T. Suppose that $\deg w = t \le \Delta$ and $c(w) = b \in S$. Choose $b_1, b_2, \ldots, b_\Delta \in S - \{b\}$ such that $|b - b_1|, |b - b_2|, \ldots, |b - b_\Delta|$ are distinct. Let $u \in V_{i-1}$ such that $uw \in E(T)$. We may assume, without loss of generality, that $b_j \ne c(u)$ and $b_j \ne 2c(w) - c(u)$ for $1 \le j \le t - 1 \le \Delta - 1$. Color the vertices in $N(w) - \{u\} \subseteq V_{i+1}$ such that $\{c(w) : w \in N(w) - \{u\}\} = \{b_1, b_2, \ldots, b_{t-1}\}$. Continue this procedure for each non-end-vertex in V_i to define the color of each vertex in V_{i+1}. Therefore, T has a graceful coloring using colors from the set $S \subseteq \left[\lceil \frac{5\Delta}{3} \rceil\right]$ and so $\chi_g(T) \le \lceil \frac{5\Delta}{3} \rceil$.

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