On the local genus distribution of graph embeddings

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Abstract

The 2-cell embeddings of graphs on closed surfaces have been widely studied. It is well known that (2-cell) embedding a given graph G on a closed orientable surface is equivalent to cyclically ordering the darts incident to each vertex of G. In this paper, we study the following problem: given a genus g embedding ϵ of the graph G and a vertex of G, how many different ways of reembedding the vertex such that the resulting embedding ϵ' is of genus $g+\Delta g$? We give formulas to compute this quantity and the local minimal genus achieved by reembedding. In the process we obtain miscellaneous results. In particular, if there exists a one-face embedding of G, then the probability of a random embedding of G to be one-face is at least $\prod_{\nu \in V(G)} \frac{2}{deg(\nu)+2}$, where $deg(\nu)$ denotes the vertex degree of ν. Furthermore we obtain an easy-to-check necessary condition for a given embedding of G to be an embedding of minimum genus.

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1 Introduction

Graph embedding is one of the most important topics in topological graph theory. In particular, 2-cell embeddings of graphs (loops and multiple edges allowed) have been widely studied. A *2-cell embedding* or *map* of a given graph G on a closed surface of genus g, S_g, is an embedding on S_g such that the complement of any face is homeomorphic to an open disk. The closed surfaces could be either orientable or nonorientable. In this paper, we restrict ourselves to the orientable case.

Let $g_{min}(G)$ and $g_{max}(G)$ denote the minimum and the maximum genus g of the embeddings of G, respectively. There are many studies on determining these quantities and related problems [2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 17, 18, 19, 20, 22]. Assume G has e edges and v vertices, and that G is embedded in S_g via ϵ. In view of Euler's characteristic formula,

$$v - e + f = 2 - 2g \iff 2g = \beta(G) + 1 - f, \tag{1}$$

where $f \geq 1$ is the number of faces of ϵ and $\beta(G) = e - v + 1$ is the *Betti number* of G. Thus, the largest possible value of g is $\lfloor \frac{\beta(G)}{2} \rfloor$.

It is well known that any embedding of G on a closed orientable surface can be equivalently represented by a combinatorial map generated by G, see for instance [3, 6, 14]. We call an end of an edge a *dart*. A *combinatorial map* generated by G is the graph G with a specified cyclic order of darts around (i.e., incident to) each vertex of G, i.e. the topological properties of the embedding are implied in the cyclic orderings of darts. Any variation of the local topological structure around a vertex, i.e., the cyclic order of darts around the vertex, may change the topological properties of the whole embedding, e.g. the genus of the embedding. We will interchangeably use the terms embedding, map and combinatorial map in the following.

Plane permutations [1] were recently used to study hypermaps, in particular to enumerate hypermaps with one face, the transposition and block-interchange distance of permutations as well as the reversal distance of signed permutations. Since maps are particular hypermaps, we can employ plane permutations in order to study graph embeddings.

The paper is organized as follows: in Section 2, we establish some basic facts on hypermaps. In Section 3, we study embeddings with one face. We ask which local embeddings (reembeddings) of a fixed vertex do not affect the topological genus. By changing the local embedding, we mean changing the cyclic order of darts around the vertex. We shall show that the probability of preserving the genus is at least $\frac{2}{deg(\nu)+2}$ for reembedding any vertex ν of degree (i.e., valence) $deg(\nu)$. In addition we show that there exists at least one alternative way to reembed a vertex ν preserving the genus if $deg(\nu) \geq 4$.

In Section 4, we study embeddings with more than one face: given a genus g embedding ϵ of the graph G and a vertex of G, how many different ways of reembedding the vertex such that the resulting embedding ϵ' is of genus $g + \Delta g$? We give a formula to compute this quantity and the local minimal genus achieved by reembedding, as well as an easy-to-check necessary condition for an embedding of G to be of minimum genus.

2 Maps, hypermaps and plane permutations

Let S_n denote the group of permutations, i.e. the group of bijections from $[n] = \{1, \dots, n\}$ to $[n]$, where the multiplication is the composition of maps. We shall use the following two representations of a permutation π:
two-line form: the top line lists all elements in $[n]$, following the natural order. The bottom line lists the corresponding images of elements on the top line, i.e.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n-2) & \pi(n-1) & \pi(n) \end{pmatrix}.$$

cycle form: regarding $\langle \pi \rangle$ as a cyclic group, we represent π by its collection of orbits (cycles). The set consisting of the lengths of these disjoint cycles is called the *cycle-type* of π. We can encode this set into a non-increasing integer sequence $\lambda = \lambda_1 \lambda_2 \cdots$, where $\sum_i \lambda_i = n$, or as $1^{a_1} 2^{a_2} \cdots n^{a_n}$, where we have a_i cycles of length i. A cycle of length k will be called a *k-cycle.*

For a permutation π on $[n]$, we denote its total number of cycles by $C(\pi)$, and we denote Par_π the partition of $[n]$ induced by the cycles of π, i.e., every set of elements in a same cycle of π contributes a part (or block) in Par_π.

A combinatorial map having n edges can be represented as a triple of permutations (α, β, γ) on $[2n]$ where α is a fixed-point free involution and $\gamma = \alpha\beta$. This can be seen as follows: we label the darts of the combinatorial map using the labels from the set $[2n]$ so that each label appears exactly once. This induces two permutations α and β, where α is a fixed point free involution, whose cycles consist of the labels of the two darts of the same (untwisted) edge and β-cycles represent the counterclockwise cyclic arrangement of all darts incident to the same vertex. $\gamma = \alpha\beta$-cycles are called the *faces*. The topological genus of a map (α, β, γ) satisfies

$$C(\beta) - C(\alpha) + C(\gamma) = 2 - 2g. \tag{2}$$

Any map induces a unique graph G, via (α, Par_β), where each block corresponds to a G-vertex and each α-cycle determines a G-edge.

Hypermaps represent a generalization of maps by allowing hyper-edges, i.e., triples (α, β, γ), where $\gamma = \alpha\beta$ and α is not necessarily fixed point free. We can also define genus g of a hypermap (α, β, γ) on $[n]$, e.g., [21], by

$$C(\alpha) + C(\beta) + C(\gamma) - n = 2 - 2g. \tag{3}$$

We will also call the pair $G = (\alpha, Par_\beta)$ the underlying graph of the hypermap although it does not induce a conventional graph. Furthermore, any hypermap $(\alpha', \beta', \gamma')$ having G as the underlying graph is called an embedding of G.

Definition 2.1 (Cyclic plane permutation). A *cyclic plane permutation* on $[n]$ is a pair $\mathfrak{p} = (s, \pi)$ where $s = (s_i)_{i=0}^{n-1}$ is an n-cycle and π is an arbitrary permutation on $[n]$. The permutation $D_\mathfrak{p} = s \circ \pi^{-1}$ is called the *diagonal* of $\mathfrak{p}$.

Given $s = (s_0 s_1 \cdots s_{n-1})$, a cyclic plane permutation $\mathfrak{p} = (s, \pi)$ can be represented by two aligned rows:

$$(s, \pi) = \begin{pmatrix} s_0 & s_1 & \cdots & s_{n-2} & s_{n-1} \\ \pi(s_0) & \pi(s_1) & \cdots & \pi(s_{n-2}) & \pi(s_{n-1}) \end{pmatrix}. \quad (4)$$

Indeed, $D_\mathfrak{p}$ is determined by the *diagonal-pairs* (cyclically) in the two-line representation here, i.e., $D_\mathfrak{p}(\pi(s_{i-1})) = s(s_{i-1}) = s_i$ for $0 < i < n$, and $D_\mathfrak{p}(\pi(s_{n-1})) = s_0$. W.l.o.g. we shall assume $s_0 = 1$ and by "the cycles of $\mathfrak{p} = (s, \pi)$" mean the cycles of π. We will refer to the blocks in Par_π as $\mathfrak{p}$-*vertices* or *vertices* for short, and elements in a vertex *darts.*

Hypermaps having one face, (α, β, γ), can be represented as cyclic plane permutations (γ, β). In particular, one-face maps are cyclic plane permutations where the diagonals are fixed-point free involutions. Any cyclic plane permutation having a fixed-point free involution diagonal encodes a one-face embedding of some graph.

A combinatorial map with one face is displayed in Figure 1. The two presentations are showing the same one-face map, whose corresponding cyclic plane permutation reads

$$\mathfrak{p} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 6 & 7 & 8 & 3 & 4 & 5 & 2 \end{pmatrix}.$$

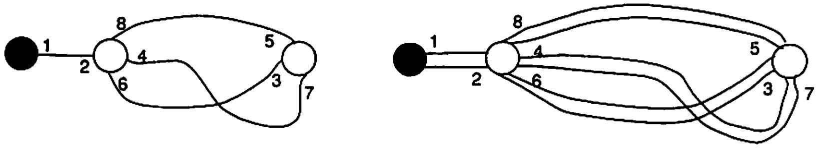

Figure 1: A one-face map with 4 edges.

Definition 2.2 (Localization). Given a cyclic plane permutation $\mathfrak{p} = (s, \pi)$ on $[n]$ and a $\mathfrak{p}$-vertex ν, the *localization* of $\mathfrak{p}$ at ν, $\mathrm{loc}_\nu(\mathfrak{p}) = (s_\nu, \pi_\nu)$ is the cyclic plane permutation

$$(s_\nu, \pi_\nu) = \begin{pmatrix} s_{i1} & s_{i2} & \cdots & s_{i(k-1)} & s_{ik} \\ \pi(s_{i1}) & \pi(s_{i2}) & \cdots & \pi(s_{i(k-1)}) & \pi(s_{ik}) \end{pmatrix},$$

which is obtained by deleting all columns not containing darts incident to ν in the two-line representation of $\mathfrak{p} = (s, \pi)$.

Let D_ν denote the diagonal of $\mathrm{loc}_\nu(\mathfrak{p})$, i.e., $D_\nu = s_\nu \circ \pi_\nu^{-1}$. Note that even if (s, π) is a map, D_ν is not necessarily a fixed-point free involution. For example, given (s, π)

$$\begin{pmatrix} 1 & 3 & 2 & 5 & 7 & 4 & 6 & 9 & 8 & 10 & 11 & 12 \\ 5 & 8 & 3 & 4 & 7 & 10 & 12 & 2 & 6 & 1 & 9 & 11 \end{pmatrix},$$

where $\pi = (1,5,4,10)(2,3,8,6,12,11,9)(7)$, let $\nu = \{2,3,8,6,12,9,11\}$. Then,

$$\mathrm{loc}_\nu(\mathfrak{p}) = \begin{pmatrix} 3 & 2 & 6 & 9 & 8 & 11 & 12 \\ 8 & 3 & 12 & 2 & 6 & 9 & 11 \end{pmatrix}$$

and we arrive at $D_\nu = (2,8)(3,6,11)(9,12)$.

A set of consecutive diagonal-pairs in $\mathfrak{p} = (s, \pi)$ is called a *diagonal block.* In above example, $\begin{matrix} & 2 & 5 & 7 & 4 \\ 8 & 3 & 4 & 7 & \end{matrix}$ is a diagonal block. It is completely determined by its corners, in this case, the lower left corner, 8, as well as the upper right corner, 4. The diagonal block is denoted by $< 8, 4 >$.

Given a cyclic plane permutation $\mathfrak{p} = (s, \pi)$ on $[n]$ and a sequence $h = h_1 h_2 \cdots h_{n-1}$ on $[n-1]$, let $s^h = (s_0, s_{h_1}, s_{h_2}, \ldots s_{h_{n-1}})$, i.e. s is acted upon by h via translation its indices, and $\pi^h = D_{\mathfrak{p}}^{-1} \circ s^h$. This induces the new cyclic plane permutation (s^h, π^h) having by construction the same diagonal as (s, π). Equivalently, the two-line representation of (s^h, π^h) can be obtained by permuting the diagonal-pairs of (s, π). In the following, a cyclic plane permutation written (s^h, π^h), as it is obtained from (s, π) by permuting diagonal-pairs by h.

Lemma 2.3 (Localization lemma). *Let $(s, \pi), (s', \pi') = (s^H, \pi^H)$ be cyclic plane permutations such that $Par_\pi = Par_{\pi'}$ and π and π' exclusively differ at the vertex ν. Then, there exists some h such that $(s'_\nu, \pi'_\nu) = (s^h_\nu, \pi^h_\nu)$ and furthermore $(D_\nu, Par_{\pi_\nu}) = (D'_\nu, Par_{\pi'_\nu})$.*

Proof. Assume $\mathfrak{p} = (s, \pi)$ and $\mathfrak{p}' = (s', \pi')$ are respectively

$$\begin{pmatrix} \cdots s_{i_0} & s_{i_0+1} & \cdots & s_{i_1} & \cdots & s_{i_{k-2}} & \cdots & s_{i_{k-1}} & \cdots \\ \cdots \pi(s_{i_0}) & \cdots & \pi(s_{i_1-1}) & \pi(s_{i_1}) & \cdots & \pi(s_{i_{k-2}}) & \cdots & \pi(s_{i_{k-1}}) \cdots & \end{pmatrix},$$

$$\begin{pmatrix} \cdots s'_{i_0} & s'_{i_0+1} & \cdots & s'_{i_1} & \cdots & s'_{i_{k-2}} & \cdots & s'_{i_{k-1}} & \cdots \\ \cdots \pi'(s'_{i_0}) & \cdots & \pi'(s'_{i_1-1}) & \pi'(s'_{i_1}) & \cdots & \pi'(s'_{i_{k-2}}) & \cdots & \pi'(s'_{i_{k-1}}) \cdots & \end{pmatrix}$$

where we assume $\nu = \{s_{i_0}, s_{i_1}, \ldots s_{i_{k-1}}\} = \{s'_{i_0}, s'_{i_1}, \ldots s'_{i_{k-1}}\}$ and $s'_0 = s_0$. Since by assumption π and π' only differ at the vertex ν, we have $s_j = s'_j$

for $0 \leq j \leq i_0$. Furthermore $Par_\pi = Par_{\pi'}$ implies $Par_{\pi_\nu} = Par_{\pi'_\nu}$.
Claim. $D_\nu = D'_\nu$.
We observe that for fixed j, each diagonal block $< \pi'(s'_{i_j}), s'_{i_{j+1}} >$ equals the diagonal block $< \pi(s_{i_l}), s_{i_{l+1}} >$ for some $l(j)$, i.e.,

$$\left(\begin{array}{cccc} & s'_{i_j+1} & s'_{i_j+2} \quad \cdots & s'_{i_{j+1}} \\ & / & & / \\ / & & / & \\ \pi'(s'_{i_j}) & \pi'(s'_{i_j+1}) & \cdots \quad \pi'(s'_{i_{j+1}-1}) & \end{array}\right)$$

$$= \left(\begin{array}{cccc} & s_{i_l+1} & s_{i_l+2} \quad \cdots & s_{i_{l+1}} \\ & / & & / \\ / & & / & \\ \pi(s_{i_l}) & \pi(s_{i_l+1}) & \cdots \quad \pi(s_{i_{l+1}-1}) & \end{array}\right)$$

To prove this, we observe that by construction for fixed j there exists some index l such that $\pi(s_{i_l}) = \pi'(s'_{i_j})$ holds. This implies,

$$s'_{i_j+1} = D_\mathfrak{p} \circ \pi'(s'_{i_j}) = D_\mathfrak{p} \circ \pi(s_{i_l}) = s_{i_l+1}.$$

In case of $s'_{i_j+1} \notin \nu$, we have $\pi'(s'_{i_j+1}) = \pi(s'_{i_j+1}) = \pi(s_{i_l+1})$ and derive

$$s'_{i_j+2} = D_\mathfrak{p} \circ \pi'(s'_{i_j+1}) = D_\mathfrak{p} \circ \pi(s_{i_l+1}) = s_{i_l+2}.$$

Iterating this we arrive at $s'_{i_{j+1}} = s_{i_{l+1}}$, whence the two diagonal blocks are equal, the Claim follows and the proof of the lemma is complete. $\square$

Definition 2.4. Given a cyclic plane permutation $\mathfrak{p} = (s, \pi)$ on $[n]$ and its localization at ν, $\mathrm{loc}_\nu(\mathfrak{p}) = (s_\nu, \pi_\nu)$. Suppose (s^h_ν, π^h_ν) is such that $Par_{\pi_\nu} = Par_{\pi^h_\nu}$. Then the *inflation* of (s^h_ν, π^h_ν) w.r.t. $\mathfrak{p}$ is the cyclic plane permutation $\mathrm{inf}_\mathfrak{p}((s^h_\nu, \pi^h_\nu))$, obtained from (s^h_ν, π^h_ν) by substituting each diagonal-pair with the diagonal block in $\mathfrak{p}$ having the diagonal-pair as its corners.

Let $(s^h_\nu, \pi^h_\nu) = \begin{pmatrix} 3 & 9 & 8 & 11 & 2 & 6 & 12 \\ 12 & 2 & 6 & 8 & 3 & 9 & 11 \end{pmatrix}$, then the inflation of (s^h_ν, π^h_ν) w.r.t. (s, π) is

$$\mathrm{inf}_\mathfrak{p}((s^h_\nu, \pi^h_\nu)) = \begin{pmatrix} 1 & 3 & 9 & 8 & 10 & 11 & 2 & 5 & 7 & 4 & 6 & 12 \\ 5 & 12 & 2 & 6 & 1 & 8 & 3 & 4 & 7 & 10 & 9 & 11 \end{pmatrix}.$$

Lemma 2.5 (Inflation lemma). *Let* $\mathrm{inf}_\mathfrak{p}((s^h_\nu, \pi^h_\nu)) = (s', \pi')$. *Then we have*

$$(D_\mathfrak{p}, Par_\pi) = (D_{\mathfrak{p}'}, Par_{\pi'})$$

and π' *differs from* π *only at the vertex* ν.

Proof. By construction, $D_{\mathfrak{p}'} = D_{\mathfrak{p}}$. Any dart not contained in ν, is located inside the respective diagonal blocks, whence π and π' are equal on these darts. All ν-darts contribute exactly one block in Par_π and $Par_{\pi'}$, since $Par_{\pi_\nu} = Par_{\pi^h_\nu}$. Accordingly, we have $Par_\pi = Par_{\pi'}$, completing the proof of the lemma. □

Combining Lemma 2.3 and 2.5, we obtain

Theorem 2.6. *Let $\mathfrak{p} = (s, \pi)$ be a cyclic plane permutation. Let $X = \{H \mid Par_\pi = Par_{\pi^H} \;\wedge\; \pi(i) = \pi^H(i),\ i \notin \nu\}$ and let $Y = \{h \mid Par_{\pi_\nu} = Par_{\pi^h_\nu}\}$. Then there is a bijection between X and Y and we have the commutative diagram:*

$$
\begin{array}{ccc}
(s,\pi) & \xrightarrow{\;H:\quad Par_\pi = Par_{\pi^H}\;} & (s',\pi') \\
\Big\downarrow \mathrm{loc}_\nu & & \Big\uparrow \mathrm{inf}_{\mathfrak{p}} \\
(s_\nu,\pi_\nu) & \xrightarrow[\;h:\quad Par_{\pi_\nu} = Par_{\pi^h_\nu}\;]{} & (s^h_\nu,\pi^h_\nu)
\end{array}
$$

3 Embeddings having one face

Lemma 3.1. *Let $\mathfrak{p} = (s, \pi)$ be a cyclic plane permutation with the underlying graph $G = (D_{\mathfrak{p}}, Par_\pi)$. Then, $\mathfrak{p}' = (s', \pi')$ is an embedding of G iff $(s', \pi') = (s^h, \pi^h)$ for some h and $(D_{\mathfrak{p}'}, Par_{\pi'}) = (D_{\mathfrak{p}}, Par_\pi)$.*

Proof. If $\mathfrak{p}' = (s', \pi')$ is an embedding of $G = (D_{\mathfrak{p}}, Par_\pi)$, then, by definition, $(D_{\mathfrak{p}'}, Par_{\pi'}) = (D_{\mathfrak{p}}, Par_\pi)$. Thus, $D_{\mathfrak{p}} = D_{\mathfrak{p}'} = s' \circ \pi'^{-1}$. Clearly, there exists some h such that $s' = s^h$. By construction we have $\pi^h = D_{\mathfrak{p}}^{-1} \circ s^h = D_{\mathfrak{p}'}^{-1} \circ s' = \pi'$. Hence, $(s', \pi') = (s^h, \pi^h)$ for some h. The converse is clear, whence the lemma. □

This lemma shows that any one-face embedding is originated by the action of some h on a cyclic plane permutation. Explicitly, by permuting diagonal-pairs based on a fixed one-face embedding (s, π) of the graph such that $Par_\pi = Par_{\pi^h}$.

Corollary 3.2. *Let $\mathfrak{p} = (s, \pi)$ be a one-face map with the underlying graph $G = (D_{\mathfrak{p}}, Par_\pi)$. Fixing the (local) embedding of all vertices of G as in $\mathfrak{p}$ but the vertex ν, each local embedding of ν leading to a one-face map $\mathfrak{p}' = (s', \pi')$ corresponds to a H such that $(s', \pi') = (s^H, \pi^H)$ and π' differs from π only at the vertex ν.*

According to the bijection between H and h in Theorem 2.6, the number of different embeddings of ν keeping one face is equal to the number of different h such that (s^h_ν, π^h_ν) and (s_ν, π_ν) having the same underlying graph. We denote this number by R_ν. Moreover, the darts contained in ν split the cyclic plane permutation into $|\nu|$ diagonal blocks. We can view these diagonal blocks to be arranged in a circle, as displayed in Figure 2. To reembed ν means to permute these diagonal blocks.

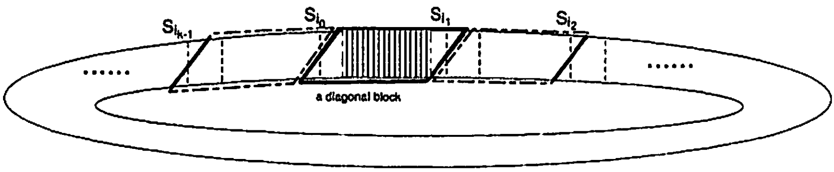

Figure 2: Circular arrangement of diagonal blocks determined by v

Let U_D denote the set of cyclic plane permutations having diagonal D, where D is a fixed permutation. Note $\mathfrak{p} = (s, \pi) \in U_D$ iff $D = D_{\mathfrak{p}} = s \circ \pi^{-1}$. Thus, $|U_D|$ enumerates the ways to write D as a product of an n-cycle with another permutation.

A partition λ of n is written as $\lambda \vdash n$. Let $\mu \vdash n, \eta \vdash n$. We write $\mu \rhd_{2i+1} \eta$ if μ can be obtained by splitting one η-block into $(2i+1)$ non-zero parts. Let furthermore $\kappa_{\mu,\eta}$ denote the number of different ways to obtain η from μ by merging $\ell(\mu)-\ell(\eta)+1$ μ-blocks into one, where $\ell(\mu)$ and $\ell(\eta)$ denote the number of blocks in the partitions μ and η, respectively.

Theorem 3.3. *[1] Let $p_k^\lambda(n)$ denote the number of $\mathfrak{p} \in U_D$ having k cycles, where D is of cycle-type λ. Let q^λ denote the number of permutations of cycle-type λ. Then,*

$$p_k^\lambda(n) = \frac{\sum_{i=1}^{\lfloor\frac{n-k}{2}\rfloor} \binom{k+2i}{k-1} p_{k+2i}^\lambda(n) q^\lambda + \sum_{i=1}^{\lfloor\frac{n-\ell(\lambda)}{2}\rfloor} \sum_{\mu \rhd_{2i+1} \lambda} \kappa_{\mu,\lambda} p_k^\mu(n) q^\mu}{q^\lambda [n+1-k-\ell(\lambda)]}. \tag{5}$$

Note that the localizations (s_ν^h, π_ν^h) and (s_ν, π_ν) have the same underlying graph G, iff both of them belong to U_D where $D = D_\nu$ and $C(\pi_\nu^h) = C(\pi_\nu) = 1$. Therefore, if D_ν has cycle-type λ, $R_\nu = p_1^\lambda(|\nu|)$. As a result we obtain

Theorem 3.4. *Let ϵ be a one-face embedding of G, and ν be a vertex of G with $deg(\nu) \geq 4$. Then there exists at least one additional way to reembed ν such that the obtained embedding ϵ' has the same genus as ϵ.*

Proof. Assume $d \geq 4$ and ϵ is localized at ν

$$(s_\nu, \pi_\nu) = \begin{pmatrix} v_1 & v_2 & \cdots & v_{d-1} & v_d \\ v_{i,1} & v_{i,2} & \cdots & v_{i,d-1} & v_{i,d} \end{pmatrix},$$

where $\pi_\nu = (v_1, V_2, \cdots, V_{d-1}, V_d)$. Firstly, if $V_l = v_p, V_m = v_q$ and $1 < l < m \leq d, 1 < p < q \leq d$, i.e. Case 3 of [1, Lemma 3], then there exists at least one additional way to reembed ν preserving genus. Otherwise, we have $\pi_\nu = (v_1, v_d, v_{d-1}, \ldots, v_2)$. In this case,

$$(s_\nu, \pi_\nu) = \begin{pmatrix} v_1 & v_2 & \cdots & v_{d-1} & v_d \\ v_d & v_1 & \cdots & v_{d-2} & v_{d-1} \end{pmatrix},$$

whence

$$D_\nu = \begin{cases} (v_1, v_3, \ldots, v_d, v_2, v_4, \ldots, v_{d-1}), & d \in odd, \\ (v_1, v_3, \ldots, v_{d-1})(v_2, v_4, \ldots, v_d), & d \in even. \end{cases}$$

It remains to show that if $d \geq 4$ we have $R_\nu \geq 2$ in all cases. To this end, we apply a formula for $p_1^\lambda(k)$ due to Stanley [15]. If $\lambda = (1^{a_1}, 2^{a_2}, \ldots, k^{a_k})$, then

$$p_1^\lambda(k) = \sum_{i=0}^{k-1} \frac{i!(k-1-i)!}{k} \sum_{r_1,\ldots,r_i} \binom{a_1 - 1}{r_1}\binom{a_2}{r_2} \cdots \binom{a_i}{r_i}(-1)^{r_2+r_4+r_6+\cdots}, \tag{6}$$

where $r_1, \ldots, r_i$ ranges over all non-negative integer solutions of the equation $\sum_j jr_j = i$. Applying Stanley's formula we can compute R_ν, if $d \in odd$ as

$$R_\nu = \frac{(d-1)!}{d} \sum_{i=0}^{d-1} (-1)^i \binom{d-1}{i}^{-1} = \frac{2(d-1)!}{d+1}.$$

The simplification of the summation stems from the following formula [16]

$$\sum_{i=0}^{n} (-1)^i \binom{x}{i}^{-1} = \frac{x+1}{x+2}(1 + (-1)^n \binom{x+1}{n+1}^{-1}).$$

It is not hard to see that $R_\nu \geq 2$ if $d \geq 4$. Similarly, if $4|d$ and $d \geq 4$, we have

$$R_\nu = \sum_{i=0}^{\frac{d}{2}-1} (-1)^i \frac{i!(d-1-i)!}{d} + \sum_{i=\frac{d}{2}}^{d-1} (-1)^i \frac{i!(d-1-i)!}{d}$$
$$\cdot [(-1)^i + (-1)^{i-\frac{d}{2}} \binom{2}{1}(-1)] = \frac{2(d-1)!}{d+1}(1 - \binom{d}{\frac{d}{2}}^{-1}).$$

If $d \in even$ and $4 \nmid d$, we have

$$R_\nu = \sum_{i=0}^{\frac{d}{2}-1}(-1)^i\frac{i!(d-1-i)!}{d} + \sum_{i=\frac{d}{2}}^{d-1}(-1)^i\frac{i!(d-1-i)!}{d}$$

$$\cdot [(-1)^i + (-1)^{i-\frac{d}{2}}\binom{2}{1}] = \frac{2(d-1)!}{d+1}(1+\binom{d}{\frac{d}{2}}^{-1}).$$

In both cases, if $d \geq 4$, it is straightforward to show that $R_\nu \geq 2$, since both $\frac{2(d-1)!}{d+1}$ and $(1-\binom{d}{\frac{d}{2}}^{-1})$ are increasing functions of d. Accordingly, in all cases, if $d \geq 4$, then $R_\nu \geq 2$, completing the proof. □

Note in case of a vertex ν having degree 1 or 2, the situation is clear. Thus it remains to consider the case $deg(\nu) = 3$. For such a vertex, a reembedding preserving genus can be impossible. For example, $D_\nu = (132) = (123)(312)$ is the unique decomposition of D_ν.

Corollary 3.5. *Any even permutation on $[n]$ with $n \geq 4$ has at least two different factorizations into two n-cycles.*

Proof. Since $D_\nu = s_\nu \circ \pi_\nu^{-1}$ and both s_ν as well as π_ν have only one cycle, D_ν is an even permutation. Theorem 3.4 implies that D_ν has at least 2 factorizations into two n-cycles. □

Corollary 3.6. *Let G be a graph having m vertices of degree no less than 4. If there exists a one-face embedding of G, then there are at least 2^m one-face embeddings of G.*

Theorem 3.7. *Let (s, π) be a one-face embedding of G and (s_ν, π_ν) its localization at ν. Suppose D_ν has cycle-type $\lambda = (1^{a_1}, 2^{a_2}, \dots, k^{a_k})$ where $k = deg(\nu)$, then the probability $prob_1(\nu)$ of a reembedding of ν to be one-face satisfies*

$$\frac{2}{deg(\nu)-a_1+2} \leq prob_1(\nu) \leq \frac{2}{deg(\nu)-a_1+\frac{19}{29}}. \tag{7}$$

In particular, for any vertex ν, $prob_1(\nu) \geq \frac{2}{deg(\nu)+2}$.

Proof. In Zagier [23], it was proved that

$$\frac{2(k-1)!}{k-a_1+2} \leq p_1^\lambda(k) \leq \frac{2(k-1)!}{k-a_1+\frac{19}{29}}.$$

Since there are $(k-1)!$ different ways to embed ν, eq. (7) immediately follows in view of $p_1^\lambda(k) = R_\nu$. Clearly, we have $\frac{2}{deg(\nu)-a_1+2} \geq \frac{2}{deg(\nu)+2}$, whence the second assertion. □

Corollary 3.8. *If there exists a one-face embedding of G, then the probability of a random embedding of G to have one face is at least* $\prod_{\nu \in V(G)} \frac{2}{deg(\nu)+2}$.

4 Embeddings with multiple faces

In this section, we generalize cyclic plane permutations to general plane permutations. This puts us in position to study graph embeddings having k faces. Although it is hard to determine g_{min} and g_{max}, as well as the genus distribution for a given graph G, we will show that locally these quantities can be more easily obtained.

Definition 4.1. A *plane permutation* on $[n]$ is a pair, $\mathfrak{p}$, of permutations s and π on $[n]$. The permutation $D_{\mathfrak{p}} = s \circ \pi^{-1}$ is called the *diagonal* of $\mathfrak{p}$. If s has k cycles, we write $\mathfrak{p} = (s, \pi)_k$.

Assume $s = (s_{11}, \dots s_{1m_1})(s_{21}, \dots s_{2m_2}) \cdots (s_{k1}, \dots s_{km_k})$, such that the summation $\sum_i m_i = n$. A plane permutation $(s, \pi)_k$ can be represented by two aligned rows:

$$\begin{pmatrix} \boxed{s_{11}} & s_{12} & \cdots & s_{1m_1} & \cdots & \boxed{s_{k1}} & \cdots & s_{km_k} \\ \pi(s_{11}) & \pi(s_{12}) & \cdots & \boxed{\pi(s_{1m_1})} & \cdots & \pi(s_{k1}) & \cdots & \boxed{\pi(s_{km_k})} \end{pmatrix}.$$

The diagonal $D_{\mathfrak{p}}$ can be defined as follows:

- For $1 \le i \le k$, $D_{\mathfrak{p}}(\pi(s_{ij})) = s_{i(j+1)}$ if $j \neq m_i$;
- For $1 \le i \le k$, $D_{\mathfrak{p}}(\pi(s_{im_i})) = s_{i1}$.

We call blocks

$$\begin{pmatrix} \boxed{s_{i1}} & s_{i2} & \cdots & s_{im_i} \\ \pi(s_{i1}) & \pi(s_{i2}) & \cdots & \boxed{\pi(s_{im_i})} \end{pmatrix}$$

the *cycles* of the plane permutation. If the face $(s_{i1}, \dots, s_{im_i})$ is incident to a $\mathfrak{p}$-vertex ν, the corresponding cycle is said to be incident to ν. Since every embedding having k faces can be represented by a triple (α, β, γ), where $\gamma = \alpha\beta$ and γ has k cycles, any embedding can be expressed via a plane permutation $(\gamma, \beta)_k$. Let $H(f)$ denote the set of darts contained in the face f.

Lemma 4.2. *Let ν be a vertex of the graph G and ϵ be an embedding of G, where ν is incident to q faces, f_i, for $1 \le i \le q$. Let ϵ' be an* *embedding, obtained by reembedding ν such that ν is incident to q' faces, f'_i, for $1 \leq i \leq q'$. Then we have*

$$\bigcup_{i=1}^{q} H(f_i) = \bigcup_{i=1}^{q'} H(f'_i), \qquad q \equiv q' \pmod 2.$$

Proof. Let ϵ, ϵ' be two embeddings represented by $\mathfrak{p} = (s,\pi)_k$ and $\mathfrak{p}' = (s',\pi')_{k'}$, respectively, such that $D_\mathfrak{p} = D_{\mathfrak{p}'}$ and $Par_\pi = Par_{\pi'}$. Note that ϵ and ϵ' only differ w.r.t. the cyclic order of the darts around ν. Thus, for $z \notin \nu$, we have $\pi(z) = \pi'(z)$. Clearly, any face f of ϵ can be expressed as the sequence $(D_\mathfrak{p}\pi(z), (D_\mathfrak{p}\pi)^2(z), \ldots)$ for any $z \in H(f)$. The lemma is implied by the following

Claim. Any ϵ'-face either intersects some ϵ-face f_i for $1 \leq i \leq q$ and is incident to ν or it coincides with an ϵ-face f not incident to ν.

Suppose f' does not intersect any f_i for $1 \leq i \leq q$. For any $z \in H(f')$, $z \notin \nu$ holds and by construction $\pi'(z) = \pi(z)$. As a result, $D_\mathfrak{p}(\pi(z)) = D_\mathfrak{p}(\pi'(z)) = D_{\mathfrak{p}'}(\pi'(z))$, i.e. f' coincides with an ϵ-face f that is not incident to ν.

If f' intersects some ϵ-face f_i for $1 \leq i \leq q$, we shall prove that f' is incident to ν. Assume the dart u is contained in the ϵ-face f_j as well as in the face f' of ϵ'. Then,

$$\begin{aligned} f_j &= (D_\mathfrak{p}\pi(u), (D_\mathfrak{p}\pi)^2(u), \ldots, v_i, D_\mathfrak{p}(\pi(v_i)), \ldots) \\ f' &= (D_{\mathfrak{p}'}\pi'(u), (D_{\mathfrak{p}'}\pi')^2(u), \ldots), \end{aligned}$$

where v_i is the first dart of ν that appears in f_j. Since $\pi(z) = \pi'(z)$ if $z \notin \nu$, we have $D_\mathfrak{p}\pi(u) = D_\mathfrak{p}\pi'(u) = D_{\mathfrak{p}'}\pi'(u)$, whence the entire subsequence from $D_\mathfrak{p}(\pi(u))$ to v_i in f_j appears also in f'. In particular we have $v_i \in H(f')$, which means that f' is incident to ν and the Claim follows. □

Let ϵ be an embedding of the graph G and ν be a vertex of G, where ν is incident to q faces in ϵ. Assume ϵ is represented by $\mathfrak{p} = (s,\pi)_k$. Similar to the situation of one-face maps, we can define the *localization* at ν which is a plane permutation having q cycles, $(s_\nu, \pi_\nu)_q$, and that is obtained as follows: the q cycles of $(s_\nu, \pi_\nu)_q$ are obtained from the q cycles of $\mathfrak{p}$ incident to ν by deleting all columns having no darts of ν. Let D_ν denote the diagonal of $(s_\nu, \pi_\nu)_q$. By construction, we have $s_\nu = D_\nu \circ \pi_\nu$, having q cycles.

Given a plane permutation $(s'_\nu, \pi'_\nu)_{q'}$, where $(D'_\nu, Par_{\pi'_\nu}) = (D_\nu, Par_{\pi_\nu})$, we can inflate w.r.t. $\mathfrak{p}$ into an embedding of G as in the case of cyclic plane permutations. Namely, we substitute each diagonal-pair with the corresponding diagonal block in $\mathfrak{p}$ and then add any $\mathfrak{p}$-cycles containing darts not incident to ν.

Fix an embedding ϵ, represented by the plane permutation $(s,\pi)_k$, of genus $g(\epsilon)$. We compute in the following the distribution of genera resulting from reembedding a vertex ν incident to q faces in ϵ. Let $R_\nu(\Delta g)$ denote the number of different embeddings, ϵ', coming from reembedding ν such that $g(\epsilon') = g(\epsilon) + \Delta g$ and denote the cycle-type of D_ν as $\lambda(D_\nu)$. Then we have

Theorem 4.3.

$$R_\nu(\Delta g) = p_{q+2\Delta g}^{\lambda(D_\nu)}(deg(\nu)), \tag{8}$$

Proof. Let $(s,\pi)_k$ represent ϵ and $(s',\pi')_{k+2\Delta g}$ represent ϵ', respectively. Here the index $k+2\Delta g$ stems from $g(\epsilon') = g(\epsilon) + \Delta g$, which implies that ϵ' differs by $2\Delta g$ faces from ϵ.

Suppose ν is incident to q ϵ-faces, $f_1, \dots f_q$. According to Lemma 4.2, we have the following situation: $\bigcup_{i=1}^{q} H(f_i)$ is reorganized into $q+2\Delta g$ ϵ'-faces, $f_1', \dots, f_{q+2\Delta g}'$ and any other ϵ'-face coincides with some ϵ-face not incident to ν.

Let $(s_\nu', \pi_\nu')_{q+2\Delta g}$ be the localization of $(s',\pi')_{k+2\Delta g}$ at ν, having the diagonal D_ν'. By definition, $s_\nu' = D_\nu' \circ \pi_\nu'$ has $(q+2\Delta g)$ cycles.

Claim 1. Given ϵ represented by $(s,\pi)_k$, any reembedding of ν, ϵ' represented by $(s',\pi')_{k+2\Delta g}$ satisfies $D_\nu' = D_\nu$.

Suppose the ϵ'-cycle of face f_i' reads:

$$\begin{pmatrix} \boxed{v_{i1}'} & x_1 & \cdots & v_{i2}' & x_2 & \cdots & v_{i3}' & \cdots & v_{it_i}' & \cdots & y \\ v_{ij_1}' & \cdots & x_1' & v_{ij_2}' & \cdots & x_2' & v_{ij_3}' & \cdots & v_{ij_{t_i}}' & \cdots & \boxed{z} \end{pmatrix},$$

where $v_{ik}', v_{ij_k}' \in \nu \ \wedge \ v_{ij_k}' = \pi_\nu'(v_{ik}')$. Then, by the same argument as in the proof for the Lemma 2.3, the diagonal block

$$\begin{matrix} & x_l & \cdots & v_{i(l+1)}' \\ v_{ij_l}' & \cdots & x_l' & \end{matrix}$$

is also a diagonal block in ϵ, which in turn implies $D_\nu' = D_\nu$.

Claim 2. Suppose $(s_\nu', \pi_\nu')_{q+2\Delta g}$ is a plane permutation such that $D_\nu' = D_\nu$ and $C(\pi_\nu') = 1$. Then $(s_\nu', \pi_\nu')_{q+2\Delta g}$ can be inflated into an embedding ϵ' such that $g(\epsilon') = g(\epsilon) + \Delta g$.

Suppose $(s_\nu', \pi_\nu')_{q+2\Delta g}$ is given by:

$$\begin{pmatrix} \boxed{v_{11}'} \cdots & v_{1t_1}' & \cdots & \boxed{v_{(q+2\Delta g)1}'} \cdots & v_{(q+2\Delta g)t_{q+2\Delta g}}' \\ \pi_\nu'(v_{11}') \cdots & \boxed{\pi_\nu'(v_{1t_1}')} & \cdots & \pi_\nu'(v_{(q+2\Delta g)1}') \cdots & \boxed{\pi_\nu'(v_{(q+2\Delta g)t_{q+2\Delta g}}')} \end{pmatrix}$$

Inflating every diagonal-pair into a diagonal block w.r.t. ϵ and adding the ϵ-cycles which are not incident to ν, we obtain an embedding ϵ' with $2\Delta g$ more faces than ϵ, i.e., $g(\epsilon') = g(\epsilon) + \Delta g$. By construction, ϵ and ϵ' only differ by cyclic rearrangement of the darts around ν. □

We proceed by studying the values of Δg in Theorem 4.3, i.e. the set $\{k | p_k^\lambda(n) \neq 0\}$. According to [1] we have:

Proposition 4.4. *[1] For a cyclic plane permutation* $\mathfrak{p} = (s, \pi)$ *on* $[n]$, *the sum of the numbers of cycles in* π *and in* $D_\mathfrak{p}$ *is smaller than* $n + 2$. *Equivalently,*

$$\max\{k \mid p_k^\lambda(n) \neq 0\} \leq n + 1 - \ell(\lambda).$$

Next we show that the maximum can be always achieved.

Proposition 4.5. *Let* $\lambda \vdash n$ *and* $n \geq 1$. *Then,*

$$\max\{k \mid p_k^\lambda(n) \neq 0\} = n + 1 - \ell(\lambda). \tag{9}$$

Proof. For $n = 1$, the assertion is clear, whence we can assume w.l.o.g. $n \geq 2$. For any permutation α on $[n]$ of cycle type λ and $\ell(\lambda) = 1$, we have $\alpha = \alpha e_n$ where e_n is the identity permutation on $[n]$ which obviously has n cycles. Therefore, in case of $\ell(\lambda) = 1$, $\max\{k \mid p_k^\lambda(n) \neq 0\} = n = n + 1 - \ell(\lambda)$. Suppose for any λ with $1 \leq \ell(\lambda) = m < n$ holds

$$\max\{k \mid p_k^\lambda(n) \neq 0\} = n + 1 - m.$$

Let α' be a permutation on $[n]$ of cycle type λ' and $\ell(\lambda') = m + 1$. Since $m + 1 \geq 2$, we can always find a and b such that a and b are in different cycles of α'. Let $\alpha = \alpha'(a, b)$. Thus, α must be of cycle type μ for some μ such that $\ell(\mu) = m$. By assumption, there exists a relation $\alpha = s\pi$ such that s has only one cycle and π has $n + 1 - m$ cycles. Then,

$$\alpha' = \alpha(a, b) = s\pi(a, b).$$

Note $\pi(a, b)$ has the number of cycles either $n+1-m-1$ or $n+1-m+1$. The latter is impossible because it would contradict the bound established in Proposition 4.4. Hence, for any λ' with $\ell(\lambda') = m + 1$,

$$\max\{k \mid p_k^{\lambda'}(n) \neq 0\} = n + 1 - m - 1 = n + 1 - \ell(\lambda'),$$

which completes the proof of the proposition. □

Corollary 4.6. *Let* ϵ *be an embedding of the graph* G *where a vertex* ν *has its localization* $(s_\nu, \pi_\nu)_q$. *Then, there exists for any*

$$-\lfloor \frac{deg(\nu) + 1 - \ell(\lambda(D_\nu)) - q}{2} \rfloor \leq \Delta g \leq \lfloor \frac{q-1}{2} \rfloor$$

an embedding ϵ' *of* G *such that* $g(\epsilon') = g(\epsilon) + \Delta g$.

Proof. According to Theorem 3.3, we have $p_k^\lambda(n) \neq 0$ as long as $p_{k+2i}^\lambda(n) \neq 0$ for some $i > 0$ holds. Furthermore, Proposition 4.5 implies

$$p^\lambda_{deg(\nu)+1-\ell(\lambda(D_\nu))}(deg(\nu)) \neq 0.$$

Therefore, for any

$$1 \leq d \leq deg(\nu) + 1 - \ell(\lambda(D_\nu)), \quad d \equiv q \pmod 2,$$

reembedding ν can lead to an embedding where ν is incident to d faces. Accordingly, Euler's characteristic formula, implies

$$-\lfloor \frac{deg(\nu) + 1 - \ell(\lambda(D_\nu)) - q}{2} \rfloor \leq \Delta g \leq \lfloor \frac{q-1}{2} \rfloor,$$

completing the proof of the corollary. □

This result is similar to the result in [2], where it was shown for any $g_{min}(G) \leq g \leq g_{max}(G)$, there exists an embedding of G on S_g. However, while it is very hard to obtain g_{min} and g_{max}, we obtain easily the local minimum and the local maximum.

Suppose we are given two vertices, such that there exists no face of ϵ incident to both, then we call these two vertices *ϵ-face disjoint*. In view of Lemma 4.2, Corollary 4.6 has the following implication:

Corollary 4.7. *Let ϵ be an embedding of the graph G. If the vertices $\nu_i = (s_{\nu_i}, \pi_{\nu_i})_{q_i}$, $1 \leq i \leq m$, are mutually ϵ-face disjoint, then there exists an embedding ϵ' of G for any*

$$\sum_{i=1}^{m} -\lfloor \frac{deg(\nu_i) + 1 - \ell(\lambda(D_{\nu_i})) - q_i}{2} \rfloor \leq \Delta g \leq \sum_{i=1}^{m} \lfloor \frac{q_i - 1}{2} \rfloor,$$

such that $g(\epsilon') = g(\epsilon) + \Delta g$.

The following corollary provides a necessary condition for an embedding of G to be of maximum genus as well as a necessary condition for an embedding of G to be of minimum genus.

Corollary 4.8. *If ϵ is an embedding of the graph G with genus $g_{max}(G)$, then every vertex is incident to at most 2 faces in ϵ. Furthermore, if ϵ is an embedding of the graph G with genus $g_{min}(G)$, and $(s_\nu, \pi_\nu)_{q_\nu}$ is the localization at ν, then*

$$\ell(\lambda(D_\nu)) + q_\nu = deg(\nu) + 1. \tag{10}$$

Proof. The assertions are implied by Corollary 4.6. □

The fact that if there exists a vertex incident to at least 3 faces in an embedding, an embedding with higher genus always exists, is well known, see e.g., in [8, 22]. However, to the best of our knowledge, given an embedding ϵ, there is no simple characterization in order to determine if there exists an embedding of lower genus. Corollary 4.8 gives a sufficient condition, i.e., if $\ell(\lambda(D_\nu)) + q_\nu \neq deg(\nu) + 1$ for some vertex ν, then there exists an embeding of lower genus.

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