Total Domination of Grid Graphs

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Abstract

The size of a minimum total dominating set in the $m \times n$ grid graph is denoted by $\gamma_t(P_m \Box P_n)$. Here a dynamic programming algorithm that computes $\gamma_t(P_m \Box P_n)$ for any m and n is presented, and it is shown how properties of the algorithm can be used to derive formulae for a fixed, small value of m. Using this method formulae for $\gamma_t(P_m \Box P_n)$, $m \leq 28$ are obtained. Formulae for larger m are further conjectured, and a new general upper bound on $\gamma_t(P_m \Box P_n)$ is proved. **Keywords:** total domination, grid graph

1 Introduction

The Cartesian product graph $G = P_m \Box P_n$, where P_i denotes a path with i vertices (and length $i-1$), is said to be an *$m \times n$ grid graph*. We here label the vertices and edges of the graph $G = (V, E)$ as follows:

$$\begin{aligned} V &= \{v_{i,j} : 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\}, \\ E &= \{\{v_{i,j}, v_{i',j'}\} : (i = i' \text{ and } |j - j'| = 1) \text{ or } (j = j' \text{ and } |i - i'| = 1)\}. \end{aligned}$$

A *total dominating set* of a graph $G = (V, E)$ is a subset $V' \subseteq V$ such that every vertex in V is adjacent to at least one vertex in V'. The *total domination number* $\gamma_t(G)$ of a graph G is the minimum size of a total dominating set in G. The total domination number of the $m \times n$ grid graph is denoted by $\gamma_t(P_m \Box P_n)$. By symmetry, $\gamma_t(P_m \Box P_n) = \gamma_t(P_n \Box P_m)$. Two related types of sets are *dominating sets*, where every vertex in $V \backslash V'$ is adjacent to at least one vertex in the dominating set

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V', and *independent dominating sets*, which are simultaneously dominating sets and independent sets.

For dominating sets we say that a vertex v *dominates* a vertex v' if v is adjacent to v' or $v = v'$. Similarly, for total dominating sets we say that v *dominates* v' if v is adjacent to v'. A (total) dominating set is said to be *perfect* if every vertex of the graph is dominated by exactly one vertex.

In early studies of the domination number of grid graphs, domination numbers for small grids were obtained [1], general bounds were improved [2, 4, 5, 7, 13], and formulae with one of the parameters fixed were proved [3, 15, 17, 16, 18, 21, 22, 23, 25, 26]. Recently, the entire problem of determining the domination number of grid graphs was finally settled [12]. As a result, the problem of determining the independent domination number of grid graphs could also be settled [9].

The concept of total domination number was introduced quite early, but the first study devoted to the total domination number of grid graphs did not appear until 2002 [14], where among other things $\gamma_t(P_m \Box P_n)$ was determined for $m \leq 3$ and a formula was claimed for $m = 4$ without proof. The seminal work was later extended to $m \leq 6$ in [19, 24], which unfortunately contain some incorrect results. Correct results for $m \leq 8$ are presented in [10]. In 2005, all grid graphs that have perfect total dominating sets were characterized [20]. The total domination number of those grid graphs is easy to get from the constructions provided and gives us

$$\gamma_t(P_m \Box P_n) = \begin{cases} \frac{(n-1)(m^2+2m)}{4(m+1)} + \frac{m}{2}, & \text{if } m \equiv 0 \pmod 4 \text{ and } n \equiv 1 \pmod{m+1} \\ \frac{(n-1)(m^2+2m)}{4(m+1)} + \frac{m}{2} + 1, & \text{if } m \equiv 2 \pmod 4 \text{ and } n \equiv 1 \pmod{m+1}, \end{cases}$$

$$\gamma_t(P_m \Box P_n) = \frac{(n+1)(m^2+2m)}{4(m+1)}, \quad \text{if } m \equiv 0 \pmod 2 \text{ and } n \equiv m \pmod{m+1},$$

$$\gamma_t(P_m \Box P_n) = \begin{cases} \frac{(n+3)(m^2+2m)}{4(m+1)} - \frac{m}{2}, & \text{if } m \equiv 0 \pmod 4 \text{ and } n \equiv m-2 \pmod{m+1} \\ \frac{(n+3)(m^2+2m)}{4(m+1)} - \frac{m}{2} - 1, & \text{if } m \equiv 2 \pmod 4 \text{ and } n \equiv m-2 \pmod{m+1}. \end{cases}$$

In Section 2 new upper bounds for $\gamma_t(P_m \Box P_n)$ are obtained. In Section 3 an algorithm that can be used to obtain formulae for $\gamma_t(P_m \Box P_n)$ with one of the parameters (say m) fixed and small is developed. Such formulae are later derived and presented for all $m \leq 28$. These formulae further inspire a general conjecture regarding the exact value of $\gamma_t(P_m \Box P_n)$.

2 An Upper Bound

The set of vertices dominated by a vertex naturally leads to a formulation of the problem of (total) domination of a grid graph in terms of covering an $m \times n$ rectangle by different shapes. In Figure 1 we show the shapes for domination and perfect domination, and—since the middle square in the latter shape is outside the shape and can only be covered in one way up to symmetry—the shape coming from two adjacent vertices in a total dominating set. The first and the last shape in Figure 1 are known as polyominoes [11]. Perfect domination and perfect total domination can then be viewed as *tilings* of an $m \times n$ rectangle (allowing shapes to go outside the rectangle) with these polyominoes.

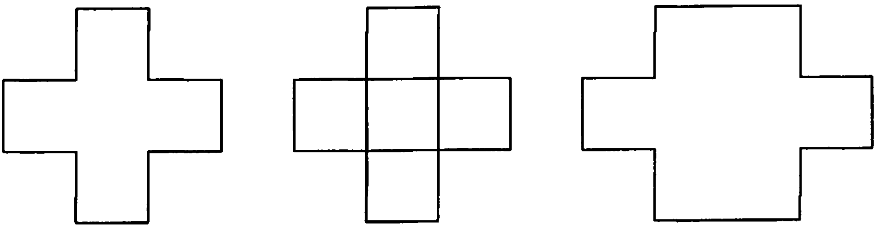

Figure 1: Shapes for geometric packings

One way of getting small (total) dominating sets is to consider a rectangle in a tiling of the plane (or some infinite part of the plane). With the first shape in Figure 1 there is, up to symmetry, only one tiling of the plane.

For the last shape in Figure 1, however, there are an infinite number of symmetry classes of tilings of the plane. One example of such a tiling is shown in Figure 2 as a total dominating set, with vertices at the intersections of lines and vertices in the dominating set drawn as dots. Diagonals with adjacent pairs of vertices in the dominating set can be shifted to get further dominating sets.

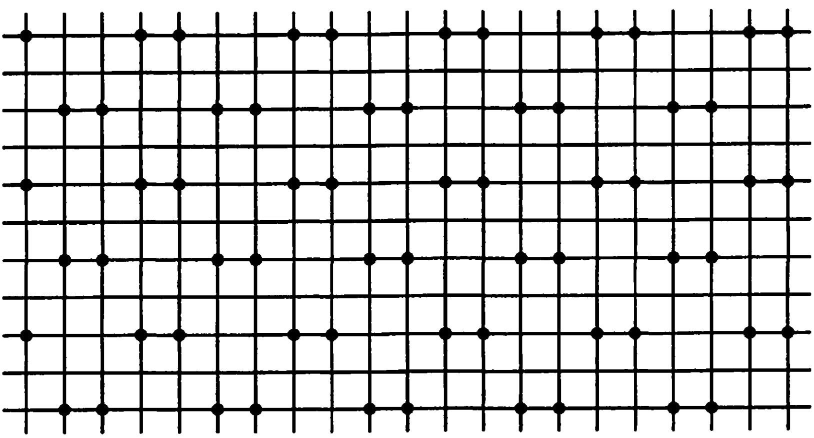

Figure 2: Perfect total dominating set in the plane

Work aiming at improving the best known general upper bound on the total domination number of a grid graph can now be carried out by considering different infinite tilings with polyominos, and extracting different rectangles out of these. The best known general lower and upper bound on the total domination number for grid graphs [14] are

$$\frac{3mn+2(m+n)}{12}-1 \leq \gamma_t(P_m\Box P_n) \leq \left\lfloor \frac{(m+2)(n+2)}{4} \right\rfloor -4. \tag{1}$$

The upper bound in (1) will be improved in Theorem 1, the result of which is divided into four different cases.

In figures throughout this paper, we denote the vertex in the lower left corner $v_{0,0}$, with the first index corresponding to columns and the second to rows.

Theorem 1.

$$\gamma_t(P_m\Box P_n) \leq \begin{cases} \left\lfloor \frac{(n+2)(m^2+2m)-4m}{4(m+1)} \right\rfloor, & \textit{if } m \equiv 0 \pmod 4 \\ \left\lfloor \frac{(n+2)(m^2+2m)-2m+4}{4(m+1)} \right\rfloor, & \textit{if } m \equiv 2 \pmod 4 \\ \left\lfloor \frac{(m+1)(n+2)}{4} \right\rfloor, & \textit{if } m \equiv 1 \pmod 4 \textit{ and } n \not\equiv 0 \pmod 4 \\ & \textit{or if } m \equiv 3 \pmod 4 \\ \left\lfloor \frac{(m+1)(n+2)}{4} -1 \right\rfloor, & \textit{if } m \equiv 1 \pmod 4 \textit{ and } n \equiv 0 \pmod 4. \end{cases}$$

Proof. We will give a proof by construction, considering the cases of odd and even m separately.

***m* odd:** Consider the perfect total dominating set in the plane $\{v_{i,j} : j \equiv 0 \pmod 2, i+j \equiv 1 \text{ or } 2 \pmod 4\}$ (Figure 2) and extract the $m \times n$ grid induced by $v_{i,j}$, $0 \leq i \leq n-1$, $0 \leq j \leq m-1$.

The vertices $v_{0,2+4i}$ are not dominated in the extracted grid, and neither are the vertices $v_{n-1,2+4i}$ when $n \equiv 0,3 \pmod 4$ nor the vertices $v_{n-1,4i}$ when $n \equiv 1,2 \pmod 4$. For each of these vertices, one additional vertex is required in the dominating set. When $m \equiv 3 \pmod 4$ this means we get $(m+1)(n+2)/4$ dominating vertices. For $m \equiv 1 \pmod 4$ we also get $(m+1)(n+2)/4$ dominating vertices when $n \equiv 2 \pmod 4$, when $n \equiv 3,0$, or $1 \pmod 4$ we get $(m+1)(n+1)/4+m-1$, $(m+1)n/4+m-1+m+3$, and $(m+1)(n-1)/4+m-1+2(m+3)$, respectively. Summing up the number of vertices in the dominating set in each case gives the following upper bounds:

$$\gamma_t(P_m\Box P_n) \leq \begin{cases} \frac{(m+1)(n+2)}{4} - 1, & \text{if } m \equiv 1 \pmod 4 \text{ and } n \equiv 0 \pmod 4 \\ \frac{(m+1)(n+2)-2}{4}, & \text{if } m \equiv 1 \pmod 4 \text{ and } n \equiv 1,3 \pmod 4 \\ \frac{(m+1)(n+2)}{4}, & \text{if } m \equiv 1 \pmod 4 \text{ and } n \equiv 2 \pmod 4 \\ & \text{or if } m \equiv 3 \pmod 4. \end{cases}$$

***m* even:** We treat the cases $m \equiv 0 \pmod 4$ and $m \equiv 2 \pmod 4$ separately. For $m \equiv 0 \pmod 4$, consider the following perfect total dominating set for the

$m \times (m+2)$ grid [24]:

$$\begin{array}{l}\{v_{2a,2a+4b+i} : 0 \le a \le m/4, 0 \le b \le m/4 - a, i \in \{1,2\}\} \cup \\ \{v_{m+1-2a,2a+4b+i} : 0 \le a \le m/4, 0 \le b \le m/4 - a, i \in \{1,2\}\} \cup \\ \{v_{2a+4b+i,2a} : 0 \le a \le m/4, 0 \le b \le m/4 - a, i \in \{2,3\}\} \cup \\ \{v_{2a+4b+i,m-1-2a} : 0 \le a \le m/4, 0 \le b \le m/4 - a, i \in \{2,3\}\}.\end{array}$$

The instance for $m = 8$ is shown in Figure 3. By shifting and copying this graph and the dominating set according to $v_{i,j} \to v_{i+m+1,j}$, we get a perfect total dominating set for the $m \times (2m+3)$ grid, and this shifting can be repeated to give perfect total dominating sets for the $m \times n$ grid when $n \equiv 1 \pmod{m+1}$. A part of such a graph is shown in Figure 4 for $m = 8$.

Again we extract the $m \times n$ grid induced by $v_{i,j}$, $0 \le i \le n-1$, $0 \le j \le m-1$ leading to a perfect total dominating set in the $m \times n$ grid of size $((n+1)(m^2+2m)-2m)/(4(m+1))$ when $n \equiv 1 \pmod{m+1}$ and a total dominating set of size $((n+2)(m^2+2m)-4m)/(4(m+1))$ when $n \equiv 2 \pmod{m+1}$. The remaining $m-1$ cases $n \equiv 1+i \pmod{m+1}$, $2 \le i \le m+1$, can be handled by the following argument. For even i (the case odd i is similar) and $n \equiv 1 \pmod{m+1}$, there is a dominating set in the $n' \times m = (n+i) \times n$ grid of size at most

$$\begin{array}{l}\dfrac{(n'-i+1)(m^2+2m)-2m}{4(m+1)} + \dfrac{im}{4} = \\ \dfrac{(n'+2)(m^2+2m)-4m}{4(m+1)} - \dfrac{i(m^2+2m)+m^2}{4(m+1)} + \dfrac{im}{4} = \\ \dfrac{(n'+2)(m^2+2m)-4m}{4(m+1)} + \dfrac{i(m^2+m-m^2-2m)-m^2}{4(m+1)} = \\ \dfrac{(n'+2)(m^2+2m)-4m}{4(m+1)} - \dfrac{m(m+i)}{4(m+1)} < \\ \dfrac{(n'+2)(m^2+2m)-4m}{4(m+1)}.\end{array}$$

A similar calculation when i is odd and the inequality $((n+2)(m^2+2m)-4m)/(4(m+1)) > ((n+1)(m^2+2m)-2m)/(4(m+1))$ implies that the former expression in this inequality is an upper bound in the case $m \equiv 0 \pmod 4$.

For $m \equiv 2 \pmod 4$, consider the following perfect total dominating set for the $m \times (m+2)$ grid:

$$\begin{array}{l}\{v_{2a,2a+4b+i} : 0 \le a \le (m-2)/4, 0 \le b \le (m-2)/4 - a, i \in \{0,1\}\} \cup \\ \{v_{m+1-2a,2a+4b+i} : 0 \le a \le (m-2)/4, 0 \le b \le (m-2)/4 - a, i \in \{0,1\}\} \cup \\ \{v_{2a+4b+i,2a} : 0 \le a \le (m-6)/4, 0 \le b \le (m-6)/4 - a, i \in \{3,4\}\} \cup \\ \{v_{2a+4b+i,m-1-2a} : 0 \le a \le (m-6)/4, 0 \le b \le (m-6)/4 - a, i \in \{3,4\}\}.\end{array}$$

The instance for $m = 8$ is shown in Figure 5. We can repeat this structure in the same way as for $m \equiv 0 \pmod 4$. Once more we extract the $m \times n$ grid induced by $v_{i,j}, 0 \le i \le n-1, 0 \le j \le m-1$ to obtain a perfect total dominating set in the $m \times n$ grid of size $((n+1)(m^2+2m)+2m+4)/(4(m+1))$ when $n \equiv 1 \pmod{m+1}$ and a total dominating set of size $((n+2)(m^2+2m)-4m)/(4(m+1))$ when $n \equiv 2 \pmod{m+1}$. All other cases modulo $m+1$ can be handled as we saw for $m \equiv 0 \pmod 4$, to arrive at an overall upper bound of $((n+2)(m^2+2m)-2m+4)/(4(m+1))$ dominating vertices when $m \equiv 2 \pmod 4$.

To summarize, when m is even we have

$$\gamma_t(P_m \square P_n) \le \begin{cases} \frac{(n+2)(m^2+2m)-4m}{4(m+1)}, & \text{if } m \equiv 0 \pmod 4 \\ \frac{(n+2)(m^2+2m)-2m+4}{4(m+1)}, & \text{if } m \equiv 2 \pmod 4. \end{cases}$$

□

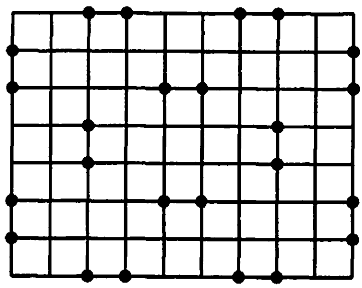

Figure 3: Perfect total dominating set in the 8×10 grid.

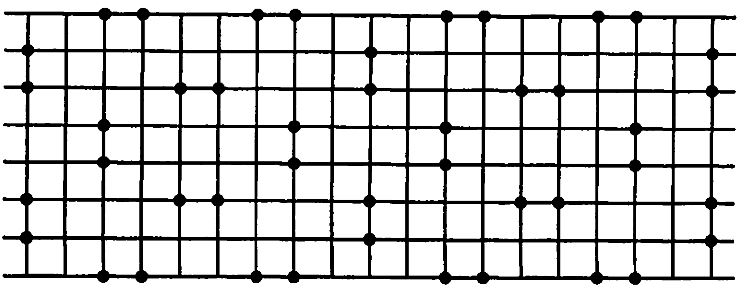

Figure 4: Perfect total dominating set in an infinite graph.

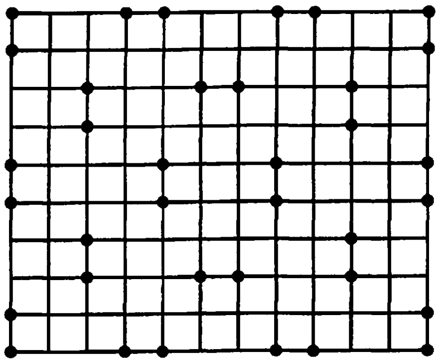

Figure 5: Perfect total dominating set for the 10×12 grid.

3 Formulae for $m \leq 28$

3.1 Preliminaries

Here we present some theorems for total dominating sets that are analogous to those presented for dominating sets in [1]; consult that paper for proofs.

Theorem 2. *Every minimum total dominating set can be constructed by an exhaustive search where in each step any undominated vertex is picked, after which all possible ways of dominating this vertex are considered in turn.*

We now define an order of the vertices of an $m \times n$ grid graph with vertices $v_{i,j}$, $0 \leq i \leq m-1$, $0 \leq j \leq n-1$, as defined in the introduction. This notation gives a lexicographic order of the vertices, where $v_{i,j}$ is smaller than $v_{k,l}$ if $i < k$ or if $i = k$ and $j < l$. As we can pick the vertices to be dominated in any order, we have chosen to always consider the lexicographically smallest undominated vertex.

We will next introduce some notations that are useful in the following theorems and in the description of the algorithm. Consider a grid graph $G = (V, E)$.

- For a vertex $v \in V$, the set of vertices dominated by v is denoted by $D(v)$.
- For a set $V' \subseteq V$, the set of vertices dominated by (the vertices in) V' is denoted by $D(V')$. In other words, $D(V') = \cup_{v \in V'} D(v)$.
- For a set $S \subseteq V$, the lexicographically smallest vertex in $V \setminus S$ is denoted by $s(S)$.

We can now present the remaining theorems that will help us in developing the algorithm.

Theorem 3. *Consider an $m \times n$ grid graph $G = (V,E)$, and let $V_1 \subseteq V$ and $V_2 \subseteq V$ such that $|V_1| = |V_2|$ and $D(V_1) \subseteq D(V_2)$. To find a minimum total dominating set in G, one may ignore V_1 and only consider total dominating sets that extend V_2.*

The automorphism group Aut(G) of an $m \times n$ grid graph has order 4 if $m \neq n$ and order 8 if $m = n$. It turns out that in particular the subgroup of order 2 generated by the mapping of $v_{i,j}$ to $v_{i,n+1-j}$ will be useful.

Theorem 4. *Consider an $m \times n$ grid graph $G = (V,E)$ and a mapping $f : V \to V$ such that $f(v_{i,j}) = v_{i,n+1-j}$ for all i, j. Let $V_1 \subseteq V$ and $V_2 \subseteq V$ such that f maps the set V_1 to V_2. To find a minimum total dominating set in G, one may ignore V_1 and only consider total dominating sets that extend V_2.*

3.2 The Algorithm

The algorithm to be presented is an exhaustive breadth-first search (BFS) algorithm with the features of dynamic programming [8, Chapter 15] and is similar to that in [1]. The input parameter is the value m, for which we will construct a formula for the total domination number in terms of n.

On each level of the BFS, we have a collection $\mathscr{S}$ of sets of dominated vertices (starting from the empty set), and for each $S \in \mathscr{S}$ we consider all vertices that dominate $s(S)$. When we form a new collection $\mathscr{S}'$ of sets of dominated vertices from an old collection $\mathscr{S}$, we use Theorem 3 whenever possible to prune the search. Also a combination of Theorems 3 and 4 can be used for pruning.

By applying the algorithm described so far, we calculate $\gamma_t(P_m \Box P_n)$ for all n in increasing order. We determine $\gamma_t(P_m \Box P_n)$ simply by checking at what level of the BFS the first set S is created for which $s(S)$ is larger than $v_{m-1,n-1}$. Implementation details of this algorithm can be found in [1].

To give further details of the developed algorithm, the following theorems are necessary. Given a fixed m, we first define

$$n_i := \max\{n : \gamma_t(P_m \Box P_n) \leq i\}.$$

Theorem 5. *Every set of dominated vertices on level i of the BFS is completely defined by its intersection with the set $\{v_{k,l} : n_i - 1 \leq k \leq n_i + 3, 0 \leq l \leq m-1\}$.*

Proof. Let $\mathscr{S}$ be a collection of sets of dominated vertices and let $v_{k,l}$ be the smallest undominated vertex for some fixed set in $\mathscr{S}$ on level i. By definition, $k \leq n_i + 1$.

On each earlier level of the search, a vertex $v_{k',l'}$ with $k' \leq n_i + 1$ has been dominated by some vertex v. Such a vertex v does not dominate any vertices $v_{k'',l''}$ with $k'' \geq n_i + 4$.

Assume that there is a set $S \in \mathscr{S}$ for which there is an undominated vertex $v_{k',l'}$ with $k' \leq n_i - 2$. Then on all earlier levels of the search, extension of a substructure has been carried out by dominating some vertex $v_{k'',l''}$ with $k'' \leq n_i - 2$. It then follows that necessarily $S \subset \{v_{k',l'} : 0 \leq k' \leq n_i, 0 \leq l' \leq m-1\}$, and would have been excluded by Theorem 3, a contradiction.

To sum up, the sets in $\mathscr{S}$ contain all $v_{k',l'}$ with $0 \leq k' \leq n_i - 2$, $0 \leq l' \leq m-1$, and no sets contain vertices $v_{k',l'}$ with $k' \geq n_i + 4$. □

When saving a collection of sets of vertices of an $m \times 5$ subgraph, we relabel the vertices to get the same labelling for all such subgraphs and make it possible to compare sets of vertices on different levels. Such a collection on level i is denoted by $\mathscr{S}_i$. Obviously, we need to detect when $n_i > n_{i-1}$ and the $m \times 5$ subgraph to be considered has to be changed.

Theorem 6. *For any $k \geq 0$, if $\mathscr{S}_i = \mathscr{S}_j$, then $\mathscr{S}_{i+k} = \mathscr{S}_{j+k}$ and $n_{i+k} - n_i = n_{j+k} - n_j$.*

Proof. The algorithm is deterministic for a given m, and n_i and the collection $\mathscr{S}_i$ contains sufficient information to continue from an intermediate stage. Moreover, $n_{i+1} - n_i$ depends only on $\mathscr{S}_i$.

By induction this means that if $\mathscr{S}_i$ equals $\mathscr{S}_j$, then $\mathscr{S}_{i+k}$ will equal $\mathscr{S}_{j+k}$ for all $k \geq 0$, and since $n_{i+k} - n_i$ depends only on $\mathscr{S}_i$ and k, we get that $n_{i+k} - n_i = n_{j+k} - n_j$. □

We now have a procedure for determining a formula for $\gamma_t(P_m \Box P_n)$ when m is fixed.

Theorem 7. *If $\mathscr{S}_i$ equals $\mathscr{S}_j$ and $n_j = n_i + k$, with $k \geq 0$, then $\gamma_t(P_m \Box P_n) = \lfloor ((j-i)n + x_{n \bmod k})/k \rfloor$ for $n \geq n_i$, where $x_{n \bmod k}$ is to be determined by the specific values of $\gamma_t(P_m \Box P'_n)$ for $n_i \leq n' < n_j$.*

Proof. For each $n_i \leq n < n_j$ we choose $x_{n \bmod k}$ such that the formula holds for these values of n.

By Theorem 6 we know that $n_{i+k} - n_i = n_{j+k} - n_j$ for any $k \geq 0$. In particular this means that $\gamma_t(P_m \Box P_{n_j}) = \gamma_t(P_m \Box P_{n_i}) + (j-i)$ and $\gamma_t(P_m \Box P_{n_j+l}) = \gamma_t(P_m \Box P_{n_i+l}) + (j-i)$.

Assume that $\gamma_t(P_m \Box P_n) = \lfloor ((j-i)n + x_{n \bmod k})/k \rfloor$ holds for $n_j \leq n' < n$. Then

$$\begin{aligned}
\gamma_t(P_m \Box P_n) &= \gamma_t(P_m \Box P_{n_i+l+k}) \\
&= \gamma_t(P_m \Box P_{n_j+l}) \\
&= \gamma_t(P_m \Box P_{n_i+l}) + (j-i) \\
&= \gamma_t(P_m \Box P_{n-k}) + (j-i) \\
&= \left\lfloor \frac{(j-i)(n-k) + x_{n \bmod k}}{k} \right\rfloor + (j-i) \\
&= \left\lfloor \frac{(j-i)n + x_{n \bmod k}}{k} \right\rfloor .
\end{aligned}$$

□

The floor function makes it possible to choose k possible integer values for each $x_{n \bmod k}$. We aim at choosing values that minimize the number of cases in the final formula for $\gamma_t(P_m \Box P_n)$. After the formula is obtained, we look at the computed values for $\gamma_t(P_m \Box P_n)$, $m \le n < n_i$ to see whether there will be any exceptions amongst those.

In the implementation of the algorithm one has to take into account that we do not a priori know the start and length of the repeating pattern. We therefore repeatedly update the parameter of the collection $\mathscr{S}_i$ to which future comparisons are made. We have used $i = 10, 20, 40, \ldots, 10 \cdot 2^j, \ldots$.

3.3 Results

In the following we list all formulae obtained by our algorithm. We will always assume that $m \le n$. The formulae can be presented as an expression that is rounded down or up. We will use the former way, as is common.

$$\gamma_t(P_1 \Box P_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor, & \text{if } n \equiv 0 \pmod 4 \\ \lfloor \frac{n+2}{2} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_2 \Box P_n) = \begin{cases} \lfloor \frac{2n+5}{3} \rfloor, & \text{if } n \equiv 1 \pmod 3 \\ \lfloor \frac{2n+2}{3} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_3 \Box P_n) = n$$

$$\gamma_t(P_4 \Box P_n) = \begin{cases} \lfloor \frac{6n+13}{5} \rfloor, & \text{if } n \equiv 0,3 \pmod 5 \\ \lfloor \frac{6n+8}{5} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_5 \Box P_n) = \begin{cases} \lfloor \frac{6n+11}{4} \rfloor, & \text{if } n \equiv 0 \pmod 4 \\ \lfloor \frac{6n+7}{4} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_6 \Box P_n) = \begin{cases} \lfloor \frac{12n+28}{7} \rfloor, & \text{if } n \equiv 5 \pmod 7 \\ \lfloor \frac{12n+21}{7} \rfloor, & \text{if } n \equiv 1,2,3 \pmod 7 \\ \lfloor \frac{12n+14}{7} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_7 \Box P_n) = \begin{cases} 2n+2, & \text{if } n \equiv 0 \pmod 2 \text{ or } n \in \{9,11,15,21\} \\ 2n+1, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_8\Box P_n) = \begin{cases} \lfloor \frac{20n+42}{9} \rfloor, & \text{if } n \equiv 0,7 \pmod 9 \text{ and } n \notin \{9,16\} \\ \lfloor \frac{20n+33}{9} \rfloor, & \text{if } n \equiv 2,3,4,5 \pmod 9 \\ \lfloor \frac{20n+24}{9} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_9\Box P_n) = \begin{cases} \lfloor \frac{10n+15}{4} \rfloor, & \text{if } n \equiv 2 \pmod 4 \\ \lfloor \frac{10n+11}{4} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_{10}\Box P_n) = \begin{cases} \lfloor \frac{30n+67}{11} \rfloor, & \text{if } n \equiv 9 \pmod{11} \text{ and } n \neq 20 \\ \lfloor \frac{30n+56}{11} \rfloor, & \text{if } n \equiv 2,5,7 \pmod{11} \text{ and } n \notin \{13,18\} \\ \lfloor \frac{30n+45}{11} \rfloor, & \text{if } n \equiv 0,1,3,6 \pmod{11} \text{ or } n = 20 \\ \lfloor \frac{30n+34}{11} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_{11}\Box P_n) = \begin{cases} \lfloor \frac{6n+9}{2} \rfloor, & \text{if } n \in \{12,22\} \\ \lfloor \frac{6n+7}{2} \rfloor, & \text{if } n \in \{13,15,17,19,23,27,29,33,37,43,47,57\} \\ \lfloor \frac{6n+5}{2} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_{12}\Box P_n) = \begin{cases} \lfloor \frac{42n+87}{13} \rfloor, & \text{if } n \equiv 0,11 \pmod{13} \text{ and } n \notin \{13,24,26,37\} \\ \lfloor \frac{42n+74}{13} \rfloor, & \text{if } n \equiv 2,4,7,9 \pmod{13} \text{ and } n \notin \{15,17,20\} \\ \lfloor \frac{42n+61}{13} \rfloor, & \text{if } n \equiv 3,5,6,8 \pmod{13} \text{ or } n \in \{13,24,26,37\} \\ \lfloor \frac{42n+48}{13} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_{13}\Box P_n) = \begin{cases} \lfloor \frac{14n+23}{4} \rfloor, & \text{if } n \in \{14,26\} \\ \lfloor \frac{14n+19}{4} \rfloor, & \text{if } n \equiv 0 \pmod 4 \text{ or } n = 19 \\ \lfloor \frac{14n+15}{4} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_{14}\Box P_n) = \begin{cases} \lfloor \frac{56n+122}{15} \rfloor, & \text{if } n \equiv 13 \pmod{15} \text{ and } n \notin \{28,43\} \\ \lfloor \frac{56n+107}{15} \rfloor, & \text{if } n \equiv 2,9,11 \pmod{15} \text{ and } n \notin \{17,24,26,32,41\} \\ \lfloor \frac{56n+92}{15} \rfloor, & \text{if } n \equiv 0,5,6,7 \pmod{15} \text{ and } n \notin \{15,21,22,30\} \\ & \text{or } n \in \{28,43\} \\ \lfloor \frac{56n+77}{15} \rfloor, & \text{if } n \equiv 1,3,4,10 \pmod{15} \text{ or } n \in \{17,24,26,32,41\} \\ \lfloor \frac{56n+62}{15} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_{15}\Box P_n) = \begin{cases} \lfloor \frac{8n+13}{2} \rfloor, & \text{if } n \in \{16,30\} \\ \lfloor \frac{8n+11}{2} \rfloor, & \text{if } n \in \{21,23\} \\ \lfloor \frac{8n+9}{2} \rfloor, & \text{if } n \equiv 0 \pmod 2 \text{ and } n \notin \{16,30\} \\ & \text{or } n \in \{17,19,25,27,31,35,37,39,41,45,49,53,55\} \\ & \text{or } n \in \{59,63,67,73,77,81,91,95,109\} \\ \lfloor \frac{8n+7}{2} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_{16}\Box P_n) = \begin{cases} \lfloor \frac{72n+148}{17} \rfloor, & \text{if } n \equiv 0,15 \pmod{17} \text{ and } n \notin \{17,32,34,49,51,66\} \\ \lfloor \frac{72n+131}{17} \rfloor, & \text{if } n \equiv 2,4,11,13 \pmod{17} \\ & \text{and } n \notin \{19,21,28,30,36,38,45,53\} \\ \lfloor \frac{72n+114}{17} \rfloor, & \text{if } n \equiv 6,7,8,9 \pmod{17} \text{ and } n \notin \{23,24,25\} \\ & \text{or } n \in \{34,49,51,66\} \\ \lfloor \frac{72n+97}{17} \rfloor, & \text{if } n \equiv 3,5,10,12 \pmod{17} \\ & \text{or } n \in \{19,21,28,30,36,38,45,53\} \\ \lfloor \frac{72n+80}{17} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_{17}\Box P_n) = \begin{cases} \lfloor \frac{18n+31}{4} \rfloor, & \text{if } n \in \{34\} \\ \lfloor \frac{18n+27}{4} \rfloor, & \text{if } n \in \{20,32,36,52\} \\ \lfloor \frac{18n+23}{4} \rfloor, & \text{if } n \equiv 2 \pmod{4} \text{ and } n \notin \{34\} \text{ or } n \in \{23,25,27,43\} \\ \lfloor \frac{18n+19}{4} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_{18}\Box P_n) = \begin{cases} \lfloor \frac{90n+193}{19} \rfloor, & \text{if } n \equiv 17 \pmod{19} \text{ and } n \notin \{36,55,74\} \\ \lfloor \frac{90n+174}{19} \rfloor, & \text{if } n \equiv 2,13,15 \pmod{19} \\ & \text{and } n \notin \{21,32,34,40,51,53,59,72\} \\ \lfloor \frac{90n+155}{19} \rfloor, & \text{if } n \equiv 0,6,9,11 \pmod{19} \\ & \text{and } n \notin \{19,25,28,30,38,44,49,57\} \text{ or } n \in \{55,74\} \\ \lfloor \frac{90n+136}{19} \rfloor, & \text{if } n \equiv 4,5,7,10 \pmod{19} \text{ and } n \notin \{23,26,29\} \\ & \text{or } n \in \{32,40,51,53,59,72\} \\ \lfloor \frac{90n+117}{19} \rfloor, & \text{if } n \equiv 1,3,8,14 \pmod{19} \\ & \text{or } n \in \{19,25,28,30,36,38,44,49,57\} \\ \lfloor \frac{90n+98}{19} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_{19}\Box P_n) = \begin{cases} \lfloor \frac{10n+17}{2} \rfloor, & \text{if } n \in \{38\} \\ \lfloor \frac{10n+13}{2} \rfloor, & \text{if } n \in \{20,22,24,25,27,29,31,34,36,40,42,47,49,56\} \\ & \text{or } n \in \{58,60,78\} \\ \lfloor \frac{10n+11}{2} \rfloor, & \text{if } n \in \{21,23,33,35,39,43,45,51,53,57,61,65,67,69\} \\ & \text{or } n \in \{71,75,79,83,87,89,93,97,101,105,111,115,119\} \\ & \text{or } n \in \{123,133,137,141,155,159,177\} \\ \lfloor \frac{10n+9}{2} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma(P_{20}\Box P_n)=\begin{cases}\lfloor\frac{110n+225}{21}\rfloor, & \text{if } n\equiv 0,19 \pmod{21} \text{ and } n\notin\{21,40,42,61,63,82,84,103\}\\ \lfloor\frac{110n+204}{21}\rfloor, & \text{if } n\equiv 2,4,15,17 \pmod{21}\\ & \text{and } n\notin\{23,25,36,38,44,46,57,59,65,67,78,86\}\\ \lfloor\frac{110n+183}{21}\rfloor, & \text{if } n\equiv 6,8,11,13 \pmod{21} \text{ and } n\notin\{27,29,32,34,48,50,53\}\\ & \text{or } n\in\{63,82,84,103\}\\ \lfloor\frac{110n+162}{21}\rfloor, & \text{if } n\equiv 7,9,10,12 \pmod{21} \text{ and } n\notin\{28,31\}\\ & \text{or } n\in\{38,44,46,57,59,65,67,78,86\}\\ \lfloor\frac{110n+141}{21}\rfloor, & \text{if } n\equiv 3,5,14,16 \pmod{21}\\ & \text{or } n\in\{21,27,29,32,34,40,42,48,50,53,61\}\\ \lfloor\frac{110n+120}{21}\rfloor, & \text{otherwise,}\end{cases}$$

$$\gamma(P_{21}\Box P_n)=\begin{cases}\lfloor\frac{22n+35}{4}\rfloor, & \text{if } n\in\{40,44,64\}\\ \lfloor\frac{22n+31}{4}\rfloor, & \text{if } n\in\{22,26,38,42,46,62,66,86\}\\ \lfloor\frac{22n+27}{4}\rfloor, & \text{if } n\equiv 0 \pmod{4} \text{ and } n\notin\{40,44,64\}\\ & \text{or } n\in\{27,29,31,33,35,51,53,55,75\}\\ \lfloor\frac{22n+23}{4}\rfloor, & \text{otherwise,}\end{cases}$$

$$\gamma(P_{22}\Box P_n)=\begin{cases}\lfloor\frac{132n+280}{23}\rfloor, & \text{if } n\equiv 21 \pmod{23} \text{ and } n\notin\{44,67,90,113\}\\ \lfloor\frac{132n+257}{23}\rfloor, & \text{if } n\equiv 2,17,19 \pmod{23}\\ & \text{and } n\notin\{25,40,42,48,63,65,71,86,88,94,111\}\\ \lfloor\frac{132n+234}{23}\rfloor, & \text{if } n\equiv 0,6,13,15 \pmod{23}\\ & \text{and } n\notin\{23,29,36,38,46,52,59,61,69,75,84,92\}\\ & \text{or } n\in\{90,113\}\\ \lfloor\frac{132n+211}{23}\rfloor, & \text{if } n\equiv 4,9,10,11 \pmod{23} \text{ and } n\notin\{27,32,33,34,50,56,57\}\\ & \text{or } n\in\{63,71,86,88,94,111\}\\ \lfloor\frac{132n+188}{23}\rfloor, & \text{if } n\equiv 5,7,8,14 \pmod{23} \text{ and } n\notin\{31\}\\ & \text{or } n\in\{23,36,44,46,52,59,61,67,69,75,84,92\}\\ \lfloor\frac{132n+165}{23}\rfloor, & \text{if } n\equiv 1,3,12,18 \pmod{23}\\ & \text{or } n\in\{25,27,32,33,34,40,42,48,50,56,57,65\}\\ \lfloor\frac{132n+142}{23}\rfloor, & \text{otherwise,}\end{cases}$$

$$\gamma(P_{23}\Box P_n)=\begin{cases}\lfloor\frac{12n+17}{2}\rfloor, & \text{if } n\in\{24,26,28,42,44,46,48,50,68,70,72,94\}\\ \lfloor\frac{12n+15}{2}\rfloor, & \text{if } n\in\{29,31,33,35,37,39,55,57,59,61,81,83\}\\ \lfloor\frac{12n+13}{2}\rfloor, & \text{if } n\equiv 0 \pmod{2} \text{ and } n\notin\{24,26,28,42,44,46,48,50,68,70,72,94\}\\ & \text{or } n\in\{25,27,41,43,47,51,53,63,65,69,73,77,79,85,87,91,95\}\\ & \text{or } n\in\{99,103,105,107,109,113,117,121,125,129,131,135,139\}\\ & \text{or } n\in\{143,147,151,157,161,165,169,173,183,187,191,195,209\}\\ & \text{or } n\in\{213,217,235,239,261\}\\ \lfloor\frac{12n+11}{2}\rfloor, & \text{otherwise,}\end{cases}$$

$$\gamma_t(P_{24}\Box P_n) = \begin{cases} \lfloor \frac{156n+318}{25} \rfloor, & \text{if } n \equiv 0,23 \pmod{25} \\ & \text{and } n \notin \{25,48,50,73,75,98,100,123,125,148\} \\ \lfloor \frac{156n+293}{25} \rfloor, & \text{if } n \equiv 2,4,19,21 \pmod{25} \\ & \text{and } n \notin \{27,29,44,46,52,54,69,71,77,79,94\} \\ & \text{or } n \in \{96,102,104,119,127\} \\ \lfloor \frac{156n+268}{25} \rfloor, & \text{if } n \equiv 6,8,15,17 \pmod{25} \\ & \text{and } n \notin \{31,33,40,42,56,58,65,67,81,83,90\} \\ & \text{or } n \in \{100,123,125,148\} \\ \lfloor \frac{156n+243}{25} \rfloor, & \text{if } n \equiv 10,11,12,13 \pmod{25} \\ & \text{and } n \notin \{35,36,37,38,60,61,62\} \\ & \text{or } n \in \{71,77,79,94,96,102,104,119,127\} \\ \lfloor \frac{156n+218}{25} \rfloor, & \text{if } n \equiv 7,9,14,16 \pmod{25} \\ & \text{or } n \in \{25,42,48,50,56,58,65,67,73,75,81,83,90,98\} \\ \lfloor \frac{156n+193}{25} \rfloor, & \text{if } n \equiv 3,5,18,20 \pmod{25} \\ & \text{or } n \in \{27,29,35,36,37,38,44,46,52,54,60,61,62,69\} \\ \lfloor \frac{156n+168}{25} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_{25}\Box P_n) = \begin{cases} \lfloor \frac{26n+43}{4} \rfloor, & \text{if } n \in \{76\} \\ \lfloor \frac{26n+39}{4} \rfloor, & \text{if } n \in \{46,50,54,74,78,102\} \\ \lfloor \frac{26n+35}{4} \rfloor, & \text{if } n \in \{28,32,44,48,52,56,63,72,80,100,104,128\} \\ \lfloor \frac{26n+31}{4} \rfloor, & \text{if } n \equiv 2 \pmod{4} \text{ and } n \notin \{46,50,54,74,78,102\} \\ & \text{or } n \in \{31,33,35,37,39,41,43,59,61,65,67,87,89,91,115\} \\ \lfloor \frac{26n+27}{4} \rfloor, & \text{otherwise,} \end{cases}$$

$$\gamma_t(P_{26}\Box P_n) = \begin{cases} \lfloor \frac{182n+383}{27} \rfloor, & \text{if } n \equiv 25 \pmod{27} \text{ and } n \notin \{52,79,106,133,160\} \\ \lfloor \frac{182n+356}{27} \rfloor, & \text{if } n \equiv 2,21,23 \pmod{27} \\ & \text{and } n \notin \{29,48,50,56,75,77,83,102,104,110,129\} \\ & \text{or } n \in \{131,137,158\} \\ \lfloor \frac{182n+329}{27} \rfloor, & \text{if } n \equiv 0,6,17,19 \pmod{27} \text{ and } n \notin \{27,33,44,46,54,60\} \\ & \text{and } n \notin \{71,73,81,87,98,100,108,114,127,135\} \\ & \text{or } n \in \{133,160\} \\ \lfloor \frac{182n+302}{27} \rfloor, & \text{if } n \equiv 4,10,13,15 \pmod{27} \\ & \text{and } n \notin \{31,37,40,42,58,64,67,69,85,91,96,112\} \\ & \text{or } n \in \{102,110,129,131,137,158\} \\ \lfloor \frac{182n+275}{27} \rfloor, & \text{if } n \equiv 8,9,11,14 \pmod{27} \text{ and } n \notin \{35,38,41,62,65,68\} \\ & \text{or } n \in \{27,52,71,79,87,98,100,106,108,114,127,135\} \\ \lfloor \frac{182n+248}{27} \rfloor, & \text{if } n \equiv 5,7,12,18 \pmod{27} \text{ and } n \notin \{39\} \\ & \text{or } n \in \{29,40,48,56,58,64,67,69,75,77,83,85,91,96,104,112\} \\ \lfloor \frac{182n+221}{27} \rfloor, & \text{if } n \equiv 1,3,16,22 \pmod{27} \\ & \text{or } n \in \{33,35,38,41,44,46,54,60,62,65,68,73,81\} \\ \lfloor \frac{182n+194}{27} \rfloor, & \text{otherwise,} \end{cases}$$

$$
\gamma_t(P_{27}\Box P_n) = \begin{cases}
\lfloor \frac{14n+21}{2} \rfloor, & \text{if } n \in \{50,52,54,56,58,80,82,84,110\} \\
\lfloor \frac{14n+19}{2} \rfloor, & \text{if } n \in \{67,69\} \\
\lfloor \frac{14n+17}{2} \rfloor, & \text{if } n \in \{28,30,32,33,34,35,36,37,39,41,43,45,46,47,48,60,62\} \\
 & \text{or } n \in \{63,65,71,7376,78,86,88,93,95,97,99,106,108,112,114\} \\
 & \text{or } n \in \{123,125,136,138,140,166\} \\
\lfloor \frac{14n+15}{2} \rfloor, & \text{if } n \in \{29,31,49,51,55,59,61,75,77,81,85,89,91,101,103,107\} \\
 & \text{or } n \in \{111,115,119,121,127,129,133,137,141,145,149,151\} \\
 & \text{or } n \in \{153,155,159,163,167,171,175,179,181,185,189,193\} \\
 & \text{or } n \in \{197,201,205,211,215,219,223,227,231,241,245,249\} \\
 & \text{or } n \in \{253,257,271,275,279,283,301,305,309,331,335,361\} \\
\lfloor \frac{14n+13}{2} \rfloor, & \text{otherwise.}
\end{cases}
$$

$$
\gamma_t(P_{28}\Box P_n) = \begin{cases}
\lfloor \frac{210n+427}{29} \rfloor, & \text{if } n \equiv 0,27 \pmod{29} \\
 & \text{and } n \notin \{29,56,58,85,87,114,116,143,145,172,174,201\} \\
\lfloor \frac{210n+398}{29} \rfloor, & \text{if } n \equiv 2,4,23,25 \pmod{29} \text{ and } n \notin \{31,33,52,54,60,62\} \\
 & \text{and } n \notin \{81,83,89,91,110,112,118,120,139,141,147\} \\
 & \text{and } n \notin \{149,168,176\} \\
\lfloor \frac{210n+369}{29} \rfloor, & \text{if } n \equiv 6,8,19,21 \pmod{29} \text{ and } n \notin \{35,37,48,50,64,66\} \\
 & \text{and } n \notin \{77,79,93,95,106,108,122,124,135,151\} \\
 & \text{or } n \in \{145,172,174,201\}, \\
\lfloor \frac{210n+340}{29} \rfloor, & \text{if } n \equiv 10,12,15,17 \pmod{29} \\
 & \text{and } n \notin \{39,41,44,46,68,70,73,75,97,99,102\} \\
 & \text{or } n \in \{112,120,139,141,147,149,168,176\} \\
\lfloor \frac{210n+311}{29} \rfloor, & \text{if } n \equiv 11,13,14,16 \pmod{29} \text{ and } n \notin \{40,42,43,69,72,74\} \\
 & \text{or } n \in \{29,56,79,87,93,95,106,108,114,116,122,124,135\} \\
 & \text{or } n \in \{143,151\} \\
\lfloor \frac{210n+282}{29} \rfloor, & \text{if } n \equiv 7,9,18,20 \pmod{29} \text{ or } n \in \{31,46,54,60,62,68,70\} \\
 & \text{or } n \in \{73,75,81,83,89,91,97,99,102,110,118\} \\
\lfloor \frac{210n+253}{29} \rfloor, & \text{if } n \equiv 3,5,22,24 \pmod{29} \\
 & \text{or } n \in \{35,37,40,42,43,48,50,58,64,66,69,72,74,77,85\} \\
\lfloor \frac{210n+224}{29} \rfloor, & \text{otherwise.}
\end{cases}
$$

By the current results—corroborating those in [10], which considers the cases $m \leq 8$—there are errors in [19, 24]. Among other things, the claim in [19] that $\gamma_t(P_5\Box P_n) = \lfloor \frac{3n+4}{2} \rfloor$ when $n \leq 6$ is incorrect and so is the claim in [24] that $\gamma_t(P_8\Box P_9) = 24$.

3.4 A Conjecture

The formulae listed earlier can be combined to obtain the following conjecture for a general formula.

Conjecture 8. *Let* $N = n \bmod (m+1)$. *If* $m \equiv 0 \pmod 4$ *and* $n > (m^2 - m - 8)/4$, *then*

$$\gamma_t(P_m \Box P_n) = \begin{cases} \left\lfloor \frac{(n+1)(m+2)m+2m}{4(m+1)} \right\rfloor + \left\lfloor \frac{m-N}{4} \right\rfloor, & \text{if } n \equiv 0 \pmod 2 \\ \left\lfloor \frac{(n+1)(m+2)m+2m}{4(m+1)} \right\rfloor + \left\lfloor \frac{N+2}{4} \right\rfloor, & \text{if } n \equiv 1 \pmod 2). \end{cases}$$

If $m \equiv 1 \pmod 4$ *and* $n > (m^2 - 4m - 13)/4$, *then*

$$\gamma_t(P_m \Box P_n) = \begin{cases} \left\lfloor \frac{(m+1)(n+1)+5}{4} \right\rfloor, & \text{if } m \equiv 1 \pmod 8 \text{ and } n \equiv 2 \pmod 4 \\ & \text{or if } m \equiv 5 \pmod 8 \text{ and } n \equiv 0 \pmod 4 \\ \left\lfloor \frac{(m+1)(n+1)+1}{4} \right\rfloor, & \text{otherwise.} \end{cases}$$

If $m \equiv 2 \pmod 4$ *and* $n > (m^2 - m - 10)/4$, *then*

$$\gamma_t(P_m \Box P_n) = \begin{cases} \left\lfloor \frac{(n+1)(m+2)m+2(m-2)}{4(m+1)} \right\rfloor + \max\left(\frac{m-N-6}{4}, 0\right), & \text{if } n \equiv 0 \pmod 4 \\ \left\lfloor \frac{(n+1)(m+2)m+2m}{4(m+1)} \right\rfloor + \left\lfloor \frac{N+3}{4} \right\rfloor, & \text{if } n \equiv 1 \pmod 2 \\ \left\lfloor \frac{(n+1)(m+2)m+2(m-2)}{4(m+1)} \right\rfloor + \left\lfloor \frac{m-N}{4} \right\rfloor, & \text{if } n \equiv 2 \pmod 4). \end{cases}$$

If $m \equiv 3 \pmod 4$ *and* $n > (m^2 - 7)/2$, *then*

$$\gamma_t(P_m \Box P_n) = \begin{cases} \left\lfloor \frac{(m+1)(n+1)+2}{4} \right\rfloor, & \text{if } m \equiv 5 \pmod 8 \text{ and } n \equiv 0 \pmod 2 \\ \left\lfloor \frac{(m+1)(n+1)-2}{4} \right\rfloor, & \text{otherwise.} \end{cases}$$

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