The rainbow index of complementary graphs *

Fengnan Yanling,† Chengfu Ye, Yaping Mao, Zhao Wang

Department of Mathematics, Qinghai Normal University, Xining, Qinghai 810008, China

E-mails: fengnanyanlin@yahoo.com; yechf@qhnu.edu.cn; maoyaping@ymail.com; wangzhao380@yahoo.com.

Abstract

The k-rainbow index $rx_k(G)$ of a connected graph G was introduced by Chartrand, Okamoto and Zhang in 2010. Let G be a nontrivial connected graph with an edge-coloring $c : E(G) \rightarrow \{1, 2, \ldots, q\}$, $q \in \mathbb{N}$, where adjacent edges may be colored the same. A tree T in G is called a *rainbow tree* if no two edges of T receive the same color. For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a such subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. For $S \subseteq V(G)$ and $|S| \geq 2$, an S-Steiner tree T is said to be a *rainbow S-tree* if no two edges of T receive the same color. The minimum number of colors that are needed in an edge-coloring of G such that there is a rainbow S-tree for every k-set S of $V(G)$ is called the *k-rainbow index* of G, denoted by $rx_k(G)$. In this paper, we consider when $|S| = 3$. An upper bound of complete multipartite graphs is obtained. By this upper bound, for a connected graph G with $diam(G) \geq 3$, we give an upper bound of its complementary graph.

Keywords: rainbow S-tree, k-rainbow index.

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†Corresponding author

1 Introduction

The rainbow connections of a graph which are applied to measure the safety of a network are introduced by Chartrand, Johns, McKeon and Zhang [5]. Readers can see [5, 6, 7] for details. Consider an edge-coloring (not necessarily proper) of a graph $G = (V, E)$. We say that a path of G is *rainbow*, if no two edges on the path have the same color. An edge-colored graph G is *rainbow connected* if every two vertices are connected by a rainbow path. The minimum number of colors required to rainbow color a graph G is called *the rainbow connection number*, denoted by $rc(G)$. For more results on the rainbow connection, we refer to the survey paper [13] of Li, Shi and Sun and a new book [14] of Li and Sun. All graphs considered in this paper are finite, undirected and simple. We follow the notation and terminology of Bondy and Murty [1], unless otherwise stated.

For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a such subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. A tree T in G is a *rainbow tree* if no two edges of T are colored the same. For $S \subseteq V(G)$, a *rainbow S-Steiner tree* (or simply, *rainbow S-tree*) is a rainbow tree connecting S. For a fixed integer k with $2 \leq k \leq n$, the edge-coloring c of G is called a *k-rainbow coloring* if for every k-subset S of $V(G)$ there exists a rainbow S-tree. In this case, G is called *rainbow k-tree-connected.* The minimum number of colors that are needed in a k-rainbow coloring of G is called the *k-rainbow index* of G, denoted by $rx_k(G)$. When $k = 2$, $rx_2(G)$ is the rainbow connection number $rc(G)$ of G. For more details on k-rainbow index, we refer to [2, 3, 8, 11, 12, 17, 18].

Chartrand, Okamoto and Zhang [7] obtained the following result.

Lemma 1 [7] (1) *For every integer $n \geq 6$, $rx_3(K_n) = 3$.*
(2) *For $3 \leq n \leq 5, rx_3(K_n) = 2$.*

For every connected graph G of order n, it is easy to see that

$$rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_n(G).$$

Chakraborty et showed that computing the rainbow connection number of a graph is NP-hard. So it is also NP-hard to compute k-rainbow index of graph. If G' is a connected spanning subgraph of G, then $rx_k(G) \leq rx_k(G')$. In an edge-colored graph G, we use $c(e)$ denotes the color of an edge e and for a subgraph H of G, $c(H)$ denotes the set of colors of edges in H. For a subset X, of $V(G)$, we use $E[X]$ to denote edge set of the induced subgraph $G[X]$. The *distance* between two vertices u and v in an connected graph G, denoted by $d_G(u,v)$, which is the shortest path between them in G. The *eccentricity* of a vertex v in G is defined as $ecc_G = \max_{x\in V(G)} d_G(v,x)$.

Let k be a positive integer. A subset $D \subseteq V(G)$ is a *k-dominating set* of the graph G if $|N_G(v) \cap D| \geq k$ for every $v \in V \backslash D$. A subset D is a *connected k-dominating set* if it is a k-dominating set and the graph induced by D is connected.

Chandran et al. [4] used a strengthened connected dominating set (connected 2-way dominating set) to prove $rc(G) \leq rc(G[D]) + 3$.

Recently, Li et al. [11] obtained some result.

Lemma 2 *[11] Let G be a 2-connected graph of order n $(n \geq 4)$. Then $rx_3(G) \leq n-2$, with equality if and only if*

- *$G = C_n$;*
- *G is a spanning subgraph of 3-sun, where a 3-sun is a graph which is defined from $C_6 = v_1v_2\ldots v_6v_1$ by adding three edges v_2v_4, v_2v_6 and v_4v_6;*
- *G is a spanning subgraph of $K_5\backslash e$ or G is a spanning subgraph of K_4.*

In [15, 16], Liu and Hu obtained the following theorem.

Lemma 3 *[15] Let G be a connected graph with minimal degree $\delta \geq 3$. If D is a connected 2-dominating set of G, then $rx_3(G) \leq rx_3(G[D]) + 4$.*

Lemma 4 *[16] For any integer s and t with $3 \leq s \leq t$, $rx_3(K_{s,t}) \leq \min\{6, s+t-3\}$. Moreover the bound is tight.*

Lemma 5 *[16] For any integer $t \geq 1$,*

$$rx_3(K_{2,t}) = \begin{cases} 2, & \text{if } t = 1,2; \\ 3, & \text{if } t = 3,4; \\ 4, & \text{if } 5 \leq t \leq 8; \\ 5, & \text{if } 9 \leq t \leq 20; \\ \ell, & \text{if } (\ell-1)(\ell-2)+1 \leq t \leq \ell(\ell-1) \text{ and } \ell \geq 6. \end{cases} \tag{1.1}$$

In Section 2, we obtain an upper bound of complete multipartite graphs by the 2-connected dominating set result, which is obtained by Liu and Hu [15].

Theorem 1 *Let $K_{n_1,n_2,\ldots,n_k}$ be a complete multipartite graphs. If $k \geq 4$, then*

$$rx_3(K_{n_1,n_2,\ldots n_k}) \leq \min\{6, n_1 + n_2 + \ldots + n_k - 2\}.$$

If $k = 3$, then $rx_3(K_{n_1,n_2,n_3}) = 2$ for $n_1 = n_2 = n_3 = 1$ and

$$rx_3(K_{n_1,n_2,n_3}) \leq \begin{cases} \lceil \frac{(n_1+n_2+n_3)}{2} \rceil, & n_1 = n_3 = 1, n_2 \geq 2; \\ \min\{6, n_1 + n_2 + n_3 - 2\}, & n_1 \geq 1, n_2, n_3 \geq 2. \end{cases} \tag{1.2}$$

By this upper bound, for a connected graph G with $diam(G) \geq 3$, we give an upper bound of its complementary graph in Section 3.

Before the state of the next theorem, we give some symbols. For the graph G, we choose a vertex x with $ecc_G(x) = diam(G) = d$. Let $N_G^i(x) = \{v : d_G(x, v) = i\}$ where $0 \leq i \leq d$. So $N_G^0(x) = \{x_0\}$, $N_G^1(x) = N_G(x_0)$ as usual. Then $\cup_{o \leq i \leq d} N_G^i(x)$is a vertex partition of $V(G)$ with $|N_G^i| = n_i$. Let $A = \cup_{i \text{ is even}} N_G^i, B = \cup_{i \text{ is odd}} N_G^i$. So, if $d = 2k$ $(k \geq 2)$, then $A = \cup_{0 \leq i \leq d \text{ is even}} N_G^i$, $B = \cup_{0 \leq i \leq d-1 \text{ is odd}} N_G^i$; if $d = 2k + 1$ $(k \geq 2)$, then $A = \cup_{0 \leq i \leq d-1 \text{ is even}} N_G^i$, $B = \cup_{0 \leq i \leq d \text{ is odd}} N_G^i$. Then by the definition of complement graphs, we know that $\bar{G}[A]$ $(\bar{G}[B])$ contains a spanning complete k_1-partite subgraph (complete k_2-partite subgraph) where $k_1 = \lceil \frac{d+1}{2} \rceil (k_2 = \lceil \frac{d}{2} \rceil)$; see Figure 1.

Theorem 2 *Let G be a connected graph. Then*

(1) *If $diam(G) \geq 7$, then $rx_3(\bar{G}) \leq 7$;*

(2) *If $diam(G) = 6$, then $rx_3(\bar{G}) \leq \max\{7, (\lceil \frac{n_1+n_3+n_5}{2} \rceil + 1\}$;*

(3) *If $diam(G) = 5$, then $rx_3(\bar{G}) \leq \max\{7, (\lceil \frac{n_1+n_3+n_5}{2} \rceil + 1, (\lceil \frac{1+n_2+n_4}{2} \rceil + 1\}$;*

(4) *If $diam(G) = 4$, then $rx_3(\bar{G}) \leq \max\{\ell + 7, n_3 + 7, \ell + \lceil \frac{n_0+n_2+n_4}{2} \rceil + 1, n_3 + \lceil \frac{n_0+n_2+n_4}{2} \rceil + 1\}$;*

(5) *If $diam(G) = 3$, then $rx_3(\bar{G}) \leq \max\{\ell + n_2 + 1, n_2 + n_3 + 1\}$, where ℓ is the same as in Lemma 5 .*

2 Proof of Theorem 1

In this section, we prove Theorem 1 by the 2-connected dominating set.

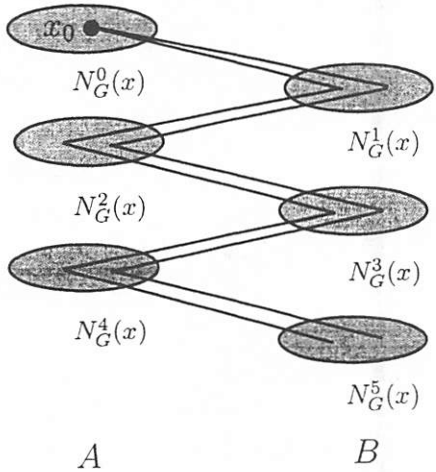

Figure 1: Graphs for the proof of Theorem 2.

Proof of Theorem 1: Set $G = K_{n_1,n_2,\ldots,n_k}$. Suppose $k \geq 4$. Since $K_{n_1,n_2,\ldots n_k}$ with $k \geq 4$ is a 2-connected graph, it follows from Lemma2 that

$$rx_3(K_{n_1,n_2,\ldots n_k}) \leq n_1 + n_2 + \ldots + n_k - 2.$$

Thus, to complete our proof, it suffices to show $rx_3(K_{n_1,n_2,\ldots n_k}) \leq 6$, $k \geq 4$. Let $U_1, U_2 \ldots, U_k$ be all the partite sets of $K_{n_1,n_2,\ldots n_k}$. Note that $K_{1,1,1,1} = K_4$. From Lemma 1, $rx_3(K_4) = 2$. Now we assume that $k \geq 5$, or $k = 4$ and there exist some n_i $(1 \leq i \leq 4)$ such that $n_i \geq 2$. It is clearly that $\delta(G) \geq 3$. Pick up $u_1 \in U_1, u_2 \in U_2, u_3 \in U_3, u_4 \in U_4$ such that $\{u_1, u_2, u_3, u_4\}$ is a connected 2-dominating set. Note that $rx_3(G[D]) = rx_3(K_4)$. By lemma 3, $rx_3(K_{n_1,n_2,\ldots n_k}) \leq rx_3(G[D]) + 4 = 6$.

Suppose $k = 3$. If $n_1, n_2, n_3 = 1$, then it follows from Lemma 1 that $rx_3(K_3) = 2$. If $n_1 \geq 1, n_2, n_3 \geq 2$, then we can find a connected 2-dominating set. Let U_1, U_2, U_3 be the partite sets of K_{n_1,n_2,n_3} and $|U_1| \geq 1, |U_2| \geq 2, |U_3| \geq 2$. Suppose $u_2, u_2' \in U_2$ and $u_3, u_3' \in U_3$. Let $D = \{u_2, u_2', u_3, u_3'\}$. Then D is a connected 2-dominating set. Since $rx_3(G[D]) = rx_3(C_4) = 2$, we have $rc_3(G) \leq rx_3(G[D]) + 4 = 6$. It follows from Lemma 2 that $rx_3(G) \leq n_1 + n_2 + n_3 - 2$. So $rx_3(G) \leq min\{6, n_1 + n_2 + n_3 - 2\}$ as desired. We now assume $n_1 = n_3 = 1, n_2 \geq 2$.

At first, we consider the case n_2 is even. Set $n_2 = 2\ell$. Let U_1, U_2, U_3 be the three parts of complete multipartite graph G such that $|U_1| = 1$, $|U_2| = 2\ell$ and $|U_3| = 1$. Set $U_1 = \{u\}$, $U_2 = \{u_1, u_2, \dots u_\ell, v_1, v_2, \dots v_\ell\}$ and $U_3 = \{v\}$. To show that $rx_3(G) \leq \lceil\frac{(n_1+n_2+n_3)}{2}\rceil = \ell + 1$, we provide a rainbow $(\ell+1)$-edge-coloring c: $E(G) \to (1, 2, \dots, \ell+1)$ of G defined by

$$\begin{cases} c(uu_i) = i, & 1 \leq i \leq \ell; \\ c(uv_i) = i, & 1 \leq i \leq \ell; \\ c(vu_i) = (i+1) \equiv \mathit{mod}\ (\ell), & 1 \leq i \leq \ell; \\ c(vv_i) = (i+1) \equiv \mathit{mod}\ (\ell), & 1 \leq i \leq \ell; \end{cases}$$

see Figure 2.

Now, we prove that it is a 3-rainbow coloring. Set $X = \{v_1, v_2, \dots, v_\ell\}$ and $Y = \{u_1, u_2, \dots, u_\ell\}$. Clearly, $V(G) = \{u, v\} \cup X \cup Y$.

Suppose $|S \cap \{u, v\}| = 2$. Then $|S \cap X| = 1$ or $|S \cap Y| = 1$. Without loss of generality, let $|S \cap X| = 1$. Then $S = \{u, v, v_i\}$ $(1 \leq i \leq \ell)$. Obviously, the tree induced by the edges in $\{uv_i, v_iv\}$ $(1 \leq i \leq \ell)$ is a rainbow S-tree.

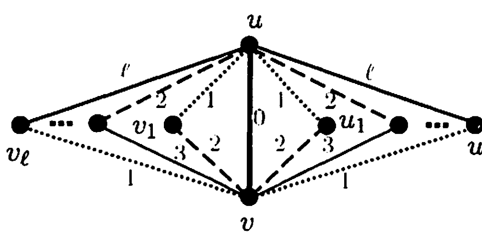

Figure 2: Graphs for the proof of Theorem 2.

Suppose $|S \cap \{u, v\}| = 1$. Without loss of generality, let $u \in S$. Then $|S \cap X| = 2$ or $|S \cap Y| = 2$ or $|S \cap X| = 1$ and $|S \cap Y| = 1$. If $|S \cap X| = 2$, then the tree induced by the edges in $\{uv_i, uv_j\}$ $(1 \leq i, j \leq \ell \ and \ i \neq j)$ is a rainbow S-tree. If $|S \cap X| = 1$ and $|S \cap Y| = 1$, then the tree induced by the edges in $\{uv_i, uu_j\}$ for $i \neq j$ or $\{v_iu, uv, vu_j\}$ for $i = j$ is a rainbow S-tree.

Suppose $|S \cap \{u, v\}| = 0$. Then $|S \cap X| = 3$ or $|S \cap Y| = 3$ or $|S \cap X| = 2$ and $|S \cap Y| = 1$ or $|S \cap X| = 1$ and $|S \cap Y| = 2$. If $|S \cap X| = 3$, then the tree induced by edges in $\{uv_i, uv_j, uv_h\}$ $(1 \leq i, j, h \leq \ell)$is a rainbow S-tree. If $|S \cap X| = 2$, then $|S \cap Y| = 1$ and hence the tree induced by the edges in $\{uv_i, uv_j, uu_h\}$ or $\{vv_i, vv_j, uv, vu_h\}$ (for $h = i$, or $h = j$) is a rainbow S-tree.

Next, we consider the case n_2 is odd. Set $n_2 = 2\ell - 1$. We delete the vertex u_ℓ in G. Then one can also check that the above edge-coloring c is a 3-rainbow coloring, as desired. ∎

3 Proof of Theorem 2

Proof of Theorem 2: If $diam(G) \leq 2$, then $\bar{G}$ is disconnected. Since we only consider the connected graphs for rainbow index. So we assume that $diam(G) \geq 3$.

(1) Suppose $diam(G) \geq 7$. Then $k_1 \geq 4$ and $k_2 \geq 4$. From Theorem 1, we have $rx_3(\bar{G}[A]) \leq 6$, and $rx_3(\bar{G}[B]) \leq 6$. We now give $\bar{G}$ an edge-coloring as follow.At first we first give the subgraph $\bar{G}[A]$ a rainbow edge-coloring using six colors, and then we give the subgraph $\bar{G}[B]$ a rainbow coloring using the same colors as that of the subgraph $\bar{G}[A]$; Next we give a fresh color to all edges between the subgraph $\bar{G}[A]$ and the subgraph $\bar{G}[B]$. Let us now prove the edge-coloring c is a 3-rainbow coloring. It is sufficient to show that there is a rainbow S-tree for any $|S| = 3$ and $S \subseteq V(G)$. Say $S = \{x, y, z\}$. If $S \subseteq A$, then there is a rainbow S-tree since $rx_3(\bar{G}(A)) \leq 6$; If $S \subseteq B$, then there is also a rainbow S-tree since $rx_3(\bar{G}(B)) \leq 6$. Suppose $|S \cap A| = 2$ or $|S \cap B| = 2$. Without loss of generality, let $|S \cap A| = 2$. Assume $x, y \in A, andz \in B$. For any $z \in B$, we can find a vertex z_0 in A which is adjacent to z. Since we can find a rainbow tree connecting $\{x, y, z_0\}$, say T', it follows that the tree induced by the edges in $zz_0 \cup E(T')$ is a rainbow tree connecting $\{x, y, z\}$.

(2) Suppose that $diam(G) = 6$. Firstly, we consider the case $n_0 = n_2 = n_4 = n_6 = 1$. Clearly, there is a spanning subgraph K_4 in $\bar{G}[A]$. By Lemma 1, we have $rx_3(\bar{G}[A]) \leq 2$. If $n_1 = n_3 = n_5 = 1$, then there is a spanning subgraph K_3 in $\bar{G}[B]$. By Lemma 1, we have $rx_3(\bar{G}[B])) \leq 2$. By the method shown in Case 1, one can prove that $rx_3(\bar{G}) \leq 3$. If there exists only one element in $\{n_1, n_3, n_5\}$, say n_i, such that $n_i \geq 2$ $(i \in \{1, 3, 5\})$. From Theorem 1, we have $rx_3(\bar{G}[B]) \leq \lceil \frac{n_1+n_3+n_5}{2} \rceil$. We now give $\bar{G}$ an edge-coloring as follow: We first give the subgraph $\bar{G}[A]$ a rainbow edge-coloring using two colors, and then we give the subgraph $\bar{G}[B]$ a rainbow coloring using $\lceil \frac{n_1+n_3+n_5}{2} \rceil$ colors. Since $\lceil \frac{n_1+n_3+n_5}{2} \rceil \geq 2$, it follows that $c(\bar{G}(A)) \subset c(\bar{G}(B))$. Next, we give a fresh color to all edges between the subgraph $\bar{G}[A]$ and the subgraph $\bar{G}[B]$. So we have $rx_3(\bar{G}) \leq \lceil \frac{n_1+n_3+n_5}{2} \rceil + 1$, as desired. Suppose that there exists two elements in $\{n_1, n_3, n_5\}$, say n_i, n_j, such that $n_i \geq 2, n_j \geq 2$ $(i \in \{1, 3, 5\})$. Then $\delta(G) \geq 3$ and we can find a connected 2-dominating set of G. By Lemma

3, we have $rx_3(\bar{G}[B]) \leq 6$. One can prove that $rx_3(\bar{G}) \leq 7$.

Next, we consider the case that there exists some element in $\{n_0, n_2, n_4, n_6\}$, say n_i, such that $n_i \geq 2$ ($i \in \{0, 2, 4, 6\}$). By Theorem 1, we have $rx_3(\bar{G}[A]) \leq 6$. If $n_1 = n_3 = n_5 = 1$, then $rx_3(\bar{G}[B])) \leq 2$ and hence $rx_3(\bar{G}) \leq 7$. If there exists a element in $\{n_1, n_3, n_5\}$, say n_i, such that such that $n_i \geq 2$ ($i \in \{1, 3, 5\}$). Then $rx_3(\bar{G}[B]) \leq \lceil \frac{n_1+n_3+n_5}{2} \rceil$ and hence $rc_3(\bar{G}) \leq \max\{7, \lceil \frac{n_1+n_3+n_5}{2} \rceil + 1\}$. Suppose that there exists two elements in $\{n_1, n_3, n_5\}$, say n_i, n_j, such that $n_i \geq 2, n_j \geq 2$ ($i, j \in \{1, 3, 5\}$). Then $rx_3((\bar{G}[B])) \leq 6$ and hence $rx_3(\bar{G}) \leq 7$.

Above all, $rx_3(\bar{G}) \leq \max\{7, (\lceil \frac{n_1+n_3+n_5}{2} \rceil + 1\}$.

(3) Suppose that $diam(G) = 5$. Similarly to the proof of (2), the result holds.

(4) Suppose that $diam(G) = 4$. Firstly, we consider the case $n_0 = n_2 = n_4 = 1$. Observe that there is a spanning subgraph K_3 in $\bar{G}[A]$. By Lemma 1, we have $rx_3(\bar{G}[A])) \leq 2$.

If $n_1 \geq n_3 \geq 3$, then it follows by Lemma 4 that $rc_3(\bar{G}[B]) \leq 6$. We now give $\bar{G}$ an edge-coloring as follow: We first give the subgraph $\bar{G}[A]$ a rainbow edge-coloring using two colors, then we give the subgraph $\bar{G}[B]$ a rainbow coloring using another six colors, and last we give a fresh color to all edges between the subgraph $\bar{G}[A]$ and the subgraph $\bar{G}[B]$. To show $rc_3(\bar{G}) \leq 8$, it suffices to prove that there is a rainbow S-tree for any $S \subseteq V(G)$ and $|S| = 3$. If $S \subseteq A$, then there is a rainbow S-tree since $rx_3(\bar{G}(A)) \leq 2$. If $S \subseteq B$, then there is also a rainbow S-tree since $rx_3(\bar{G}(B)) \leq 6$. Suppose $|S \cap A| = 2$. Then $|S \cap B| = 1$. Let $x, y \in A$ and $z \in B$. For any $z \in \bar{G}[B]$, we can find a vertex z_0 in A such that $zz_0 \in E(\bar{G})$. Since we can find a rainbow tree connecting $\{x, y, z_0\}$, say T', it follows that the tree induced by the edges in $zz_0 \cup E(T')$ is a rainbow tree connecting $\{x, y, z\}$. Suppose $|S \cap B| = 2$. Then $|S \cap A| = 1$. Let $x, y \in B$ and $z \in A$. If $z \in N_G^0(x)$, then we can find a vertex $z_0 \in N_G^3(x)$ such that $zz_0 \in E(\bar{G})$. Note that there is a rainbow tree connecting $\{x, y, z_0\}$, say T'. The the tree induced by the edges in $zz_0 \cup E(T')$ is a rainbow tree connecting $\{x, y, z\}$. If $z \in N_G^4(x)$, then we can find a vertex $z_0 \in N_G^1(x)$ such that $zz_0 \in E(\bar{G})$. The the tree induced by the edges in $zz_0 \cup E(T')$ is a rainbow tree connecting $\{x, y, z\}$. If $z \in N_G^2(x)$, then z is adjacent to the vertex in $N_G^0(x)$, say x_0. Then there exists a vertex $x_0' \in N_G^3(x)$ such that there is a rainbow S-tree connecting $\{x, y, x_0'\}$, say T'. Furthermore, the tree induced by the edges in $\{zx_0, x_0x_0'\} \cup E(T')$ is a rainbow S-tree.

If $n_1 = 2$ and $n_3 \geq 1$, then it follows by Lemma 4 that $rc_3(\bar{G}[B]) \leq \ell$, where $(\ell - 1)(\ell - 2) + 1 \leq n_3 \leq \ell(\ell - 1)$ and $\ell \geq 6$. From the above discussion, we now give the subgraph $\bar{G}[A]$ a rainbow edge-coloring using two colors, then we give the subgraph $\bar{G}[B]$ a rainbow coloring using the other ℓ colors, and last we give a fresh color to all edges between the subgraph $\bar{G}[A]$ and the subgraph $\bar{G}[B]$. One can prove that there is a rainbow S-tree for any $S \subseteq V(G)$ and $|S| = 3$, and $rx_3(\bar{G}) \leq \ell + 3$.

If $n_1 = 1, n_3 \geq 2$, then $rx_3(\bar{G}[B]) = n_3$. The same as above, we have $rx_3(\bar{G}) \leq n_3 + 3$. We conclude that $rx_3(\bar{G}) \leq \max\{n_3 + 3, \ell + 3\}$.

Secondly, there exists only one element in $\{n_2, n_4\}$, say n_i, such that $n_i \geq 2$ $(i \in \{2, 4\})$. By Theorem 1, we have $rx_3(\bar{G}[A]) \leq \lceil \frac{n_0+n_2+n_4}{2} \rceil$. If $n_1 \geq n_3 \geq 3$, then it follows by Lemma 4 that $rc_3(\bar{G}[B]) \leq 6$. So $rx_3(\bar{G}) \leq 7 + \lceil \frac{n_0+n_2+n_4}{2} \rceil$. If $n_1 = 2$ and $n_3 \geq 1$, then it follows by Lemma 4 that $rc_3(\bar{G}[B]) \leq \ell$, where $(\ell-1)(\ell-2)+1 \leq n_3 \leq \ell(\ell-1)$ and $\ell \geq 6$. So $rx_3(\bar{G}) \leq \ell + \lceil \frac{n_0+n_2+n_4}{2} \rceil + 1$. If $n_1 = 1, n_3 \geq 2$, then $rc_3(\bar{G}[B]) = n_3$. So $rx_3(\bar{G}) \leq n_3 + \lceil \frac{n_0+n_2+n_4}{2} \rceil + 1$. We conclude that $rx_3(\bar{G}) \leq \max\{n_3 + \lceil \frac{n_0+n_2+n_4}{2} \rceil + 1, \ell + \lceil \frac{n_0+n_2+n_4}{2} \rceil + 1\}$.

In the end, we consider the case that $n_0 = 1$ and $n_2 \geq n_4 \geq 2$. By Theorem 1, we have $rx_3(\bar{G}[A]) \leq 6$, and hence $rx_3(\bar{G}) \leq \max\{\ell + 7, n_3 + 7\}$.

From all of the above argument, we know that $rx_3(\bar{G}) \leq \max\{\ell + 7, n_3 + 7, n_3 + \lceil \frac{n_0+n_2+n_4}{2} \rceil + 1, \ell + \lceil \frac{n_0+n_2+n_4}{2} \rceil + 1\}$.

(5) Suppose that $diam(G) = 3$. Since $n_0 = 1$, it follows that $rx_3(\bar{G}[A]) = n_2$. If $n_1 \geq n_3 \geq 3$, then $rx_3(\bar{G}[B]) \leq 6$ and hence $rx_3(\bar{G}) \leq n_2 + 7$. If $n_1 = 2$ and $n_3 \geq 1$, then $rx_3(\bar{G}[B]) \leq \ell$ where $(\ell-1)(\ell-2)+1 \leq n_3 \leq \ell(\ell-1)$ $(\ell \geq 6)$, and hence $rx_3(\bar{G}) \leq n_2 + \ell + 1$. If $n_1 = 1, n_3 \geq 2$, then $rx_3(\bar{G}[B]) = n_3$ and hence $rx_3(\bar{G}) \leq n_2 + n_3 + 1$. From the above argument, we know that $rx_3(\bar{G}) \leq \max\{\ell + n_2 + 1, n_2 + n_3 + 1\}$. ∎

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