# The Set Chromatic Number of a Digraph

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**Abstract.** Given a (not necessarily proper) coloring of a digraph $c : V(D) \to \mathbb{N}$, let $OC(v)$ denote the set of colors assigned to the out-neighbors of $v$. Similarly, let $IC(v)$ denote the set of colors assigned to the in-neighbors of $v$. Then $c$ is a set coloring of $D$ provided $(u, v) \in A(D)$ implies $OC(u) \neq OC(v)$. Analogous to the set chromatic number of a graph given by Chartrand, et al. [3], we define $\chi_s(D)$ as the minimum number of colors required to produce a set coloring of $D$. We find bounds for $\chi_s(D)$ where $D$ is a digraph and where $D$ is a tournament. In addition we consider a second set coloring, where $(u, v) \in A(D)$ implies $OC(u) \neq IC(v)$.

**Keywords:** set coloring, chromatic number, digraph, tournament

There are many variations of graph coloring and, to a lesser extent, digraph coloring. The idea of distinguishing vertices of a graph or distinguishing adjacent pairs of vertices of a graph is also well studied. The classic *proper* vertex coloring of a graph, in which neighboring vertices must have distinct colors, is an example of a coloring that distinguishes adjacent pairs of vertices. Another example of a vertex distinguishing coloring arises from certain proper edge colorings of the graph (see Balister et al. [1]).

In [3] Chartrand, Okamoto, Rasmussen, and Zhang introduce another way in which adjacent vertices may be distinguished by associated sets of colors. In a graph $G$, assign colors to the vertices, $c : V(G) \to \mathbb{N}$, where adjacent vertices might have the same color. Given $v \in V(G)$, the *neighborhood color set*, denoted by $NC(v)$ is the set of colors of the neighbors of $v$. Then $c$ is *set neighbor-distinguishing* provided for each edge $\{u, v\} \in E(G)$, $NC(u) \neq NC(v)$. For brevity, a set neighbor-distinguishing coloring is also known as a *set coloring*. Given a graph $G$, the minimum number of colors required to produce a set coloring is the *set chromatic number* of $G$, denoted by $\chi_s(G)$. As the authors of [3] point out, there are cases where we can distinguish adjacent pairs of vertices using fewer colors with a set coloring as compared to a proper coloring.

In this paper, we consider a similarly defined chromatic parameter for digraphs. The idea of taking a parameter defined for graphs and considering it in a directed setting is certainly nothing new (e.g., coloring, domination, connectedness). The classic way to properly color the vertices of a digraph

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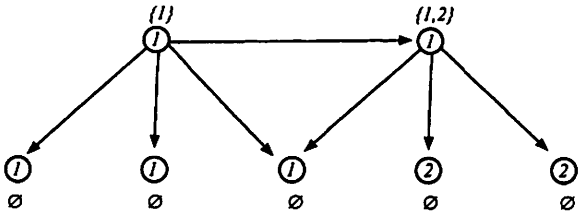


Figure 1: Each vertex of the digraph is assigned a color (labeled within each vertex) so that if two vertices are joined by an arc, the colors assigned to their out-neighbors (labeled outside each vertex) are different.

mirrors the proper coloring of a graph: assign each vertex a color so that pairs connected by an arc have different colors. For example, consider [4], wherein a connection is made between this type of coloring and an arc coloring of the digraph. In this version of coloring vertices of a digraph, the direction of the arcs has no effect on the coloring. Over time, alternative vertex colorings for digraphs have been proposed. One such vertex coloring of a digraph, wherein the color classes are acyclic, leads to nice results (see Bokal et al. [2] for example). Applying the ideas in [3] to digraphs is appealing because the colors used by out-neighbors will be different (see Figure 1). This could be meaningful when the digraph is used to model relationships (e.g., predator and prey).

Before proceeding with formal definitions, we note that graphs and digraphs are assumed to be free of multi-edges and loops. Given a vertex $v$ in digraph $D$, the *out-neighborhood* of $v$, $O(v)$, and the *in-neighborhood* of $v$, $I(v)$, are defined as

$$O(v) = \{u : (v,u) \in A(D)\} \text{ and } I(v) = \{u : (u,v) \in A(D)\}.$$

Given a (not necessarily proper) coloring of a digraph, $c : V(D) \to \mathbb{N}$, let $OC(v)$ denote the *out color set*, the set of colors assigned to the out-neighbors of $v$ and let $IC(v)$ denote the *in color set* the set of colors assigned to the in-neighbors of $v$. Note that if $O(v) = \emptyset$, then $OC(v) = \emptyset$ (see Figure 2). Similarly, if $I(v)$ is empty, then so is $IC(v)$.

We say that a coloring $c$ is a *set coloring* if for each arc $(u,v) \in A(D)$, $OC(u) \neq OC(v)$. Observe that any results derived using this definition are also true when we replace out color sets with in color sets since we change out-neighbors to in-neighbors by reversing all arcs. The minimum number of colors required to produce a set coloring for a digraph $D$ will be denoted by $\chi_s(D)$.

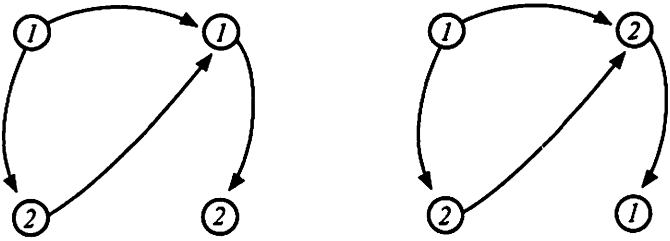


Figure 2: The coloring on the left is a set coloring, but not a set coloring of Type 2. The coloring on the right is a set coloring of Type 2, but not a set coloring. For this digraph, $\chi_s(D) = \chi_{s'}(D) = 2$.

We say that a coloring $c$ is a *set coloring of Type 2* if for each arc $(u, v) \in A(D)$, $OC(u) \neq IC(v)$. The minimum number of colors required to produce a set coloring of Type 2 for digraph $D$ will be denoted by $\chi_{s'}(D)$. Unlike the set coloring case, the set chromatic number of Type 2 is unchanged when the arcs of the digraph are reversed.

The *underlying graph* of a digraph $D$, denoted by $UG(D)$, is the graph with the same vertex set as $D$ and edge set so $\{x, y\}$ is an edge if and only if $(x, y)$ or $(y, x)$ is an arc in $D$. Notice, as illustrated by the example in Figure 2, that by using one of these set colorings it may be possible to distinguish vertices connected by an arc using fewer colors that would be needed for a proper coloring of the underlying graph.

# 1 Preliminary Observations

Observe that for symmetric digraphs, $\chi_s(D) = \chi_{s'}(D) = \chi_s(UG(D))$. Chartrand, et al. [3] observed that $\chi_s(G) = 1$ if and only if $G$ has no edges. The same can be said for set colorings of Type 2.

**Observation 1.1.** *If $D$ is a digraph, then $\chi_{s'}(D) = 1$ if and only if $D$ has no arcs.*

While the same cannot be said for set colorings (e.g., consider an oriented $K_2$), few digraphs have $\chi_s(D) = 1$.

**Observation 1.2.** *If $D$ is a digraph, then $\chi_s(D) = 1$ if and only if for every arc $(u, v) \in A(D)$, $v$ has empty outset.*

We begin with the following bound known for undirected graphs.

**Proposition 1.3.** *(Okamoto, Rasmussen, and Zhang, [5].) For every graph, $\chi_s(G) \leq \chi(G)$.*

If we use the chromatic number of the underlying graph of the digraph, we conclude the analogous bound for digraphs.

**Proposition 1.4.** *For any digraph $D$, $\chi_s(D) \leq \chi(UG(D))$.*

*Proof.* Let $c$ be a proper coloring of $UG(D)$ and let $(x,y) \in A(D)$. So $\{x,y\} \in E(UG(D))$. This means that $c(y) \in OC(x)$. But since $c$ is a proper coloring, for all $v \in O(y)$, $c(v) \neq c(y)$. So, $c(y) \notin OC(y)$. Thus $OC(x) \neq OC(y)$. Therefore $c$ is a set coloring. □

**Proposition 1.5.** *For any digraph $D$, $\chi_{s'}(D) \leq \chi(UG(D))$.*

*Proof.* Let $c$ be a proper coloring of $UG(D)$ and let $(x,y) \in A(D)$. So $\{x,y\} \in E(UG(D))$. This means that $c(x) \in IC(y)$. But since $c$ is a proper coloring, for all $v \in O(x)$, $c(v) \neq c(x)$. So $c(x) \notin OC(x)$. Thus $OC(x) \neq IC(y)$. Therefore $c$ is a set coloring of Type 2. □

**Proposition 1.6.** *(Okamoto, Rasmussen, and Zhang, [5].) For every graph $G$,*

$$\chi_s(G) \geq \lceil \log_2(\chi(G)+1) \rceil .$$

**Proposition 1.7.** *For every directed graph $D$, $\chi_s(D) \geq \lceil \log_2 \chi(UG(D)) \rceil$. Furthermore if $D$ has no vertices of out degree zero,*

$$\chi_s(D) \geq \lceil \log_2 (\chi(UG(D)) + 1) \rceil .$$

*Proof.* If we create a set coloring with $\chi_s(D) = k$ colors, we have at most $2^k$ possible distinct sets of colors, including the empty set, that might occur as an out-neighborhood color set. Since no two vertices joined by an arc may have the same out color set, we require at least $\chi(UG(D))$ distinct sets in any set coloring. So $\chi(UG(D)) \leq 2^k$ and thus, $k \geq \log_2(\chi(UG(D))$. Since $k$ is an integer we may round up. If $D$ has no vertices of out degree zero, then for all $x \in V(D)$, $OC(x) \neq \emptyset$. Thus, $\chi(UG(D)) \leq 2^k - 1$ and the result follows. □

Recall that $\omega(G)$ denotes the clique number of $G$, that is, the order of a largest complete subgraph (clique) in $G$. It is well known that $\omega(G) \leq \chi(G)$.

**Proposition 1.8.** *(Okamoto, Rasmussen, and Zhang, [5].) For every graph $G$,*

$$\chi_s(G) \geq 1 + \lceil \log_2 \omega(G) \rceil .$$

**Theorem 1.9.** *For every directed graph $D$, $\chi_s(D) \geq \lceil \log_2 \omega(UG(D)) \rceil$.*

*Proof.* Create a set coloring with $\chi_s(D) = k$ colors. From the argument given in the proof of Proposition 1.7, $\omega(UG(D)) \leq \chi(UG(D)) \leq 2^k$. Therefore $\chi_s(D) \geq \log_2(\omega(UG(D))$. Since $k$ is an integer, we may round up thereby completing the result. □

Recall that a tournament is an oriented complete graph. We can now make the following conclusion.

**Corollary 1.10.** *If $T$ is a tournament on $n$ vertices, then $\chi_s(T) \geq \lceil \log_2(n) \rceil$.*

*Proof.* This follows directly from Theorem 1.9, since the underlying graph of tournament is the complete graph, and the chromatic number of the complete graph on $n$ vertices is $n$. □

# 2 Tournaments and Set Colorings

First, we make some observations about small tournaments. Both the tournament on a single vertex and the unique tournament on two vertices have set chromatic number 1. For tournaments on three vertices, the transitive tournament has set chromatic number 2, and the cycle has set chromatic number 3. All tournaments on four vertices have set chromatic number 3. The tournament on one vertex and the cycle on three vertices are the only tournaments with $\chi_s(T) = n$ as we can see by improving the upper bound given in Proposition 1.4.

**Theorem 2.1.** *If $T$ is a tournament on $n > 3$ vertices, then $\chi_s(T) \leq n-1$.*

*Proof.* Let $x$ be any vertex of greatest out degree. Let $y$ be an arbitrary out-neighbor of $x$. Color all vertices except $x$ and $y$ distinct colors. So $c(x) = c(y)$ but any other pair of vertices have different colors. If this is a set coloring, we are finished. Suppose it is not a set coloring. Then there exists arc $(u, v) \in A(T)$ such that $OC(u) = OC(v)$. Thus, $c(v) \in OC(u)$. So $c(v) \in OC(v)$. Notice that for any vertex $w$, $c(w) \notin OC(w)$, with the exception of $x$. Therefore, $x = v$.

Therefore $(u, x) \in A(T)$ and $OC(u) = OC(x)$. Since every pair of vertices, except $x$ and $y$, have distinct colors we make the conclusion that $O(x) - \{y\} = O(u) - \{x, y\}$. Since $x$ has maximum out degree and $x \in O(u)$ it follows that $y \notin O(u)$. Therefore, since $(u, y) \notin A(T)$, $(y, u) \in A(T)$, and $O(x) = \{y\} \cup O(u) - \{x\}$.

Since $n \geq 4$ and $x$ has maximum out degree, $|O(x)| \geq 2$. Thus, there exists $z \in O(x)$, $z \neq y$. Since $O(x) = \{y\} \cup O(u) - \{x\}$, we conclude that $(u, z) \in A(T)$. Swap $c(z)$ and $c(y)$. Notice, that since $y$ is not in the outset of $u$, $OC(u) \neq OC(x)$. We claim that this new coloring is a set coloring.

Suppose not. Again, this means there exists vertex $u'$ and arc $(u', x)$ such that $OC(u') = OC(x)$. Since $u$ and $x$ no longer have the same out color set, $u' \neq u$, and by similar arguments to the preceding, we know $O(x) = \{z\} \cup O(u) - \{x\}$. Since $T$ is a tournament there must be an arc between $u$ and $u'$. However $u$ cannot be in the outset of $u'$ since $u$ is not in the outset of $x$ and $u'$ cannot be in the outset of $u$ since $u'$ is not in the outset of $x$. Thus we have reached a contradiction. Therefore we have a set coloring of $T$ using $n-1$ colors. □

Indeed, this bound is tight.

**Theorem 2.2.** *If $T_n$ is a transitive tournament on $n \geq 2$ vertices, then $\chi_s(T_n) = n - 1$.*

*Proof.* Given $T_n$, assume the vertices are labeled $v_1, v_2, \dots, v_n$ so that $(v_i, v_j) \in A(T_n)$ if and only if $i < j$. Observe that if $i < n$, then we know $O(v_i) = \{v_{i+1}, \dots, v_n\} = O(v_{i+1}) \cup \{v_{i+1}\}$ In order for the vertices to have distinct coloring $v_{i+1}$ must have a distinct color from $O(v_{i+1})$ and it follows that $v_2, \dots, v_n$ must have distinct colors. □

The transitive tournament is not strongly connected. While we do not have an example of a strongly connected tournament achieving the upper bound for $\chi_s(D)$, we can say the following.

**Theorem 2.3.** *There exists a strongly connected tournament $T$ on $n \geq 5$ vertices, $\chi_s(T) = n - 2$:*

*Proof.* Recall the upset tournament with vertices $v_1, \dots, v_n$, obtained as follows. Begin with transitive tournament with vertices $v_1, \dots, v_n$ labeled so that $(v_i, v_j)$ is an arc if and only if $i < j$. Then reverse arc $(v_1, v_n)$ so $v_n$ has an arc toward $v_1$. Since

$$O(v_{n-1}) \subseteq O(v_{n-2}) \subseteq \cdots \subseteq O(v_2),$$

we know that $c(v_3), \dots, c(v_n)$ must be distinct. So $\chi_s(T) \geq n - 2$. Let $c(v_1) = c(v_2) = c(v_3)$.

Note $c(v_n) \notin OC(v_1)$ and $c(v_n) \notin OC(v_n)$, but $c(v_n) \in O(v_{n-1}) \cap \cdots \cap O(v_2)$. Also, $c(v_{n-1}) \in OC(v_1)$ but $c(v_{n-1}) \notin OC(v_n)$. Thus we have $\chi_s(T) = n - 2$. □

Recall that Corollary 1.10 tells us that $\chi_s(T) \geq \lceil \log_2(n) \rceil$. The following theorem shows that, in a sense, this bound is not tight.

**Theorem 2.4.** *If $k > 1$, there exists no tournament $T$ with exactly $2^k$ vertices such that $\chi_s(T) = k$.*

*Proof.* Suppose $T$ is such a tournament. Since $T$ is a tournament, $OC(x)$ must be unique over all $x$. So each of the $2^k$ possible sets of colors is achieved as $OC(x)$ for some $x \in V(T)$. In particular, there must be a vertex $v$ such that $|O(v)| = 0$ which implies that $c(v) \in OC(x)$ for all $x \neq v$. This is a contradiction, provided $k > 1$, since each of the $2^k$ possible sets of colors is achieved as $OC(x)$ for some $x \in V(T)$, meaning any particular color is contained in at most half of the outsets. $\square$

While we do not have a tight lower bound for $\chi_s(T)$, we can say the following.

**Theorem 2.5.** *For each natural number $n$, there exists a tournament $T$ on $n$ vertices with $\chi_s(T) \leq 2 + \lceil \log_2 n \rceil$.*

*Proof.* For tournaments on one and two vertices, there exist only one possible tournament and in each case $\chi_s(T) = 1$, satisfying the theorem. For $n = 3$ vertices, the transitive tournament has set chromatic number 2 and likewise satisfies the formula. For $n \geq 4$, first we construct a family of tournaments $T_k$ as follows.

Begin with a transitive tournament with vertices $v_1, v_2, \ldots, v_k$, labeled so that $(v_i, v_j)$ is an arc if and only if $i < j$. Let $c(v_i) = i$. Next, add vertex $v_{k+1}$ colored $k+1$ with $O(v_{k+1}) = \{v_1 \ldots v_k\}$. Finally, add $2^k$ vertices labeled $u_i$ for $1 \leq i \leq 2^k$, each colored $k+2$. Let $U = \{u_1, \ldots, u_{2^k}\}$. Let the subtournament on $U$ be transitive where $u_1$ has maximum out degree. For each $u_i \in U$, add arc $(u_i, v_{k+1})$. Since $\{v_1, v_2, \ldots, v_k\}$ has $2^k$ subsets, we can add arcs between the $u_i$ and $v_j$ so that $O(u_i) \cap \{v_1, v_2, \ldots, v_k\}$ is unique over all $i$.

Let $1 \leq i < j \leq k$. Since $j \in OC(v_i)$ but $j \notin OC(v_j)$, it follows that $OC(v_i) \neq OC(v_j)$. Observe that $OC(v_i) \neq OC(u_j)$ since $k+1 \in OC(u_j)$, $k+1 \notin OC(v_i)$. By construction, for each $1 \leq i < j \leq 2^k$, $OC(u_i) \neq OC(u_j)$. Thus we have a set coloring.

Observe that $T_k$ has $n = 2^k + k + 1$ vertices and $\chi_s(T_k) \leq k+2$. We see that $\lceil \log_2 n \rceil = k+1$, since

$$k = \log_2 2^k < \log_2(2^k + k + 1) \leq \log_2(2^k + 2^k) = k + 1.$$

Thus, $\chi_s(T) \leq k + 2 = \lceil \log_2 n \rceil + 1$.

Next, we must provide the construction for a tournament $T$ such that $\chi_s(T) \leq \lceil \log_2 n \rceil + 2$ for $2^k + k + 1 < n < 2^{k+1} + (k+1) + 1$. Begin with the tournament $T_{k+1}$. Let $U = \{u_1, \ldots, u_{2^{k+1}}\}$. Removing vertices from $U$ will still result in a set coloring, since color $k+3$ is not required to distinguish between color sets. Create $T$ by removing the appropriate number of vertices from $U$ to achieve the desired value of $n$; the removed vertices may be chosen arbitrarily. So $\chi_s(T) \leq k+3$.

If $2^k + k + 1 < n \leq 2^{k+1}$, then $\lceil \log_2 n \rceil = k+1$ because

$$k = \log_2 k < \log_2(2^k + k + 1) < \log_2 n \leq \log_2(2^{k+1}) = k + 1.$$

Thus, $\chi_s(T) \leq (k+1)+2 = \lceil \log_2 n \rceil + 2$. On the other hand, if

$$2^{k+1} < n \leq 2^{k+1} + (k+1) + 1,$$

then $\lceil \log_2 n \rceil = k+2$ because

$$k+1 = \log_2 2^{k+1} < \log_2 n \leq \log_2(2^{k+1} + k + 2) \leq \log_2(2^{k+2}) = k+2.$$

Thus, $\chi_s(T) \leq (k+1)+2 = \lceil \log_2 n \rceil + 1$. Thus, for all $n$, we have produced a tournament $T$ with $n$ vertices such that $\chi_s(T) \leq \lceil \log_2 n \rceil + 2$. □

**Corollary 2.6.** *For all natural numbers $n \geq 3$, there exists a strongly connected tournament $T$ on $n$ vertices with $\chi_s(T) \leq 2 + \lceil \log_2 n \rceil$.*

*Proof.* In the case of $n = 3$ consider the oriented three cycle, with outset chromatic number 3. For $n > 3$, begin with the tournament constructed for this value of $n$ in Theorem 2.5. This construction begins with $T_k$ and then adding vertices so that $V(T) = \{v_1, v_2, \ldots, v_k, v_{k+1}, u_1, \ldots, u_j\}$ where $j \leq 2^k$ and $|V(T)| = n$. Recall that the subtournament on $\{v_1, v_2, \ldots, v_k\}$ is transitive and the subtournament on $\{u_1, \ldots, u_i\}$ is transitive. Furthermore, over all $u_i$, $O(u_i) \cap \{v_1, \ldots, v_k\}$ is unique. Among the arcs between $\{v_1, \ldots, v_k\}$ and $\{u_1, \ldots, u_k\}$, make sure that $(v_k, u_1)$ is an arc. Then $u_1, \ldots, u_j, v_{k+1}, v_1, \ldots, v_k, u_1$ is a hamilton cycle. Thus these tournaments are strongly connected. □

# 3 Tournaments and Set Colorings of Type 2

A vertex $v$ in a tournament $T$ is a *transmitter* if it has in-degree 0, consequently $u$ is directed toward all other vertices in $T$. Similarly a vertex $v$ is a *receiver* if it has out-degree 0. Recall that from Proposition 1.7, $\chi_s(D) \geq \lceil \log_2 (\chi(UG(D)) + 1) \rceil$. The following theorem shows us the lower bound for the set chromatic number of Type 2 is lower.

**Theorem 3.1.** *If $T$ is a tournament on $n \geq 2$ vertices that contains a transmitter or receiver, then $\chi_{s'}(T) = 2$.*

*Proof.* By Observation 1.1, we know $\chi_{s'}(T) > 1$. Let $v$ be a transmitter of $T$. Assign color 1 to $v$ and color 2 to the remaining vertices. For any vertex $u$ of $T$, other than a receiver, $OC(u) = \{2\}$. For any vertex $w$ of $T$, other than $v$, $1 \in IC(u)$. Consequently if $(u, w)$ is an arc in $V$, $OC(u) \neq IC(w)$, thus $\chi_{s'}(T) = 2$. The argument is essentially the same if $v$ is a receiver. □

**Corollary 3.2.** *For all $k \geq 2$ there is a digraph $D$ with*

$$\chi(UG(D)) = \omega(UG(D)) = k \text{ and } \chi_{s'}(\mathrm{D}) = 2.$$

Since every transitive tournament contains a transmitter, we make the following conclusion.

**Corollary 3.3.** *If $T$ is a transitive tournament on $n \geq 2$ vertices, then* $\chi_{s'}(T) = 2$.

Based on our investigation of transitive tournaments in the previous section we see that the difference between $\chi_{s'}(T)$ and $\chi_s(T)$ may be arbitrarily large.

**Corollary 3.4.** *For all $k \geq 2$ there is a digraph $D$ with*

$$\chi(UG(D)) = \omega(UG(D)) = k \text{ and } \chi_{s'}(\mathrm{D}) = 2.$$

The next result shown that even if we require the tournament to be strongly connected, the set chromatic number of Type 2 may be small.

**Theorem 3.5.** *There exists a strongly connected tournament $T$ on $n \geq 3$ vertices, $\chi_{s'}(T) \leq 3$.*

*Proof.* Select one vertex $u$ and one vertex $v$. Let $w_1, \ldots, w_{n-2}$ denote the remaining vertices . Include arcs $(v,u)$, $(w_i,v)$, and $(u,w_i)$ for $1 \leq i \leq n-2$. Arcs between the $w_i$ can be oriented in any way. Color $v$ with color 1, $u$ with color 2, and the remaining vertices with color 3. Then $2 \in OC(v)$ but $2 \notin IC(u)$. We see $2 \in IC(w_i)$, but $2 \notin OC(u)$. Observe that for each $i$, $1 \in OC(w_i)$ but $1 \notin IC(w_i)$. Finally, $1 \in OC(w_i)$, but $1 \notin IC(v)$. Thus $\chi_{s'}(T) \leq 3$. □

Finally, Theorem 2.2 tells us that for any $k$ we can find a tournament with $\chi_s(T) \geq k$ (e.g., a transitive tournament on more than $k$ vertices). While the transitive tournament is not the relevant example, the same is true for $\chi_{s'}(T)$.

**Theorem 3.6.** *For any $k$ there exists a tournament $T$ with $\chi_{s'}(T) \geq k$.*

*Proof.* By Corollary 3.3, we know there is a tournament with $\chi_{s'}(T) = 2$, so we can assume $k \geq 3$. Suppose every tournament has a Type 2 set coloring with $k-1$ or fewer colors. Let $T$ be the rotational tournament with vertex set $\{x_0, \ldots, x_{2n}\}$ with arc $(x_i, x_j)$ if and only if $1 \leq j - i \pmod{2\mathrm{n}+1} \leq \mathrm{n}$, where $2n+1 \geq 2k(k-1)$. Note that all arithmetic is modulo $2n+1$, but will not be explicitly written as such.

First, observe that $c(x_i) \neq c(x_{i+n})$ since

$$O(x_i) = \{x_{i+1}, \ldots, x_{i+n-1}, x_{i+n}\} \text{ and } I(x_{i+n}) = \{x_i, x_{i+1}, \ldots, x_{i+n-1}\}.$$

Furthermore, either $x_i$ has different color than every vertex in $O(x_i)$ or $x_{i+n}$ has a different color than every vertex in $I(x_{i+n})$, for otherwise

$$OC(x_i) = IC(x_{i+n}).$$

Since $T$ has a Type 2 set coloring with $k-1$ or fewer colors and at least $2k(k-1)$ vertices, there is a color class, call it color 1, containing at least $2k$ vertices. At least half of these vertices must be in $\{x_i, \ldots, x_{i+n}\}$ for some vertex $x_i$ since $T$ has $2n+1$ vertices. Relabel the vertices of $T$ so that $x_0$ is color 1 and there are $k$ vertices in $\{x_0, \ldots, x_n\}$ of color 1. Recall that $k \geq 3$. Let $x_{j_1}, x_{j_2}, \ldots x_{j_k}$ denote the vertices of color 1 in $\{x_0, \ldots, x_n\}$ where $0 = j_1 < j_2 < \cdots < j_k < n$. We claim that this Type 2 set coloring must use at least $k$ colors.

We know $c(x_0) = 1$ and $c(x_n) \neq c(x_0)$. Let $c(x_n) = 2$. We know $c(x_{j_2}) = 1$ and $c(x_{j_2+n}) \neq c(x_{j_2})$. Since $c(x_{j_2}) = c(x_{j_k})$ where $j_k < n < j_2 + n$, we conclude that $x_{j_2+n}$ has a different color than every vertex in $\{x_{j_2}, \ldots, x_{j_2+n-1}\}$. There is a vertex in this set with color 1, namely $x_{j_2}$, and a vertex in this set with color 2, namely $x_n$. Thus $x_{j_2+n}$ is a new color, say $c(x_{j_2+n}) = 3$.

Indeed we must use a new color for $x_{j_3+n}, \ldots, x_{j_{k-1}+n}$, meaning we must use at least $k$ colors, a contradiction since we assumed this tournament has a Type 2 set coloring with $k-1$ or fewer colors. □

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