The Exact Upper Bound of Diffy Problem

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Abstract

Given nonnegative integers, a, b, c, and d, the transition function $\bigtriangledown$ is defined by $\bigtriangledown(a,b,c,d) = (|a-b|, |b-c|, |c-d|, |d-a|)$. Diffy problem asks if it can reach $(0,0,0,0)$ after some iterations of $\bigtriangledown$ on the four numbers. If (a, b, c, d) can transfer to $(0,0,0,0)$ iterated by $\bigtriangledown$ operations, the smallest N such that $\bigtriangledown^N(a,b,c,d) = (0,0,0,0)$ is called stopping steps of Diffy problem. In this paper, we will show that it exists N such that $\bigtriangledown^N(a,b,c,d) = (0,0,0,0)$ and the loose upper bound and exact upper bound of N. In addition, we will also show that we can find a starting vector (a,b,c,d) so that it reaches the zero vector $(0,0,0,0)$ after exact k steps for any given positive integer k.

Keywords and phrases: Diffy problem, $\bigtriangledown$ function, stopping steps, the loose upper bound, the exact upper bound.

1 Diffy Problem Introduction

Let (a,b,c,d) be a quadruple of integer numbers and consider the sequence $\bigtriangledown^N(a,b,c,d)$, where $\bigtriangledown : Z^4 \to Z^4$ is defined by $\bigtriangledown(a,b,c,d) = (|a-b|, |b-c|, |c-d|, |d-a|)$.
For example, consider $(6,2,9,17)$. We have $a \to b$ means $b = \bigtriangledown(a)$. Since $(6,2,9,17) \to (4,7,8,11) \to (3,1,3,7) \to (2,2,4,4) \to (0,2,0,2) \to (2,2,2,2) \to (0,0,0,0)$, we write $\bigtriangledown^6(6,2,9,17) = (0,0,0,0)$.
It can be represented by a graph as in Figure 1.

In general, we investigate what is the smallest N such that $\bigtriangledown^N(a,b,c,d) = (0,0,0,0)$. We tested thousands and thousands of cases and found that all quadruples reach the state $(0,0,0,0)$.
The difficulty for this problem is that the terms can increase exponentially:
$\bigtriangledown^3(a,b,c,d) = \bigtriangledown^2(|a-b|, |b-c|, |c-d|, |d-a|)$
$= \bigtriangledown(||a-b|-|b-c||, ||b-c|-|c-d||, ||c-d|-|d-a||, ||d-a|-|a-b||)$
$= (|||a-b|-|b-c||-||b-c|-|c-d|||, |||b-c|-|c-d||-||c-d|-|d-a|||,$
$|||c-d|-|d-a||-||d-a|-|a-b|||, |||d-a|-|a-b||-||a-b|-|b-c|||)$

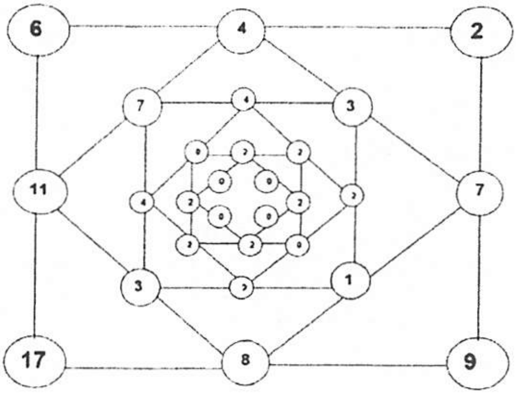

FIGURE 1. A Diffy Example

In fact, people have researched the Diffy problem for quite a long time.
In [1], Freedman introduced this problem in 1948.
In [2], Clausing used a computer to a really large class playing the Diffy game. The author found 90% of 10,000 random quadruples with values in [099] reaching to the end within three to five steps.

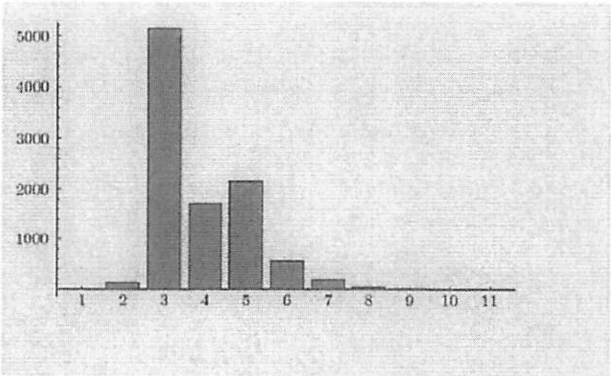

FIGURE 2. The height of 10,000 random quadruples with vailes in (0..99).

In [3], Ullman calculated the distribution of convergence times with respect to the natural probability measure on labeled squares and, thus, to explain the surprising speed of convergence.

k	$Prob\{n(\overrightarrow{v}) = k\}$	k	$Prob\{n(\overrightarrow{v}) = k\}$
0	0	9	0.6182%
1	0	10	0.1886%
2	0	11	0.0559%
3	0	12	0.0163%
4	50.0000%	13	0.0049%
5	17.7778%	14	0.0014%
6	22, 9630%	15	0.0004%
7	6.0063%	16	0.0001%
8	2.3680%	.	.

TABLE 1. The probability that the four-number game converges in k steps.

In [5], Chamberland showed some Unbounded Diffy Sequences.
In [6], Ehrlich showed the periods of for some Diffy sequences.
In [7], Webb gave a proof that the up bound of Diffy problem is 3[n/2].
In [9], Glaser and Schff found Diffy-sequences are closely related to Pascal's triangle and many properties of their cyclic structures can be found and proved considering Pascal's triangle modulo 2.
In [13], Greenwell showed that some cases only need six steps to reach to (0, 0, 0, 0) and left as one open question whether there is a simple way, given four numbers, to tell how many moves it takes go get to all zeros without actually going through all the moves.

2 Examples of Diffy Problem

We will show some interesting examples of Diffy Problem in this section
Definitions
a, b, c, and d are nonnegative integers. The transition function $\bigtriangledown$ is defined by $\bigtriangledown(a,b,c,d) = (|a-b|,|b-c|,|c-d|,|d-a|)$. The multiple operations $\bigtriangledown*$ is defined by $\bigtriangledown*(a,b,c,d) = \bigtriangledown(\bigtriangledown(....\bigtriangledown(a,b,c,d).))$. If (a, b, c, d) can transfer to (0, 0, 0, 0) iterated by $\bigtriangledown$, the smallest N such that $\bigtriangledown^N(a,b,c,d) = (0,0,0,0)$ is called stopping steps of (a, b, c, d), denoted by the stop function s(a, b, c, d) = N. If gcd(a, b, c, d) = 1, (a, b, c, d) is called the basic quadruple.

Examples
We have some simple observations.
Example 2.1 For any arithmetic progressions($d > 0$), the stopping step N = 5. N is independent on a.
Proof Suppose (a,b,c,d) = (a, a+d, a+2d, a+3d).

$\nabla^5(a, a+d, a+2d, a+3d)$
$= \nabla^4(d, d, d, 3d)$
$= \nabla^3(0, 0, 2d, 2d)$
$= \nabla^2(0, 2d, 0, 2d)$
$= \nabla(2d, 2d, 2d, 2d)$
$= (0, 0, 0, 0)$.
So N = 5.

Example 2.2 For any geometric progressions $(a > 0, r > 0), r = 1, N = 1; r = 2, N = 7; r \geq 3, N = 6$. So, N is independent on a.
Proof Suppose $(a, b, c, d) = (a, ar, ar^2, ar^3)$.
When r = 1, $\nabla(a, a, a, a) = (0, 0, 0, 0)$. Hence, N = 1.
When r = 2, $\nabla^7(a, 2a, 4a, 8a)$
$= \nabla^6(a, 2a, 4a, 7a)$
$= \nabla^5(a, 2a, 3a, 6a)$
$= \nabla^4(a, a, 3a, 5a)$
$= \nabla^3(0, 2a, 2a, 4a)$
$= \nabla^2(2a, 0, 2a, 4a)$
$= \nabla(2a, 2a, 2a, 2a)$
$= (0, 0, 0, 0)$.
That is N = 7.
When r $\geq$ 3, $\nabla^6(a, ar, ar2, ar3)$
$= \nabla^5(a(r-1), a(r^2-r), a(r^3-r^2), a(r^3-1))$
$= \nabla^4(a(r-1)(r-1), a(r-1)(r^2-r), a(r-1)(r+1), a(r-1)(r^2+r))$
$= \nabla^3(a(r-1)(r^2-2r+1), a(r-1)(r^2-2r-1), a(r-1)(r^2-1), a(r-1)(r^2+1))$
$= \nabla^2(2a(r-1), 2a(r-1)r, 2a(r-1)r, 2a(r-1)r)$
$= \nabla(2a(r-1)^2, 2a(r-1)^2, 2a(r-1)^2, 2a(r-1)^2)$
$= (0, 0, 0, 0)$.
As a result, N = 6.
Example 2.3 If a, b, c, and d are real numbers, there may exist stopping steps N such that $\nabla^N(a, b, c, d) = (0, 0, 0, 0)$. For example,
$\nabla^5(ln2, \sqrt{2}, e, \pi)$
$= \nabla^4(\sqrt{2} - ln2, e - \sqrt{2}, \pi - e, \pi - ln2)$
$= \nabla^3(e - 2\sqrt{2} + ln2, 2e - \sqrt{2} - \pi, e - ln2, \pi - \sqrt{2})$
$= \nabla^2(e+\sqrt{2}-\pi-ln2, -e+\sqrt{2}+\pi-ln2, e+\sqrt{2}-\pi-ln2, -e+\sqrt{2}+\pi-ln2)$
$= \nabla(-2e+2\pi, -2e+2\pi, -2e+2\pi, -2e+2\pi)$
$= (0, 0, 0, 0)$.

3 Results Under Some Conditions

If we put some conditions on a, b, c, d, it is easy to get the steps required to reach to (0, 0, 0, 0). For some special cases, I got the results as below.

Theorem 3.1 If $a > b > c > d, 2b > a + c, b + d > 2c$ and $a + d > b + c$, there exists a number $N, N \leq 5$, such that $\nabla^N(a, b, c, d) = (0, 0, 0, 0)$.

Proof. Since $a > b > c > d, 2b > a + c, b + d > 2c$ and $a + d > b + c$, we have

$\nabla^5(a, b, c, d)$
$= \nabla^4(a - b, b - c, c - d, a - d)$
$= \nabla^3(-a + 2b - c, b - 2c + d, a - c, b - d)$
$= \nabla^2(a - b - c + d, a - b + c - d, a - b - c + d, a - b + c - d)$
$= \nabla(2c - 2d, 2c - 2d, 2c - 2d, 2c - 2d)$
$= (0, 0, 0, 0)$.

Theorem 3.2 If $a > b > c > d, 2b > a + c, b + d > 2c$ and $a + d < b + c$, there exists a number $N, N \leq 5$, such that $\nabla^N(a, b, c, d) = (0, 0, 0, 0)$.

Proof. Since $a > b > c > d, 2b > a + c, b + d > 2c$ and $a + d < b + c$, we have

$\nabla^5(a, b, c, d)$
$= \nabla^4(a - b, b - c, c - d, a - d)$
$= \nabla^3(-a + 2b - c, b - 2c + d, a - c, b - d)$
$= \nabla^2(-a + b + c - d, a - b + c - d, -a + b + c - d, a - b + c - d)$
$= \nabla(2a - 2b, 2a - 2b, 2a - 2b, 2a - 2b)$
$= (0, 0, 0, 0)$.

Theorem 3.3 If $a > b > c > d, a + c + d > 3b, b + d > 2c$ and $a + d > b + c$, there exists a number $N, N \leq 6$, such that $\nabla^N(a, b, c, d) = (0, 0, 0, 0)$.

Proof. Since $a > b > c > d, a + c + d > 3b, b + d > 2c$ and $a + d > b + c$, we have

$\nabla^6(a, b, c, d)$
$= \nabla^5(a - b, b - c, c - d, a - d)$
$= \nabla^4(a - 2b + c, b - 2c + d, a - c, b - d)$
$= \nabla^3(a - 3b + 3c - d, a - b + c - d, a - b - c + d, a - 3b + c + d)$
$= \nabla^2(2b - 2c, 2c - 2d, 2b - 2c, 2c - 2d)$
$= \nabla(2b - 4c + 2d, 2b - 4c + 2d, 2b - 4c + 2d, 2b - 4c + 2d)$
$= (0, 0, 0, 0)$.

Theorem 3.4 If $a > b > c > d, a + c + d > 3b$ and $2c > b + d$, there exists a number $N, N \leq 5$, such that $\nabla^N(a, b, c, d) = (0, 0, 0, 0)$.

Proof. Since $a > b > c > d, a + c + d > 3b$ and $2c > b + d$, we have

$\nabla^5(a, b, c, d)$
$= \nabla^4(a - b, b - c, c - d, a - d)$
$= \nabla^3(a - 2b + c, -b + 2c - d, a - c, b - d)$
$= \nabla^2(a - b - c + d, a + b - 3c + d, a - b - c + d, a - 3b + c + d)$
$= \nabla(2b - 2c, 2b - 2c, 2b - 2c, 2b - 2c)$
$= (0, 0, 0, 0)$.

4 The Loose Upper Bound of Diffy Problem

In this section, we will show a simple proof that (a, b, c, d), in which a, b, c, and d are nonnegative integers, will terminate at (0, 0, 0, 0) by $\bigtriangledown^N$.

Lemma 4.1 For any integer $k > 0, \bigtriangledown(ka, kb, kc, kd) = k \bigtriangledown (a, b, c, d)$.
Proof. $\bigtriangledown (ka, kb, kc, kd)$
$= (|ka - kb|, |kb - kc|, |kc - kd|, |kd - ka|)$
$= k(|a - b|, |b - c|, |c - d|, |d - a|)$
$= k \bigtriangledown (a, b, c, d)$.

Lemma 4.2 $\max\{a, b, c, d\} \geq \max\{|a - b|, |b - c|, |c - d|, |d - a|\}$.
Proof. If $a \geq b, a \geq |a - b|$. If $b \geq a, b \geq |a - b|$. Therefore, $max\{a, b\} \geq |a - b|$.
Similarly, we can get
$max\{b, c\} \geq| b - c|$,
$max\{c, d\} \geq |c - d|$, and
$max\{d, a\} \geq |d - a|$.
Combine all the inequalities, so that is $max\{a, b, c, d\} \geq max\{|a - b|, |b - c|, |c - d|, |d - a|\}$.

Lemma 4.3 For any integer $t > 0, \bigtriangledown(a+t, b+t, c+t, d+t) = \bigtriangledown(a, b, c, d)$.
Proof. $\bigtriangledown (a + t, b + t, c + t, d + t)$
$= (|a - b|, |b - c|, |c - d|, |d - a|)$
$= \bigtriangledown(a, b, c, d)$.

Lemma 4.4 If $\bigtriangledown^N(a, b, c, d) = (0, 0, 0, 0)$, then $\bigtriangledown^N(b, c, d, a) = (0, 0, 0, 0)$, $\bigtriangledown^N (c, d, a, b) = (0, 0, 0, 0) and \bigtriangledown^N (d, a, b, c) = (0, 0, 0, 0)$.
Proof. If $\bigtriangledown^N(a, b, c, d) = \bigtriangledown^{N-1}(a', b', c', d')$, then $\bigtriangledown^N(b, c, d, a) = \bigtriangledown^{N-1}(b', c', d', a')$. We do the some operations N - 1 time. Then we will have $\bigtriangledown^N(a, b, c, d) = (0, 0, 0, 0)$ and $\bigtriangledown^N(b, c, d, a) = (0, 0, 0, 0)$. Similarly, $\bigtriangledown^N(c, d, a, b) = (0, 0, 0, 0)$ and $\bigtriangledown^N(d, a, b, c) = (0, 0, 0, 0)$.
This is the rotation property. Lemma 4.1 4.4 will be used in below proofs.

Lemma 4.5 The maximum steps that transfer (a, b, c, d) to $(2a_1, 2b_1, 2c_1, 2d_1)$ are $4 (N \leq 4)$.
Proof. a, b, c, and d can only be even or odd numbers. Let E stand for even numbers and O stand for odd numbers. a, b, c, and d have total sixteen combinations as below:
$(E, E, E, E), (E, E, E, O), (E, E, O, E), (E, E, O, O)$,
$(E, O, E, E), (E, O, E, O), (E, O, O, E), (E, O, O, O)$,
$(O, E, E, E), (O, E, E, O), (O, E, O, E), (O, E, O, O)$,
$(O, O, E, E), (O, O, E, O), (O, O, O, E), (O, O, O, O)$.
Since
1) $\bigtriangledown^0(E, E, E, E) = (E, E, E, E)$
2) $\bigtriangledown^4(E, E, E, O) = \bigtriangledown^3(E, E, O, O) = \bigtriangledown^2(E, O, E, O) = \bigtriangledown(O, O, O, O) = (E, E, E, E)$

3) $\nabla^4(E,E,O,E) = \nabla^2(E,O,O,E) = \nabla^2(O,E,O,E) = \nabla(O,O,O,O) = (E,E,E,E)$
4) $\nabla^3(E,E,O,O) = \nabla^2(E,O,E,O) = \nabla(O,O,O,O) = (E,E,E,E)$
5) $\nabla^4(E,O,E,E) = \nabla^3(O,O,E,E) = \nabla^2(E,O,E,O) = \nabla(O,O,O,O) = (E,E,E,E)$
6) $\nabla^2(E,O,E,O) = \nabla(O,O,O,O) = (E,E,E,E)$
7) $\nabla^3(E,O,O,E) = \nabla^2(O,E,O,E) = \nabla(O,O,O,O) = (E,E,E,E)$
8) $\nabla^4(E,O,O,O) = \nabla^3(O,E,E,O) = \nabla^2(O,E,O,E) = \nabla(O,O,O,O) = (E,E,E,E)$
9) $\nabla^4(O,E,E,E) = \nabla^3(O,E,E,O) = \nabla^2(O,E,O,E) = \nabla(O,O,O,O) = (E,E,E,E)$
10) $\nabla^4(O,E,E,O) = \nabla^3(O,E,O,E) = \nabla^2(O,E,O,E) = \nabla(O,O,O,O) = (E,E,E,E)$
11) $\nabla^2(O,E,O,E) = \nabla(O,O,O,O) = (E,E,E,E)$
12) $\nabla^4(O,E,O,O) = \nabla^3(O,O,E,E) = \nabla^2(E,O,E,O) = \nabla(O,O,O,O) = (E,E,E,E)$
13) $\nabla^3(O,O,E,E) = \nabla^2(E,O,E,O) = \nabla(O,O,O,O) = (E,E,E,E)$
14) $\nabla^4(O,O,E,O) = \nabla^3(E,E,O,O) = \nabla^2(O,E,O,E) = \nabla(O,O,O,O) = (E,E,E,E)$
15) $\nabla^4(O,O,O,E) = \nabla^3(E,O,O,E) = \nabla^2(E,O,E,O) = \nabla(O,O,O,O) = (E,E,E,E)$
16) $\nabla(O,O,O,O) = (E,E,E,E)$
As a result, $N \leq max\{0,4,4,3,4,2,3,4,4,3,2,4,3,4,4,1\} = 4$.

Theorem 4.1 There exists a number $N, N \leq 4log_2(max\{a,b,c,d\}) + 4$, such that $\nabla^N(a,b,c,d) = (0,0,0,0)$.
Proof By Lemma 4.5, the maximum steps are 4 such that $\nabla^n(a,b,c,d) = (2a_1, 2b_1, 2c_1, 2d_1)$
Therefore,

$$N \leq 4 + s(2a_1, 2b_1, 2c_1, 2d_1) \tag{1}$$

Here, s is the stop function. According to Lemma 4.1, (1) equal to

$$N \leq 4 + s(2(a_1, b_1, c_1, d_1)) \tag{2}$$

If $(a_1, b_1, c_1, d_1) \neq (0,0,0,0)$, do the same computing as (1) and (2). The result is

$$N \leq 4 + 4 + s(2(2a_2, 2b_2, 2c_2, 2d_2)) \tag{3}$$

and

$$N \leq 4 \times 2 + s(2^2(a_2, b_2, c_2, d_2)) \tag{4}$$

Continue doing steps (1) and (2) k times until final state (a_k, b_k, c_k, d_k) containing only 1 or 0, which is

$$N \leq 4 \times k + s(2^k(a_k, b_k, c_k, d_k)) \tag{5}$$

Let E = 0 and O = 1. Do the same computing as in Lemma 4.5. As a result, the maximum steps are 4 such that $\nabla^n(a_k, b_k, c_k, d_k) = (0,0,0,0)$.

Hence,

$$s(2^k(a_k, b_k, c_k, d_k)) \leq 4 \tag{6}$$

(5) becomes

$$N \leq 4 \times k + 4 \tag{7}$$

Since (a_k, b_k, c_k, d_k) contains only 1 or 0, $max\{a_k, b_k, c_k, d_k\} \leq 1$.
By Lemma 4.2, $2^k max\{a_k, b_k, c_k, d_k\} \leq 2^k \leq max\{a, b, c, d\}$.
That is

$$k \leq log_2(max\{a, b, c, d\}) \tag{8}$$

Combine (7) and (8). The stop steps

$$N \leq 4k + 4 \leq 4log_2(max\{a, b, c, d\}) + 4 \tag{9}$$

such that $\nabla^N(a, b, c, d) = (0, 0, 0, 0)$.

5 The Exact Upper Bound of Diffy Problem

Based on the studies in previous sections, the results seem like that the max stopping steps $N \leq 7$. However, the life is not so easy. In fact, there are exists infinity examples that require more than 7 steps.
For example, apply ∇^N operations on (149, 81, 44, 24). We have
$(149, 81, 44, 24) \rightarrow (68, 37, 20, 125) \rightarrow (31, 17, 105, 57) \rightarrow (14, 88, 48, 26) \rightarrow (74, 40, 22, 12) \rightarrow (34, 18, 10, 62) \rightarrow (16, 8, 52, 28) \rightarrow (8, 44, 24, 12) \rightarrow (36, 20, 12, 4) \rightarrow (16, 8, 8, 32) \rightarrow (8, 0, 24, 16) \rightarrow (8, 24, 8, 8) \rightarrow (16, 16, 0, 0) \rightarrow (0, 16, 0, 16) \rightarrow (16, 16, 16, 16) \rightarrow (0, 0, 0, 0)$.
It requires N = 15 steps to reach (0, 0, 0, 0). In fact, this is the longest sequence for $max\{a, b, c, d\} \leq 149$ and $min\{a, b, c, d\} \geq 1$ by using a computer to search. I got below results.

Theorem 5.1 If $a > b > c > d > 0$, the basic quadruple (a, b, c, d) that has the maximum steps is in the form

$$\begin{cases} a = c_1 r_1^n + c_2 r_2^n + c_3 r_3^n \\ b = \frac{1}{2}\left(c_1(1 + r_1^2)r_1^{n-2} + c_2(1 + r_2^2)r_2^{n-2} + c_3(1 + r_3^2)r_3^{n-2}\right) \\ c = c_1 r_1^{n-1} + c_2 r_2^{n-1} + c_3 r_3^{n-1} \\ d = \frac{1}{2}\left(c_1(1 + r_1^2)r_1^{n-3} + c_2(1 + r_2^2)r_2^{n-3} + c_3(1 + r_3^2)r_3^{n-3}\right) \end{cases} \tag{10}$$

where

$$r_1 = \frac{1}{3}\left(54 + 6\sqrt{33}\right)^{\frac{1}{3}} + \frac{4}{(54 + 6\sqrt{33})^{\frac{1}{3}}} + 1 \tag{11a}$$

$$\begin{aligned} r_2 = &-\frac{1}{6}\left(54 + 6\sqrt{33}\right)^{\frac{1}{3}} - \frac{2}{(54 + 6\sqrt{33})^{\frac{1}{3}}} + 1 + \\ &\frac{\sqrt{3}}{2}\left(\frac{1}{3}\left(54 + 6\sqrt{33}\right)^{\frac{1}{3}} - \frac{4}{(54 + 6\sqrt{33})^{\frac{1}{3}}}\right) i \end{aligned} \tag{11b}$$

$$r_3 = -\frac{1}{6}\left(54+6\sqrt{33}\right)^{\frac{1}{3}} - \frac{2}{(54+6\sqrt{33})^{\frac{1}{3}}} + 1 - \frac{\sqrt{3}}{2}\left(\frac{1}{3}\left(54+6\sqrt{33}\right)^{\frac{1}{3}} - \frac{4}{(54+6\sqrt{33})^{\frac{1}{3}}}\right)i \tag{11c}$$

and

$$c_1 = \frac{1}{2}\frac{1}{(54+6\sqrt{33})^{\frac{2}{3}}(99+19\sqrt{33})}\Bigg(\left(9+\sqrt{33}\right)\Big(42\left(54+6\sqrt{33}\right)^{\frac{1}{3}} + 10\left(54+6\sqrt{33}\right)^{\frac{1}{3}}\sqrt{33} + 11\left(54+6\sqrt{33}\right)^{\frac{2}{3}} + 3\left(54+6\sqrt{33}\right)^{\frac{2}{3}}\sqrt{33} + 240 + 48\sqrt{33}\Big)\Bigg) \tag{12a}$$

$$c_2 = -\frac{1}{(54+6\sqrt{33})^{\frac{2}{3}}(99+19\sqrt{33})}\Bigg(\frac{1}{6}i\left(9+\sqrt{33}\right)\sqrt{3}\Big(-360-72\sqrt{33} + 63\left(54+6\sqrt{33}\right)^{\frac{1}{3}} + 15\left(54+6\sqrt{33}\right)^{\frac{1}{3}}\sqrt{33} - 120\sqrt{3}i - 72\sqrt{11}i - 21\left(54+6\sqrt{33}\right)^{\frac{1}{3}}\sqrt{3}i - 15\left(54+6\sqrt{3}\sqrt{11}\right)^{\frac{1}{3}}\sqrt{11}i + 9\left(54+6\sqrt{33}\right)^{\frac{2}{3}}\sqrt{11}i + 11\left(54+6\sqrt{33}\right)^{\frac{2}{3}}\sqrt{3}i\Big)\Bigg) \tag{12b}$$

$$c_3 = -\frac{1}{(54+6\sqrt{33})^{\frac{2}{3}}(99+19\sqrt{33})}\Bigg(\frac{1}{6}i\left(9+\sqrt{33}\right)\sqrt{3}\Big(360+72\sqrt{33} - 63\left(54+6\sqrt{33}\right)^{\frac{1}{3}} - 15\left(54+6\sqrt{33}\right)^{\frac{1}{3}}\sqrt{33} - 120\sqrt{3}i - 72\sqrt{11}i - 21\left(54+6\sqrt{33}\right)^{\frac{1}{3}}\sqrt{3}i - 15\left(54+6\sqrt{3}\sqrt{11}\right)^{\frac{1}{3}}\sqrt{11}i + 9\left(54+6\sqrt{33}\right)^{\frac{2}{3}}\sqrt{11}i + 11\left(54+6\sqrt{33}\right)^{\frac{2}{3}}\sqrt{3}i\Big)\Bigg) \tag{12c}$$

Proof. First, we will show that the basic quadruple (a, b, c, d) in the form,

$$\begin{cases} a_n = 2(a_{n-1} + b_{n-1}) + c_{n-1} \\ b_n = a_{n-1} + b_{n-1} + c_{n-1} \\ c_n = a_{n-1} \\ d_n = b_{n-1} \end{cases} \quad (13)$$

has the maximum steps to reach $(0,0,0,0)$.
By mathematical induction, the base case is $(3,1,1,1)$ which is a basic quadruple because $gcd(a,b,c,d) = 1$. By using computer search, it has maximum steps, 4, to $(0,0,0,0)$. for any quadruple (a,b,c,d) if $max\{a,b,c,d\} \leq 3$.
Assume the basic quadruple (a_n, b_n, c_n, d_n) has the longest sequences to $(0,0,0,0)$ for $max\{a,b,c,d\} \leq a_n$.
We need to show $(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1})$ is also a basic quadruple and requires more steps then (a_n, b_n, c_n, d_n). Since $\nabla^3(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1})$
$= \nabla^3(2(a_n + b_n) + c_n, a_n + b_n + c_n, a_n, b_n)$
$= \nabla^2(a_n + b_n, b_n + c_n, a_n - b_n, 2a_n + b_n + c_n)$
$= \nabla(a_n - c_n, 2b_n + c_n - a_n, a_n + 2b_n + c_n, a_n + c_n)$
$= (2a_n - 2b_n - 2c_n, 2a_n, 2b_n, 2c_n)$
$= 2(a_n - b_n - c_n, a_n, b_n, c_n)$
$= 2(2(a_{n-1} + b_{n-1}) + c_{n-1} - a_{n-1} - b_{n-1} - c_{n-1}) - a_{n-1}, a_n, b_n, c_n)$
$= 2(b_{n-1}, a_n, b_n, c_n) = 2(d_n, a_n, b_n, c_n)$, $(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1})$ needs 3 more steps than (a_n, b_n, c_n, d_n) to $(0,0,0,0)$. On the other hand, if $gcd(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}) = k$, then $\nabla^3(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}) = 2k(d_n, a_n, b_n, c_n) = 2(d_n, a_n, b_n, c_n)$. Therefore, $k = 1$. As a result, $gcd(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}) = 1$. so $(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1})$ is also a basic quadruple.
Hence the quadruple$(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1})$ has the maximum steps to (0, 0, 0, 0) for any quadruple (a,b,c,d) with $max\{a,b,c,d\} \leq a_{n+1}$.
Next, it needs solve the recurrence equations from (13):

$$a(n) = 2(a(n-1) + b(n-1)) + c(n-1) \quad (14a)$$

$$b_(n) = a(n-1) + b(n-1) + c(n-1) \quad (14b)$$

$$c(n) = a(n-1) \quad (14c)$$

$$d(n) = b(n-1) \quad (14d)$$

From (14a) and (14c), we can have

$$a(n-1) = 2a(n-2) + 2b(n-2) + a(n-3) \quad (15)$$

From (14b) and (14c), we can have

$$b(n) - b(n-1) = a(n-1) + a(n-2) \quad (16)$$

Therefore,

$$b(n-1) - b(n-2) = a(n-2) + a(n-3) \quad (17)$$

Do (14a) - (15) and use (17). We have
$a(n) - a(n-1) = 2a(n-1) + 2b(n-1) + a(n-2) - 2a(n-2) - 2b(n-2) -$

$a(n-3) = 2a(n-1)+2(b(n-1)-b(n-2))-a(n-2)-a(n-3) = 2a(n-1)+2a(n-2)+2a(n-3)-a(n-2)-a(n-3) = 2a(n-1)+a(n-2)+a(n-3)$.So, we got the difference equation:

$$a(n) - 3a(n-1) - a(n-2) - a(n-3) = 0 \tag{18}$$

The characteristic equation for (18) is

$$r^3 - 3r^2 - r - 1 = 0 \tag{19}$$

Three roots for (19) are listed in (11a), (11b), and (11c). $a(n)$ can be expressed as

$$a(n) = c_1 r_1^n + c_2 r_2^n + c_3 r_3^n \tag{20}$$

The initial quadruple is $(3, 1, 1, 1)$. Then we can apply it in (13) to get the initial conditions $a(0) = 3, a(1) = 9$, and $a(2) = 31$. That is

$$\begin{cases} a(0) = c_1 + c_2 + c_3 = 3 \\ a(1) = c_1 r_1 + c_2 r_2 + c_3 r_3 = 9 \\ a(2) = c_1 r_1^2 + c_2 r_2^2 + c_3 r_3^2 = 31 \end{cases} \tag{21}$$

That is

$$\begin{pmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 31 \end{pmatrix} \tag{22}$$

Therefore,

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 9 \\ 31 \end{pmatrix} \tag{23}$$

Apply r_1, r_2 and r_3 that are listed in (11a), (11b), and (11c). The computing results of c_1, c_2 and c_3 are listed in (12a), (12b), and (12c).
From (14a), we have

$$a(n-1) + b(n-1) = \frac{1}{2}(a(n) - a(n-2)) \tag{24}$$

Then we can compute $b(n)$ by using (14b) and (20). $b(n) = a(n-1) + b(n-1) + a(n-2) = \frac{1}{2}(a(n) - a(n-2)) + a(n-2) = \frac{1}{2}(a(n) + a(n-2)) = \frac{1}{2}(c_1 r_1^n + c_2 r_2^n + c_3 r_3^n + c_1 r_1^{n-2} + c_2 r_2^{n-2} + c_3 r_3^{n-2})$, which is

$$b(n) = \frac{1}{2}(c_1(1 + r^2) r_1^{n-2} + c_2(1 + r_2^2) r_2^{n-2} + c_3(1 + r_3^2) r_3^{n-2}) \tag{25}$$

Combing (20), (14c), (25), and (14d), we finally got the quadruple pattern in (10).
This completed the proof.

Theorem 5.2 If $a > b > c > d > 0$, the steps N to reach (0, 0, 0, 0) are bounded by

$$N \leq \begin{cases} 4 & \text{if } a < 4 \\ 9 & \text{if } a < 10 \\ 31 & \text{if } a < 32 \\ 3n+4 & \text{if } a > 31 \text{ and } a = a(n) \\ 3n+7 & \text{otherwiese} \end{cases} \quad (26)$$

where

$$a(n) = c_1 r_1^n + c_2 r_2^n + c_3 r_3^n, n = \lfloor log_{r_1} \frac{a+\delta}{c_1} \rfloor \text{ and } \delta = 0.020698694231 \quad (27)$$

By Theorem 5.1, the quadruple patten in (10) has the longest path. The approximate values of (11a), (11b), (11c), (12a), (12b), and (12c) are

$$\begin{cases} r_1 = 3.38297576790623749412270853645 \\ r_2 = -0.191487883953118747061354268\overline{2}3+ \\ 0.508851778832737990486422439285i \\ r_3 = -0.19148788395311874706135426823- \\ 0.508851778832737990486422439285i \\ c_1 = 2.71051953391635530567198262842 \\ c_2 = 0.14474023304182234716400868578 7+ \\ 0.112203498670439075921864367695i \\ c_3 = 0.144740233041822347164008685787- \\ 0.112203498670439075921864367695i \end{cases} \quad (28)$$

Since $c_2 r_2^n \to 0$ and $c_3 r_3^n \to 0$ if $n \to \infty$, they are minor terms. The value of $a(n)$ is determined by $c_1 r_1^n$. In fact, if $n = 2$,,

$$\mid c_2 r_2^2 + c_3 r_3^2 \mid < \delta = 0.020698694231 \quad (29)$$

Therefore, we can have

$$(a+\delta) > c_1 r_1^n > (a-\delta) \quad (30)$$

That is

$$log_{r_1} \frac{(a+\delta)}{c_1} > n > log_{r_1} \frac{(a-\delta)}{c_1} \quad (31)$$

Since

$$\lceil log_{r_1} \frac{(a+\delta)}{c_1} \rceil > log_{r_1} \frac{(a+\delta)}{c_1} > \lfloor log_{r_1} \frac{(a+\delta)}{c_1} \rfloor \quad (32)$$

and

$$\lceil log_{r_1} \frac{(a+\delta)}{c_1} \rceil - \lfloor log_{r_1} \frac{(a+\delta)}{c_1} \rfloor = 1 \quad (33)$$

n will be bounded by

$$n = \lfloor log_{r_1} \frac{a+\delta}{c_1} \rfloor \text{ or } \lceil log_{r_1} \frac{(a+\delta)}{c_1} \rceil = \lfloor log_{r_1} \frac{a+\delta}{c_1} \rfloor + 1 \tag{34}$$

By Theorem 5.1, N increases 3 steps when n increases by 1, which is $N = 3n + 4$ and a(0) =3, a(1) = 9, and a(2) =31. Combining all the results, we can get (26)
Since it is possible N = 3n + 4, it is the exact upper bound.

Theorem 5.3 For any positive integer, m, there exists a basic quadruple (a, b, c, d) that just needs m steps to reach to (0, 0, 0, 0).
Proof. If $m < 5$, we can use the quadruple (3, 1, 1, 1) which covers the steps m = 0 to m = 4.
If m = 3n + 4, the quadruple (a, b, c, d), in which a = a(n), b = b(n), c = c(n), and d = d(n) from the formula of (10). Since it needs 3 steps from (a(n), b(n), c(n), d(n)) to (a(n - 1), b(n - 1), c(n - 1), d(n - 1)), the total needs 3n + 4 steps to reach (0, 0, 0, 0).
If m = 3n + 5, the quadruple $(a,b,c,d) = \frac{1}{2}\bigtriangledown^2 (a(n+1), b(n+1), c(n+1), d(n+1))$. Since (a(n + 1), b(n + 1), c(n + 1), d(n + 1)) takes 3n + 7 steps to reach to (0, 0, 0, 0), it just needs to do twice $\bigtriangledown$ operations to make it to use 3n + 5 steps. In addition, all elements in $\bigtriangledown^2(a(n+1), b(n+1), c(n+1), d(n+1))$ are even, so divide them by 2 to get the basic quadruple.
For the same reasons, the quadruple $(a,b,c,d) = \frac{1}{2}\bigtriangledown(a(n+1), b(n+1), c(n+1), d(n+1))$, which takes 3n + 6 steps to reach to (0, 0, 0, 0).
In summary,

$$(a,b,c,d) = \begin{cases} (3,1,1,1) & \text{if } m < 5 \\ (a(n), b(n), c(n), d(n)) & \text{if m = 3n + 4} \\ \frac{1}{2}\bigtriangledown^2 (a(n+1), b(n+1), c(n+1), d(n+1)) & \text{if m = 3n + 5} \\ \frac{1}{2}\bigtriangledown (a(n+1), b(n+1), c(n+1), d(n+1)) & \text{if m = 3n + 6} \end{cases} \tag{35}$$

Since (ka, kb, kc, kd) takes the same steps as (a, b, c, d) to reach to (0, 0, 0, 0), there exist infinite quadruples that take m steps to reach (0, 0, 0, 0).

Example 5.1 Let n = 10, so m = 3n + 4 = 34 steps. From Theorem 6, $a(10) = 532159, a(9) = 157305, a(8) = 46499$, and $a(7) = 13745$. So we have $b(10) = \frac{1}{2}(a(10)+a(8)) = \frac{1}{2}(532159+46499) = 289329, c(10) = a(9) = 157305$, and $d(10) = b(9) = \frac{1}{2}(a(9) + a(7)) = \frac{1}{2}(157305 + 13745) = 85525$.

$\bigtriangledown^{34}(532159, 289329, 157305, 85525)$
$= 2 \times \bigtriangledown^{33}(121415, 66012, 35890, 223317)$
$= 2 \times \bigtriangledown^{32}(55403, 30122, 187427, 101902)$
$= 2 \times \bigtriangledown^{31}(25281, 157305, 85525, 46499)$
$= 2^2 \times \bigtriangledown^{30}(66012, 35890, 19513, 10609)$
$= 2^2 \times \bigtriangledown^{29}(30122, 16377, 8904, 55403)$

$= 2^2 \times \nabla^{28}(13745, 7473, 46499, 25281)$
$= 2^3 \times \nabla^{27}(3136, 19513, 10609, 5768)$
$= 2^3 \times \nabla^{26}(16377, 8904, 4841, 2632)$
$= 2^3 \times \nabla^{25}(7473, 4063, 2209, 13745)$
$= 2^4 \times \nabla^{24}(1705, 927, 5768, 3136)$
$= 2^4 \times \nabla^{23}(778, 4841, 2632, 1431)$
$= 2^5 \times \nabla^{22}(4063, 2209, 1201, 653)$
$= 2^5 \times \nabla^{21}(927, 504, 274, 1705)$
$= 2^5 \times \nabla^{20}(423, 230, 1431, 778)$
$= 2^5 \times \nabla^{19}(193, 1201, 653, 355)$
$= 2^6 \times \nabla^{18}(504, 274, 149, 81)$
$= 2^6 \times \nabla^{17}(230, 125, 68, 423)$
$= 2^6 \times \nabla^{16}(105, 57, 355, 193)$
$= 2^7 \times \nabla^{15}(24, 149, 81, 44)$
$= (0, 0, 0, 0).$
Here, the result from the previous example, $\nabla^{15}(24, 149, 81, 44) = (0, 0, 0, 0)$ and Lemma 4.2.

Theorem 5.4 If $a > b > c > 0$ and $d = 0$, the quadruple $(a, b, c, 0)$ that has the maximum steps is in the form $(a_n, c_n + d_n, d_n, 0)$ and it needs one more step than (a_n, b_n, c_n, d_n),in which (a_n, b_n, c_n, d_n) are list in (13).

Proof. It can be proved by using the same technique as Theorem 5.1. In fact, it is a special case of the quadruple patten in Theorem 5.1.
From (13), we have $a_n = 2(a_{n-1} + b_{n-1}) + c_{n-1} = a_{n-1} + b_{n-1} + c_{n-1} + a_{n-1} + b_{n-1} = b_n + c_n + d_n$ Therefore, $\nabla(a_n, c_n + d_n, d_n, 0) = (a_n - c_n - d_n, c_n, d_n, a_n) = (b_n, c_n, d_n, a_n)$, which is the quadruple patten in (10), but it needs one more step to $(0, 0, 0, 0)$.

6 Conculsion

In this paper, all major Diffy problems for four numbers are solved. Specially, we got the exact upper bound of Diffy problem. Based on this interesting result, we also proved that if $max\{a, b, c, d\} \rightarrow \infty$, the steps $N \rightarrow \infty$ and there exists as many as quadruples we want for a given positive integer, m.

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