Revisiting Graceful Labelings of Graphs

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Abstract

For a graph G of size m, a graceful labeling of G is an injective function $f : V(G) \to \{0, 1, \ldots, m\}$ that gives rise to a bijective function $f' : E(G) \to \{1, 2, \ldots, m\}$ defined by $f'(uv) = |f(u) - f(v)|$. A graph G is graceful if G has a graceful labeling. Over the years, a number of variations of graceful labelings have been introduced, some of which have been described in terms of colorings. We look at several of these, with special emphasis on some of those introduced more recently.

Key Words: graceful labeling, graceful graph, gracefulness, edge-graceful labeling, modular edge-graceful labeling, graceful coloring, graceful chromatic numer.

AMS Subject Classification: 05C15, 05C78.

1 Introduction

Graph coloring is one of the most popular research areas in graph theory. The most studied colorings are proper vertex colorings and proper edge colorings. A *proper vertex coloring* of a graph G is an assignment of colors to the vertices of G such that adjacent vertices are assigned distinct colors. The minimum number of colors required of a proper vertex coloring of G is its *chromatic number* $\chi(G)$. A *proper edge coloring* of a graph G is an assignment of colors to the edges of G such that adjacent edges are assigned distinct colors. The minimum number of colors required of a proper edge coloring of G is its *chromatic index* $\chi'(G)$. We refer to the book [11] for graph theory notation and terminology not described in this paper.

A vertex coloring (or labeling) of a graph G is *vertex-distinguishing* if distinct vertices of G are assigned distinct colors (or labels). There are numerous occasions when an edge coloring of a graph (not necessarily a proper coloring) gives rise to a vertex-distinguishing coloring (see [11, pp. 370-385] or [28], for example). An edge coloring (or labeling) of a graph G

is *edge-distinguishing* if distinct edges of G are assigned distinct colors (or labels). There are also occasions when a vertex coloring of a graph (not necessarily a proper coloring) gives rise to an edge-distinguishing labeling (see [18, 26], [11, pp. 359-370] or [28, 29], for example).

One of best known examples of vertex-distinguishing colorings was introduced by Chartrand et al. in [10]. At the 250th Anniversary of Graph Theory Conference held at Indiana University-Purdue University Fort Wayne in 1986, a weighting of a connected graph G was introduced for the purpose of producing a weighted graph whose degrees (obtained by adding the weights of the incident edges of each vertex) were distinct. Such a weighted graph was called *irregular.* This concept could be looked at in another manner, however. In particular, let $\mathbb{N}$ denote the set of positive integers and let E_v denote the set of edges of G incident with a vertex v. An edge coloring $c : E(G) \to \mathbb{N}$, where adjacent edges may be colored the same, is said to be *vertex-distinguishing* if the coloring $s : V(G) \to \mathbb{N}$ induced by c and defined by

$$s(v) = \sum_{e \in E_v} c(e)$$

has the property that $s(x) \neq s(y)$ for every two distinct vertices x and y of G. For example, the edge coloring of the Petersen graph with the colors $1, 2, \ldots, 5$ shown in Figure 1 is vertex-distinguishing, where the color $s(v)$ of each vertex v is placed inside v.

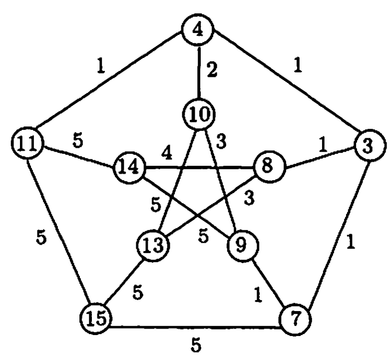

Figure 1: A vertex-distinguishing edge coloring of the Petersen graph

The main emphasis of this research dealt with minimizing the largest color assigned to the edges of a graph G to produce an irregular graph. The largest such color is referred to as the *irregularity strength* of G. In fact, the irregularity strength of the Petersen graph is 5. Much research has been done in this area of research (see [1, 14, 16, 28], for example). In recent years, a variety of edge colorings have been introduced which induce, in a number of ways, vertex colorings possessing desirable properties (see [2, 7, 8, 9, 15], for example). There has also been a variety of vertex colorings that induce edge or vertex colorings possessing desirable properties (see [3, 29] for example). We begin with vertex colorings or labelings that induce edge-distinguishing colorings or labelings of graphs.

2 Graceful Graphs

The best known example of an edge-distinguishing labeling is a graceful labeling. In 1968, Rosa [26] introduced a vertex labeling that induces an edge-distinguishing labeling defined by subtracting labels. In particular, for a graph G of size m, a vertex labeling (an injective function) $f : V(G) \to \{0, 1, \ldots, m\}$ was called a *β-valuation* by Rosa if the induced edge labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ defined by $f'(uv) = |f(u) - f(v)|$ is bijective. In 1972, Golomb [20] called a β-valuation a *graceful labeling* and a graph possessing a graceful labeling a *graceful graph*. It is this terminology that became standard. Over the past few decades the subject of graph labelings has been growing in popularity. Gallian [17] has compiled a periodically updated survey of many kinds of labelings and numerous results, obtained from well over a thousand referenced research articles.

A major problem in this area is that of determining which graphs are graceful. Among results obtained on graceful graphs are the following:

1. The cycle C_n is graceful if and only if $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$.
2. The complete graph K_n is graceful if and only if $n \leq 4$.
3. The graph $K_{s,t}$ is graceful for all positive integers s and t.
4. The n-cube Q_n is graceful for all positive integers n.
5. The path P_n is graceful for all positive integers n.
6. The grid $P_s \square P_t$ is graceful for all positive integers s and t.
7. Every caterpillar is graceful.
8. Every tree with at most four end-vertices is graceful.
9. Every tree of order at most 27 is graceful.

While the three graphs shown in Figure 2 are the only connected graphs of order 5 that are not graceful, it has been shown that almost all graphs are not graceful [12].

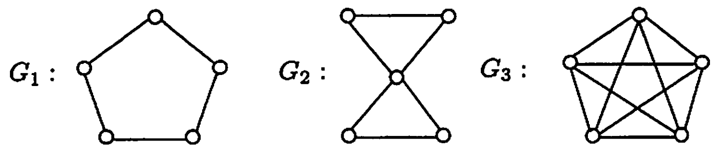

Figure 2: Three graphs that are not graceful

One of the best known conjectures dealing with graceful graphs involves trees and is due to Kotzig and Ringel (see [17]).

The Graceful Tree Conjecture *Every nontrivial tree is graceful.*

The *gracefulness* grac(G) of a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the smallest positive integer k for which it is possible to label the vertices of G with distinct elements of the set $\{0, 1, 2, \ldots, k\}$ in such a way that an edge is labeled as above and distinct edges receive distinct labels. The gracefulness of every such graph is defined, for if we label v_i by 2^{i-1} for $1 \le i \le n$, then a vertex labeling with this property exists. Thus, if G is a graph of order n and size m, then $m \le \mathrm{grac}(G) \le 2^{n-1}$. If $\mathrm{grac}(G) = m$, then G is graceful. The gracefulness of a graph G can be considered as a measure of how close G is to being graceful – the closer the gracefulness is to m, the closer the graph is to being graceful. The exact values of $\mathrm{grac}(K_n)$ were determined for $1 \le n \le 10$ in [20]. For example, $\mathrm{grac}(K_4) = 6$, $\mathrm{grac}(K_5) = 11$ and $\mathrm{grac}(K_6) = 17$. The exact value of $\mathrm{grac}(K_n)$ is not known in general, however. On the other hand, Erdős showed that $\mathrm{grac}(K_n) \sim n^2$ (see [20]).

3 Edge-Graceful Graphs

In 1985 Lo [25] introduced a dual type of graceful labeling – this one dealing with edge labelings. Let G be a connected graph of order $n \ge 2$ and size m. For a vertex v of G, let $N(v)$ denote the neighborhood of v. An *edge-graceful labeling* of G is a bijective function $f : E(G) \to \{1, 2, \ldots, m\}$ that gives rise to a bijective function $f' : V(G) \to \{0, 1, 2, \ldots, n-1\}$ given by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_n$. A graph that admits an edge-graceful labeling is called an *edge-graceful graph.* Figure 3 shows two edge-graceful graphs C_5 and $K_{1,4}$ together with an edge-graceful labeling for each of them. It is well known that C_n is graceful if and only if $n \equiv 0, 3 \pmod 4$ and so C_5 is not graceful.

It was observed in [25] that if G is an edge-graceful graph of order n and size m, then

$$\binom{n}{2} \equiv 2\binom{m+1}{2} \pmod{n}. \tag{1}$$

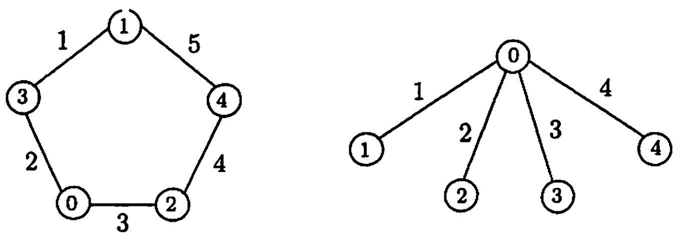

Figure 3: Two edge-graceful graphs

Since $\binom{n}{2} = 2\binom{m+1}{2}$ if G is a tree of order n, a tree satisfies (1) if and only if its order is odd. It is known that the path P_n of odd order n is edge-graceful. It was conjectured by Lee [24] that every nontrivial tree of odd order is edge-graceful. In fact, it was also conjectured by Lee [24] that every connected graph of order n with $n \not\equiv 2 \pmod 4$ is edge-graceful. Among the results obtained on edge-graceful graphs are the following:

1. The complete graph K_n is edge-graceful if and only if $n \not\equiv 2 \pmod 4$.
2. Every odd cycle is edge-graceful.
3. The Cartesian product $C_m \square C_n$ is edge-graceful if and only if m and n are both odd.

It was observed in [21] that in the definition of an edge-graceful labeling of a connected graph G of order $n \geq 2$ and size m, the edge labeling f is required to be one-to-one. Since, however, the induced vertex labels $f'(v)$ are obtained by addition in $\mathbb{Z}_n$, the function f is actually a function from $E(G)$ to $\mathbb{Z}_n$ and in general is not one-to-one. Dividing m by n, we obtain $m = nq + r$, where $q = \lfloor m/n \rfloor$ and $0 \leq r \leq n-1$. Hence, in an edge-graceful labeling of G, $q+1$ edges are labeled i for each i with $1 \leq i \leq r$ and q edges are labeled i for each i with $r+1 \leq i \leq n$ (in $\mathbb{Z}_n$). Thus, this edge labeling $f : E(G) \to \mathbb{Z}_n$ is a one-to-one function only when $m = n-1$ or $m = n$. This observation gives rise to another concept (see [21]).

4 Modular Edge-Graceful Graphs

Let G be a connected graph of order $n \geq 3$ and let $f : E(G) \to \mathbb{Z}_n$, where f need not be one-to-one. Let $f' : V(G) \to \mathbb{Z}_n$ be defined by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_n$. If f' is one-to-one, then f is called a *modular edge-graceful labeling* and G is a *modular edge-graceful graph*. Consequently, every edge-graceful graph is a modular edge-graceful graph. This concept was introduced in 1991 by Jothi [19] under the terminology of *line-graceful graphs* (also see [17]). The graphs $G_1 = C_4$ and G_2 in Figure 4 are both modular edge-graceful. Modular edge-graceful labelings are shown in Figure 4 as well. In fact, the graph G_2 is neither graceful nor edge-graceful.

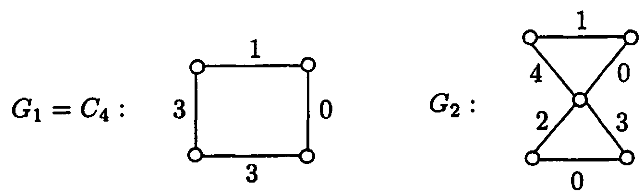

Figure 4: Two modular edge-graceful graphs

It is known that if G is a connected graph of order $n \geq 3$ for which $n \equiv 2 \pmod 4$, then G is not modular edge-graceful. Furthermore, it was conjectured that if T is a tree of order $n \geq 3$ for which $n \not\equiv 2 \pmod 4$, then T is modular edge-graceful (see [17]). This conjecture was verified in [22]. In fact, the conjecture holds not only for trees but for all connected graphs (see [22]).

Theorem 4.1 *A connected graph of order $n \geq 3$ is modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$.*

For every connected graph G of order n, there is a smallest integer $k \geq n$ for which there exists an edge labeling $f : E(G) \to \mathbb{Z}_k$ such that the induced vertex labeling $f' : V(G) \to \mathbb{Z}_k$ defined by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_k$, is one-to-one. The number k is called the *modular edge-gracefulness* $\text{meg}(G)$ of G. Thus, $\text{meg}(G) \geq n$ and $\text{meg}(G) = n$ if and only if G is a modular edge-graceful graph of order n. If G is not modular edge-graceful, then $\text{meg}(G) \geq n + 1$. As with the gracefulness of a graph, the modular edge-gracefulness of a graph G is a measure of how close G is to being modular edge-graceful. The number $\text{meg}(G)$ was determined for every connected graph G in [22].

Theorem 4.2 *If G is a nontrivial connected graph of order $n \geq 6$ that is not modular edge-graceful, then* $\text{meg}(G) = n + 1$.

If G is a modular edge-graceful spanning subgraph of a graph H where G and H are connected, then a modular edge-graceful labeling of G can be extended to a modular edge-graceful labeling of H by assigning the label 0 to each edge of H that does not belong to G. Modular edge-graceful labelings of a graph that assigns the label 0 to some edges of the graph play an important role in establishing Theorems 4.1 and 4.2.

For this reason, those modular edge-graceful labelings in which 0 is not permitted were investigated in [23]. This gives rise to another concept and to other problems. More formally, for a connected graph G of order $n \geq 3$, let $f : E(G) \to \mathbb{Z}_n - \{0\}$, where f need not be one-to-one and let $f' : V(G) \to \mathbb{Z}_n$ be defined by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_n$. If f' is one-to-one, then f is called a *nowhere-zero modular edge-graceful labeling* and G is a *nowhere-zero modular edge-graceful graph.* A characterization of connected nowhere-zero modular edge-graceful graphs was established in [23].

Theorem 4.3 *A connected graph G of order $n \geq 3$ is nowhere-zero modular edge-graceful if and only if*

(*i*) $n \not\equiv 2 \pmod 4$,

(*ii*) $G \neq K_3$ *and*

(*iii*) *G is not a star of even order.*

For every connected graph G of order n, there is a smallest integer $k \geq n$ for which there exists an edge labeling $f : E(G) \to \mathbb{Z}_k - \{0\}$ such that the induced vertex labeling $f' : V(G) \to \mathbb{Z}_k$ defined by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_k$, is one-to-one. This number k is referred to as the *nowhere-zero modular edge-gracefulness* of G and is denoted by $\text{nzg}(G)$. Thus, $\text{nzg}(G) = n$ if and only if G is nowhere-zero modular edge-graceful and so $\text{nzg}(G) \geq n + 1$ if G is not nowhere-zero modular edge-graceful. For a connected graph G of order $n \geq 3$ with $n \not\equiv 2 \pmod 4$ that is not nowhere-zero modular edge-graceful, the exact value of $\text{nzg}(G)$ has been determined (see [23]).

Theorem 4.4 *If G is a connected graph of order $n \geq 3$ that is not nowhere-zero modular edge-graceful, then* $\text{nzg}(G) \in \{n + 1, n + 2\}$. *Furthermore,*

(*i*) *if $n \not\equiv 2 \pmod 4$, then $\text{nzg}(G) = n + 1$ if and only if $G = K_3$ and $\text{nzg}(G) = n + 2$ if and only if G is a star of even order.*

(*ii*) *if $n \equiv 2 \pmod 4$, then $\text{nzg}(G) = n + 2$ if and only if G is a star.*

5 Graceful Colorings

Graceful labelings have also been looked at in terms of colorings. A *rainbow vertex coloring* of a graph G of size m is an assignment f of distinct colors to the vertices of G. If the colors are chosen from the set $\{0, 1, \ldots, m\}$, resulting in each edge uv of G being colored $f'(uv) = |f(u) - f(v)|$ such that the colors assigned to the edges of G are also distinct, then this *rainbow* vertex coloring results in a *rainbow* edge coloring $f' : E(G) \to \{1, 2, \ldots, m\}$. So, such a rainbow vertex coloring is a graceful labeling of G.

Inspired by graceful labelings and proper colorings in graphs, another type of vertex coloring was introduced in [4] that induces an edge coloring, where both colorings are proper rather than rainbow. It is useful to describe notation for certain intervals of integers. For positive integers a, b with $a \le b$, let $[a, b] = \{a, a+1, \ldots, b\}$ and $[b] = [1, b]$.

A *graceful k-coloring* of a nonempty graph G is a proper vertex coloring $c : V(G) \to [k]$, where $k \ge 2$, that induces a proper edge coloring $c' : E(G) \to [k-1]$ defined by $c'(uv) = |c(u) - c(v)|$. A vertex coloring c of a graph G is a *graceful coloring* if c is a graceful k-coloring for some $k \in \mathbb{N}$. Note that in a graceful labeling of a nonempty graph of size m, the colors are chosen from the set $\{0, 1, \ldots, m\}$ and so the color 0 can be used; while in a graceful coloring, each color is a positive integer. The minimum k for which G has a graceful k-coloring is called the *graceful chromatic number* of G, denoted by $\chi_g(G)$. For example, Figure 5 shows a graceful 5-coloring of the cube Q_3 and so $\chi_g(Q_3) \le 5$. In fact, $\chi_g(Q_3) = 5$ as we will soon see.

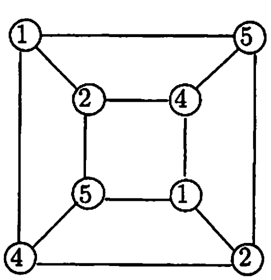

Figure 5: A graceful 5-coloring of Q_3

There are immediate lower and upper bounds for the graceful chromatic number of a graph, as observed in [4].

Observation 5.1 *If G is a nontrivial connected graph of order n, then $\chi_g(G)$ exists. Furthermore,*

$$\chi(G) \le \chi_g(G) \le \text{grac}(G) \le 2^{n-1}.$$

Figure 6 shows two graceful graphs K_4 and C_4 of order 4 together with a graceful coloring for each of these two graphs. In fact, $\chi_g(K_4) = 5 < \text{grac}(K_4) = 6$ and $\chi_g(C_4) = \text{grac}(C_4) = 4$.

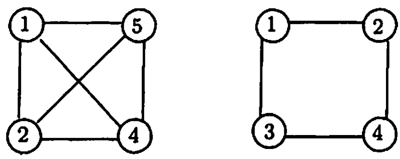

Figure 6: Graceful colorings of K_4 and C_4

For a graceful k-coloring c of a graph G, the *complementary coloring* $\bar{c} : V(G) \to [k]$ of G is a k-coloring defined by $\bar{c}(v) = k + 1 - c(v)$ for each vertex v of G. If $xy \in E(G)$, then the color $\bar{c}'(xy)$ of xy induced by $\bar{c}$ is

$$\begin{aligned} \bar{c}'(xy) &= |\bar{c}(x) - \bar{c}(y)| = |[(k+1) - c(x)] - [(k+1) - c(y)]| \\ &= |c(x) - c(y)| = c'(xy). \end{aligned}$$

Thus, as with graceful labelings, the complementary coloring of a graceful coloring of a graph is also graceful. Some useful facts about graceful colorings are described in [4].

- ⋆ If H is a subgraph of a graph G, then $\chi_g(H) \le \chi_g(G)$.
- ⋆ If G is a disconnected graph having p components $G_1, G_2, \ldots, G_p$ for some integer $p \ge 2$, then

$$\chi_g(G) = \max\{\chi_g(G_i) : 1 \le i \le p\}. \quad (2)$$

- ⋆ If G is a nontrivial connected graph, then

$$\chi_g(G) \ge \max\{\chi(G), \chi'(G)\} + 1. \quad (3)$$

By (2), it suffices to consider only nontrivial connected graphs. Since $\chi_g(K_{1,n-1}) = \chi'(K_{1,n-1}) + 1$, the bound in (3) is attained for all stars and so is sharp. There are bounds for the chromatic number and chromatic index of a graph G in terms of its maximum degree $\Delta(G)$. By Brooks' theorem [6], $\chi(G) \le \Delta(G)+1$ for every graph G and, when G is connected, $\chi(G) = \Delta(G) + 1$ if and only if G is a complete graph or an odd cycle. Furthermore, by Vizing's theorem [27], $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$ for every nonempty graph G. Thus, if G is a nonempty graph, then $\chi_g(G) \ge \Delta(G) + 1$. In fact, it was observed in [4] that if G is an r-regular graph where $r \ge 2$, then $\chi_g(G) \ge r + 2$. As we saw in Figure 5, the cubic graph Q_3 has a graceful 5-coloring and so $\chi_g(Q_3) = 5$.

If c is a graceful coloring of a nontrivial connected graph G and $v \in V(G)$, then c must assign distinct colors to the vertices in the closed neighborhood $N[v]$ of v. Thus, if $u, w \in V(G)$ such that $u \ne w$ and $d(u, w) \le 2$, then $c(u) \neq c(w)$. Furthermore, if (x, y, z) is an $x - z$ path in G, where $c(x) > c(z)$, say, then $c(x) - c(y) \neq c(y) - c(z)$ and so $c(y) \neq \frac{c(x)+c(z)}{2}$. Thus, we have the following useful observations (see [4]).

Observation 5.2 *Let $c : V(G) \to [k]$, $k \geq 2$, be a coloring of a nontrivial connected graph G. Then c is a graceful coloring of G if and only if*

(*i*) *for each vertex v of G, the vertices in the closed neighborhood of v are assigned distinct colors by c and*

(*ii*) *for each path (x, y, z) of order 3 in G, it follows that $c(y) \neq \frac{c(x)+c(z)}{2}$.*

Observation 5.2 provides a lower bound for a special class of connected graphs. The *diameter* $\text{diam}(G)$ of a connected graph G is the greatest distance between two vertices of G.

Corollary 5.3 *If G is a connected graph of order $n \geq 3$ with diameter 2, then $\chi_g(G) \geq n$.*

For example, there are exactly five connected cubic graphs with diameter 2. In fact, the graceful chromatic number of each of these graphs equals its order. Figure 7 shows a graceful coloring for each of these five graphs. This example illustrates a conjecture concerning the graceful chromatic numbers of connected cubic graphs.

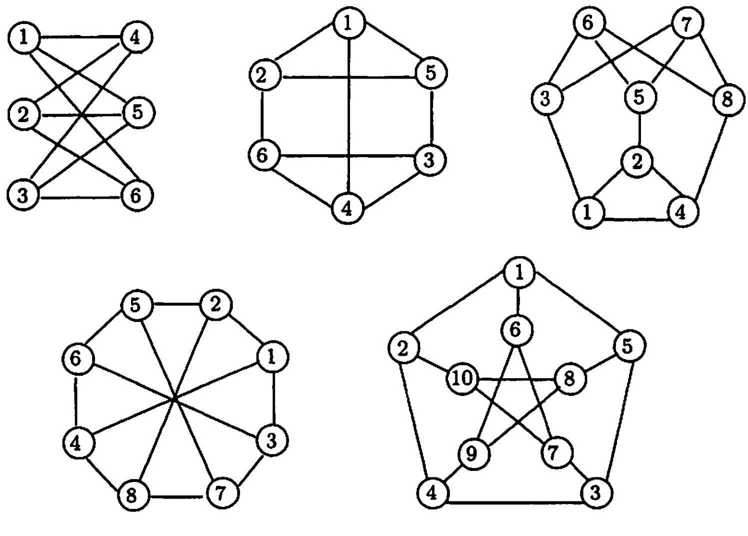

Figure 7: Graceful colorings of five connected cubic graphs with diameter 2

Conjecture 5.4 *If G is a connected cubic graph, then* $5 \leq \chi_g(G) \leq 10$.

There are infinite classes of connected graphs having diameter 2 where the graceful chromatic number of each graph equals its order, as well as an infinite classes of connected graphs having diameter 2 where the graceful chromatic number of each graph exceeds its order. The following result describes one such class of graphs in the second case.

Proposition 5.5 *If G is a nontrivial connected graph of order n such that $\delta(G) > n/2$, then $\chi_g(G) > n$.*

The graceful chromatic numbers of graphs belonging to some well-known classes of graphs were determined in [4], which are stated below.

1. If G is a complete bipartite graph of order $n \geq 3$, then $\chi_g(G) = n$.
2. For each integer $n \geq 4$, $\chi_g(C_5) = 5$ and $\chi_g(C_n) = 4$ otherwise.
3. For each integer $n \geq 4$, $\chi_g(P_4) = 3$ and $\chi_g(P_n) = 4$ for $n \geq 5$.
4. If W_n is the wheel of order $n \geq 6$, then $\chi_g(W_n) = n$.

Even though the complete graphs form a well-known class of connected graphs, it appears challenging to determine the exact value of the graceful chromatic number of complete graphs in general. Since every two vertices of K_n are adjacent and K_n is $(n-1)$-regular, it follows that $\chi_g(K_n) \geq n+1$. Furthermore, it was shown in [5] that

$$\chi_g(K_n) \leq \frac{3n^2 - n}{2} \tag{4}$$

for each integer $n \geq 5$. Since $\chi_g(K_n) \leq \text{grac}(K_n)$ and $\text{grac}(K_n) \sim n^2$, it is almost certain that there is an upper bound for $\chi_g(K_n)$ in terms of n that is superior to that described in (4).

Another class of connected graphs of diameter 2 has been studied in [4]. For integers p and k where $p \geq 2$ and $k \geq 3$, let $K_{k(p)}$ denote the regular complete k-partite graph, each of whose partite sets consists of p vertices. Thus, the order of $K_{k(p)}$ is $n = kp$ and the degree of regularity is $r = \frac{n(k-1)}{k} = (k-1)p$. The following result gives an upper bound for the graceful chromatic number of $K_{k(p)}$.

Theorem 5.6 *For integers p and k where $p \geq 2$ and $k \geq 3$,*

$$\chi_g(K_{k(p)}) \leq \begin{cases} \left(2^{\frac{k+2}{2}} - 2\right)p - 2^{\frac{k-2}{2}} + 1 & \text{if } k \text{ is even} \\ \left(2^{\frac{k+3}{2}} - 3\right)p - 2^{\frac{k-1}{2}} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

For example, $\chi_g(K_{4(4)}) \leq 23$ by Theorem 5.6. A graceful coloring $c : V(K_{4(4)}) \to [23]$ of $K_{4(4)}$ using colors from the set [23] is shown in Figure 8. Note that the seven colors $9, 10, \ldots, 15 \in [23]$ are not used in this coloring.

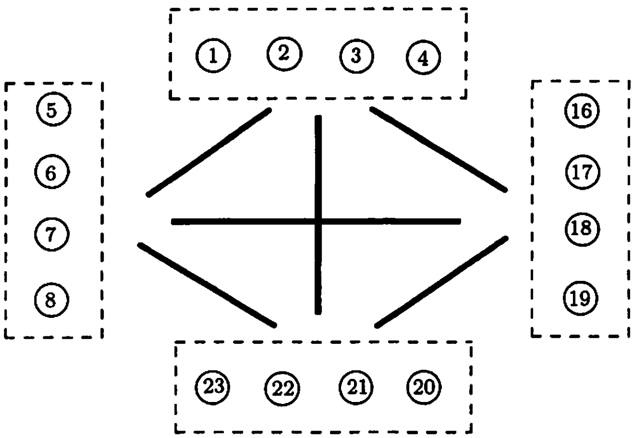

Figure 8: A graceful coloring of $K_{4(4)}$

The upper bound for $\chi_g(K_{k(p)})$ presented in Theorem 5.6 is almost certainly not sharp. While $\chi_g(K_{p,p,p}) \leq 5p - 1$ for $p \geq 2$ according to Theorem 5.6, it was shown in [4] that for each integer $p \geq 2$,

$$\chi_g(K_{p,p,p}) \leq \begin{cases} 4p-1 & \text{if } p \text{ is even} \\ 4p & \text{if } p \text{ is odd.} \end{cases} \tag{5}$$

For example, if $p = 5$, then $\chi_g(K_{5,5,5}) \leq 20$ by (5). A graceful coloring of $K_{5,5,5}$ using colors from the set [20] is shown in Figure 9. In fact, if $c : V(K_{5,5,5}) \to [20]$ is a graceful coloring of $K_{5,5,5}$, then c cannot assign any of $8, 9, 10, 11, 12$ as a color to a vertex of $K_{5,5,5}$.

In fact, it was conjectured in [4] that the upper bound in (5) is, in fact, the actual value of $\chi_g(K_{p,p,p})$ for every integer $p \geq 2$ As an illustration, we verify this for $p = 4$.

Proposition 5.7 $\chi_g(K_{4,4,4}) = 15$.

Proof. By (5), $\chi_g(K_{4,4,4}) \leq 15$. Hence, it remains to show that there is no graceful 14-coloring of $G = K_{4,4,4}$. Assume, to the contrary, that G has a graceful coloring $c : V(G) \to [14]$. Since $\text{diam}(G) = 2$, no two vertices of G are assigned the same color. Thus, 12 colors from the set [14] are used in this coloring.

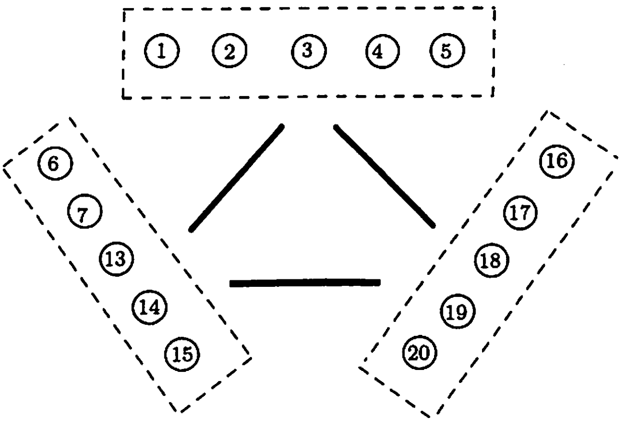

Figure 9: A graceful coloring of $K_{5,5,5}$

First, we show that no vertex of G is assigned the color 7 or 8. Assume, to the contrary, that some vertex of G is assigned one of these colors. Since the complementary coloring of c is also graceful coloring, we may assume that $c(v) = 7$ for some vertex v of G, say $v \in V_1$, one of the three partite sets of G. Now consider the six 2-element sets $\{6,8\}$, $\{5,9\}$, $\{4,10\}$, $\{3,11\}$, $\{2,12\}$, $\{1,13\}$. Since at most three of these sets contain a color assigned to a vertex in V_1 and at most two of these sets contain a color not used by the coloring c, there is a 2-element set each of whose colors is assigned to a vertex not in V_1. However then, two edges incident with v are assigned the same color, which is impossible. Hence, no vertex of G is assigned the color 7 or 8.

Therefore, the vertices of G are assigned colors from the set $[6] \cup [9,14]$. Let V_1, V_2, V_3 be the partite sets of G. We may assume that $6 \in c(V_1)$. Necessarily, one element from each of the 2-element sets $\{3,9\}$, $\{2,10\}$ and $\{1,11\}$ belongs to $c(V_1)$. We consider two cases, according to whether $3 \in c(V_1)$ or $9 \in c(V_1)$.

Case 1. $3 \in c(V_1)$. Thus, one element from each of the two sets $\{2,4\}$ and $\{1,5\}$ belongs to $c(V_1)$. Since $6 \in c(V_1)$, one element from each of the sets $\{2,10\}$ and $\{1,11\}$ belongs to $c(V_1)$. Therefore, $c(V_1) = \{1,2,3,6\}$. Since $4 \in c(V_i)$ for $i = 2,3$, the vertex colored 4 is incident with two edges colored 2, which is impossible.

Case 2. $9 \in c(V_1)$. Hence, $\{6,9\} \subseteq c(V_1)$. Since $6 \in c(V_1)$, one element from each of the sets $\{2,10\}$ and $\{1,11\}$ belongs to $c(V_1)$. Similarly, since $9 \in c(V_1)$, one element from each of the sets $\{5,13\}$ and $\{4,14\}$ belongs to $c(V_1)$. This is impossible. ∎

A lower bound for the graceful chromatic number of a connected graph was established in [13] in terms of its minimum degree.

Theorem 5.8 *If G is a connected graph with minimum degree $\delta(G) \geq 2$, then*

$$\chi_g(G) \geq \left\lceil \frac{5\delta(G)}{3} \right\rceil.$$

It was observed in [13] that the lower bound for the graceful chromatic number of a graph presented in Theorem 5.8 is best possible. For example, the graph G of Figure 10 has $\delta(G) = \delta = 2$ and graceful chromatic number $\chi_g(G) = \lceil \frac{5\delta}{3} \rceil = 4$. A graceful 4-coloring of G is shown in the figure.

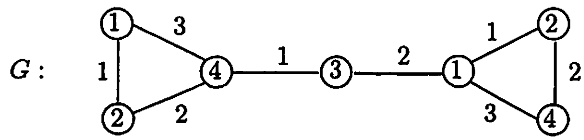

Figure 10: A graph G with $\chi_g(G) = \lceil \frac{5\delta}{3} \rceil$

The graph G of Figure 10 gives rise to the following question.

Problem 5.9 *Is there an infinite class of connected graphs G such that $\delta(G) \geq 2$ and*

$$\chi_g(G) = \left\lceil \frac{5\delta(G)}{3} \right\rceil ?$$

6 Graceful Colorings of Trees

Graceful chromatic numbers have been investigated for several classes of trees. A *caterpillar* is a tree T of order 3 or more, the removal of whose leaves produces a path (called the *spine* of T). Thus, every path, every star (of order at least 3) and every double star (a tree of diameter 3) is a caterpillar.

Theorem 6.1 *If T is a caterpillar with maximum degree $\Delta \geq 2$, then*

$$\Delta + 1 \leq \chi_g(T) \leq \Delta + 2.$$

Furthermore, $\chi_g(T) = \Delta + 2$ if and only if T has at least one vertex of degree Δ that is adjacent to two vertices of degree Δ in T.

In general, there is an upper bound for the graceful chromatic number of a tree in terms of its maximum degree.

Theorem 6.2 *If T is a nontrivial tree with maximum degree Δ, then*

$$\chi_g(T) \leq \left\lceil \frac{5\Delta}{3} \right\rceil.$$

The upper bound is Theorem 6.2 is best possible. In order to show this, the graceful chromatic numbers of trees belonging to a particular class of trees were investigated in [13]. For each integer $\Delta \geq 2$, let $T_{\Delta,1}$ be the star $K_{1,\Delta}$. The *central vertex* of $T_{\Delta,1}$ is denoted by v. Thus, $\deg v = \Delta$ and all other vertices of $T_{\Delta,1}$ have degree 1. For each integer $h \geq 2$, let $T_{\Delta,h}$ be the tree obtained from $T_{\Delta,h-1}$ by identifying each end-vertex with the central vertex of the star $K_{1,\Delta-1}$. The tree $T_{\Delta,h}$ is therefore a rooted tree (with root v) having height h. The vertex v is then the *central vertex* of $T_{\Delta,h}$. In $T_{\Delta,h}$, every vertex at distance less than h from v has degree Δ; while all remaining vertices are leaves and are at distance h from v. Thus, $T_{2,2} = P_5$, while $T_{3,2}$ and $T_{6,2}$ are shown in Figure 11.

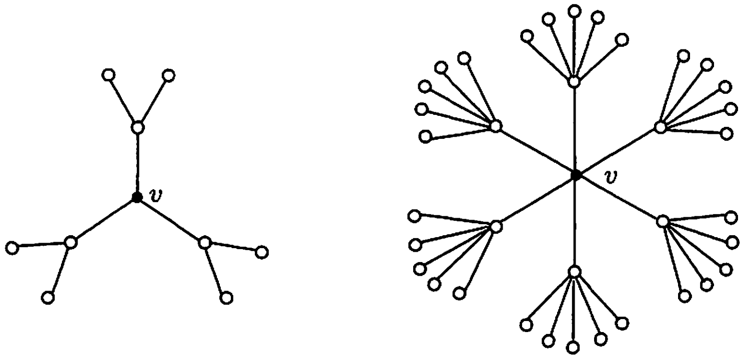

Figure 11: The trees $T_{3,2}$ and $T_{6,2}$

For integers Δ and h with $\Delta \geq 2$ and $h \in \{2, 3, 4\}$, the graceful chromatic numbers of the trees $T_{\Delta,h}$ were determined in [13].

Theorem 6.3 *For each integer $\Delta \geq 2$,*

(i) $\chi_g(T_{\Delta,2}) = \lceil \frac{3\Delta+1}{2} \rceil$.

(ii) $\chi_g(T_{\Delta,3}) = \lceil \frac{13\Delta+1}{8} \rceil$.

(iii) $\chi_g(T_{\Delta,4}) = \lceil \frac{53\Delta+1}{32} \rceil$.

The results obtained in Theorem 6.3 on $T_{\Delta,h}$ for $\Delta \geq 2$ and $h \in \{2, 3, 4\}$ suggest the following conjecture.

Conjecture 6.4 *For an integer $h \geq 2$, let $\sigma_h = 2^{2h-3} + \sum_{i=2}^{h} 2^{2i-4}$. Then*

$$\chi_g(T_{\Delta,h}) = \left\lceil \frac{\sigma_h \Delta + 1}{2^{2h-3}} \right\rceil .$$

Theorem 6.5 *Let $\Delta \geq 2$ be an integer. If h is an integer such that $h \geq 2 + \lfloor \frac{\Delta}{3} \rfloor$, then*

$$\chi_g(T_{\Delta,h}) = \left\lceil \frac{5\Delta}{3} \right\rceil .$$

The following two results are consequences of Theorem 6.5.

Corollary 6.6 *For each integer $\Delta \geq 2$,*

$$\lim_{h \to \infty} \chi_g(T_{\Delta,h}) = \left\lceil \frac{5\Delta}{3} \right\rceil .$$

Corollary 6.7 *If T is a tree with maximum degree $\Delta \geq 2$ containing a vertex v such that every vertex of T within distance $2 + \lfloor \frac{\Delta}{3} \rfloor$ of v also has degree Δ, then $\chi_g(T) = \lceil \frac{5\Delta}{3} \rceil$.*

7 Closing Statements

We saw that if G is a nontrivial connected graph, then

$$\chi_g(G) \geq \max\{\chi(G), \chi'(G)\} + 1.$$

Since there are relatively few graphs G for which $\chi_g(G) = \chi(G) + 1$ or $\chi_g(G) = \chi'(G) + 1$, this gives rise to the following natural question.

Problem 7.1 *Under what conditions does a connected graph G satisfy $\chi_g(G) = \chi(G) + 1$ or $\chi_g(G) = \chi'(G) + 1$?*

We saw that if G is a connected graph of order n with diameter 2, then $\chi_g(G) \geq n$. Thus, we have the following question.

Problem 7.2 *For each integer $k \in \mathbb{N}$, does there exist a connected graph G of order n such that $\chi_g(G) = n + k$?*

Since it is evidently challenging to determine the exact value of the graceful chromatic number of a given graph, it appears to be more practical to establish bounds for this parameter in terms of other well-known graphical parameters. We conclude with two questions related to the graceful chromatic number of a connected graph or a graceful graph in terms of the size of the graph.

Problem 7.3 *Let G be a connected graph of size m. Is there a function $f(m)$ such that $\chi_g(G) \leq f(m)$?*

Problem 7.4 *Let G be a graceful graph of order m. Is there a function $g(m)$ such that $\chi_g(G) \leq g(m)$?*

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