On Hamiltonian Cycle Extension in Cubic Hamiltonian Graphs

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November 16, 2015

Abstract

A Hamiltonian graph G is said to be ℓ-path-Hamiltonian, where ℓ is a positive integer less than or equal to the order of G, if every path of order ℓ in G is a subpath of some Hamiltonian cycle in G. The Hamiltonian cycle extension number of G is the maximum positive integer L for which G is ℓ-path-Hamiltonian for every integer ℓ with $1 \leq \ell \leq L$. Hamiltonian cycle extension numbers are determined for several well-known cubic Hamiltonian graphs. It is shown that if G is a cubic Hamiltonian graph with girth g, where $3 \leq g \leq 7$, then G is ℓ-path-Hamiltonian only if $1 \leq \ell \leq g$.

Keywords: Hamiltonian graph, ℓ-path-Hamiltonian graph, Hamiltonian cycle extension number.

AMS subject classification: 05C38, 05C45, 05C75.

1 Introduction

The Irish mathematician and physicist William Rowan Hamilton discovered a noncommutative algebra he referred to as icosian calculus. This algebra was based on three symbols i, κ and λ, all roots of unity, with $i^2 = 1$, $\kappa^3 = 1$ and $\lambda^5 = 1$, where $\lambda = i\kappa$. While this algebra is not commutative, it is associative. These elements generate a group isomorphic to the group of rotations of the regular dodecahedron. Hamilton saw that these symbols relate to journeys about a dodecahedron which led to his invention of a game he called the *Icosian Game*. One goal of this game was to discover a closed walk moving along the edges of the dodecahedron that visits each vertex exactly once. Hamilton actually envisioned this game as a two-person game, where the first player provides conditions that the second player was to follow as one proceeds about the dodecahedron. In one version of this game described by Hamilton, there are 20 markers, numbered 1 to 20. The first player is to place markers 1, 2, 3, 4, 5, in order, on five consecutive vertices of a dodecahedron. The second player is then to place the remaining markers 6, 7, ..., 20, in order, on 15 consecutive unmarked vertices, such that markers 5 and 6, and 20 and 1 appear on consecutive vertices. This is the same as beginning with any path of order 5 on the graph of the dodecahedron H and try extending the path to form a Hamiltonian cycle in H. From his icosian calculus, Hamilton knew that this could always be done, no matter which five consecutive vertices are chosen first. In terms of graphs, for every path P of order 5 (or less) in H, there always exists a Hamiltonian cycle C of H such that P is a path on C. What Hamilton observed for paths of order 5 on the graph H does not hold for all paths of order 6. As illustrated in Figure 1, the path of order 6 (drawn in bold edges) with initial vertex s cannot be extended to a Hamiltonian cycle on H, since the only way to reach y is through x and then we cannot return to s. Hamilton never mentioned this however. Hamilton's observation

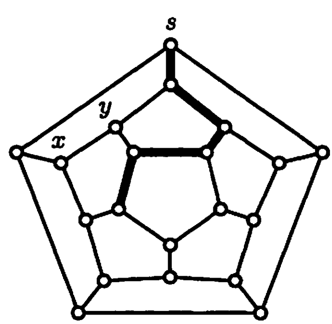

Figure 1: The graph H of the dodecahedron

led a concept that is defined for every Hamiltonian graph. This concept was introduced by Chartrand in 2013, first studied in [1] and then further studied in [3, 4]. We refer to the books [2, 5] for graph theory notation and terminology not described in this paper.

A Hamiltonian graph G of order $n \geq 3$ is called *ℓ-path-Hamiltonian,* for an integer ℓ with $1 \leq \ell \leq n$, if for every path P of order ℓ in G, there exists a Hamiltonian cycle C of G such that P lies on C. Certainly, every Hamiltonian graph is 1-path-Hamiltonian. The largest integer L for which a Hamiltonian graph G is ℓ-path-Hamiltonian for every integer ℓ with $1 \leq \ell \leq L$ is the *Hamiltonian cycle extension number* $\text{hce}(G)$. Therefore, $1 \leq \text{hce}(G) \leq n$. If G is a Hamiltonian graph for which some automorphism maps any edge of G onto any other edge of G, then $\text{hce}(G) \geq 2$. In fact, $\text{hce}(G) = 2$ if and only if every edge of G lies on some Hamiltonian cycle of G but some path of order 3 in G does not lie on any Hamiltonian cycle of G. For example, the graph G of order 5 in Figure 2 has Hamiltonian

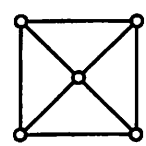

Figure 2: A graph G with $\text{hce}(G) = 2$

extension number 2 since every edge of G lies on a Hamiltonian cycle but a 3-path (u, v, w), where v is the vertex of degree 4 and $uw \notin E(G)$, cannot be extended to a Hamiltonian cycle in G. Furthermore, $\text{hce}(H) = 5$ for the graph H of the dodecahedron.

The following problem dealing with this topic is still open in general.

Problem 1.1 *If G is an ℓ-path-Hamiltonian graph for some $\ell \geq 2$, is G also $(\ell - 1)$-path-Hamiltonian?*

If the question asked in Problem 1.1 has an affirmative answer, then the Hamiltonian extension number of a Hamiltonian graph G can then be defined as the largest positive integer ℓ for which G is ℓ-path-Hamiltonian. Among the results obtained in [1, 3, 4] are the following.

Theorem 1.2 *Let G be a Hamiltonian graph of order n.*

(a) *If $n - 2 \leq \ell \leq n$, then the graph G is ℓ-path-Hamiltonian if and only if $G \in \{C_n, K_n, K_{n/2,n/2}\}$.*

(b) *The graph G is $(n-3)$-path-Hamiltonian if and only if*

(i) $G \in \{C_n, K_n, K_{n/2,n/2}\}$ *or*

(ii) $\overline{G} \in \{P_3 + P_2, C_6, 2P_3, C_4 + C_3\}$ *or*

(iii) $\delta(G) = n - 2$.

Certainly, the graph H of the dodecahedron is a cubic Hamiltonian graph. In this work, we study the Hamiltonian cycle extension numbers of cubic Hamiltonian graphs in general. First, the following is a consequence of Theorem 1.2.

Corollary 1.3 *Let G be a cubic Hamiltonian graph of even order $n \geq 4$.*

(a) *If $n = 4$, then $G = K_4$ and so G is ℓ-path-Hamiltonian if and only if $1 \leq \ell \leq 4$.*

(b) *If $n = 6$, then $G \in \{K_{3,3}, C_3 \square K_2\}$. Furthermore,*

- ⋆ *the graph $K_{3,3}$ is ℓ-path-Hamiltonian if and only if $1 \leq \ell \leq 6$ and*
- ⋆ *the graph $C_3 \square K_2$ is ℓ-path-Hamiltonian if and only if $1 \leq \ell \leq 3$.*

(c) *If $n \geq 8$, then G is ℓ-path-Hamiltonian only if $1 \leq \ell \leq n-4$.*

To illustrate these concepts, we determine the Hamiltonian cycle extension numbers of two well-known classes of cubic Hamiltonian graphs, namely Möbius ladders and prisms. A Möbius ladder M_n of even order $n \geq 4$ is obtained by joining diametrically opposite vertices of the cycle C_n. Hence, $M_4 = K_4$ and $M_6 = K_{3,3}$.

Proposition 1.4 *For the Möbius ladder M_n of even order $n \geq 8$,*

$$\mathrm{hce}(M_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 4 \\ 4 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Furthermore, M_n is ℓ-path-Hamiltonian if and only if $1 \leq \ell \leq \mathrm{hce}(M_n)$.

Proof. Let $n = 2k$ for some integer $k \geq 4$ and construct the graph $G = M_n$ from the n-cycle $C = (v_1, v_2, \ldots, v_n, v_1)$ by adding the edge $v_i v_{i+k}$ for $1 \leq i \leq k$. It is straightforward to verify that every ℓ-path in M_n is on a Hamiltonian cycle in G for $1 \leq \ell \leq 3$. Also, if $n \equiv 2 \pmod 4$, then every 4-path is on a Hamiltonian cycle in G. Thus, $\mathrm{hce}(M_n) \geq 3$ if $n \equiv 0 \pmod 4$ and $\mathrm{hce}(M_n) \geq 4$ if $n \equiv 2 \pmod 4$.

Consider the path P given by

$$P = \begin{cases} (v_1, v_{k+1}, v_{k+2}, v_2) & \text{if } n \equiv 0 \pmod 4 \\ (v_1, v_2, v_3, v_{k+3}, v_{k+4}) & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Then no Hamiltonian cycle in G contains P as a subpath. Since P is on a Hamiltonian path in G, the result now follows. ■

Similarly, it can be shown that for prisms $C_{n/2} \square K_2$ of even order $n \geq 6$,

$$\mathrm{hce}(C_{n/2} \square K_2) = \begin{cases} 3 & \text{if } n \equiv 2 \pmod 4 \\ 4 & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Furthermore, $C_{n/2} \square K_2$ is ℓ-path-Hamiltonian if and only if $1 \leq \ell \leq \text{hce}(C_{n/2} \square K_2)$. Therefore, we have the following result on these two classes of cubic Hamiltonian graphs.

Proposition 1.5 *If G is either a prism $C_{n/2} \square P_2$ or a Möbius ladder M_n of even order $n \geq 6$, then*

$$\text{hce}(G) = \begin{cases} 6 & \text{if } G = M_6 = K_{3,3} \\ 4 & \text{if } n \geq 8 \text{ and } G \text{ is bipartite} \\ 3 & \text{otherwise.} \end{cases}$$

Furthermore, G is ℓ-path-Hamiltonian if and only if $1 \leq \ell \leq \text{hce}(G)$.

2 Cubic Hamiltonian Graphs of Small Order

In this section, we investigate the Hamiltonian cycle extension numbers of several cubic Hamiltonian graphs whose order is 20 or less. All five connected cubic graphs of order 8 are shown in Figure 3, each of which is Hamiltonian. Every edge belongs to a Hamiltonian cycle. Thus, each

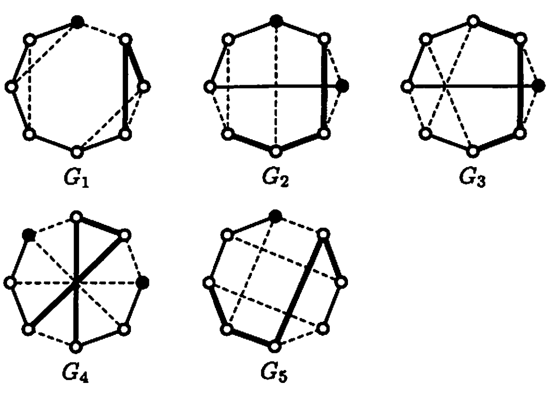

Figure 3: The five connected cubic graphs of order 8

of these graphs is ℓ-path-Hamiltonian for $\ell = 1, 2$. The Hamiltonian path whose edges are represented by solid line segments in each G_i shows that G_i is ℓ-path-Hamiltonian only if $\ell \leq 2$ for $i = 1$, $\ell \leq 3$ for $i \in \{2, 3, 4\}$ and $\ell \leq 4$ for $i = 5$ since the path in bold and the shaded vertices cannot belong to a common cycle. In fact, $\text{hce}(G_1) = 2$, $\text{hce}(G_2) = \text{hce}(G_3) = \text{hce}(G_4) = 3$ and $\text{hce}(G_5) = 4$.

There are exactly nineteen connected cubic graphs of order 10, seventeen of which are Hamiltonian. By examining each of them, the twenty two cubic Hamiltoian graphs of order 8, 10 can be classified as follows.

Observation 2.1 *Let G be a cubic Hamiltonian graph of order $n \in \{8, 10\}$. Then G is ℓ-path-Hamiltonian if and only if $1 \leq \ell \leq \text{hce}(G)$, where*

- $\text{hce}(G) = 1$ *if and only if* $\kappa(G) = 2$ *and G contains an edge xy such that $\{x, y\}$ is a cut-set of G;*
- $\text{hce}(G) = 2$ *if and only if* $\kappa(G) = 2$ *and $xy \notin E(G)$ whenever $\{x, y\}$ is a cut-set of G;*
- $\text{hce}(G) = 3$ *if and only if* $\kappa(G) = 3$ *and G is not bipartite;*
- $\text{hce}(G) = 4$ *if and only if* $\kappa(G) = 3$ *and G is bipartite.*

Regarding the connectivity of a graph, let us state a useful observation.

Observation 2.2 *If G is a Hamiltonian graph that is not 3-connected, then either G itself is a cycle or G is not ℓ-path-Hamiltonian for $\ell \geq 3$.*

For this reason, we only consider cubic Hamiltonian graphs that are 3-connected. Note also that there exists a cubic Hamiltonian graph that is 3-connected but not ℓ-path-Hamiltonian for $\ell \geq 3$. For example, the graph shown in Figure 4 is 3-connected and 2-path-Hamiltonian while no Hamiltonian cycle contains (v_1, v_{12}, v_8) as a subpath, which implies that Observation 2.1 does not hold for $n = 12$.

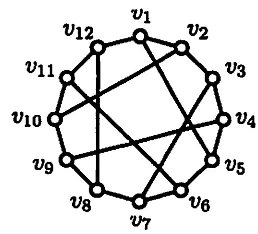

Figure 4: A 3-connected cubic Hamiltonian graph of order 12

We saw that the graph H of the dodecahedron of order 20 is ℓ-path-Hamiltonian if and only if $1 \leq \ell \leq 5$. Thus, $\text{hce}(H) = 5$. For quite some time, it was not known whether there are other cubic Hamiltonian graphs G of order 20 with $\text{hce}(G) = 5$. We now present an affirmative answer to this question.

Theorem 2.3 *The Hamiltonian extension number of the cubic Hamiltonian graph G of order 20 shown in Figure 5 equals 5.*

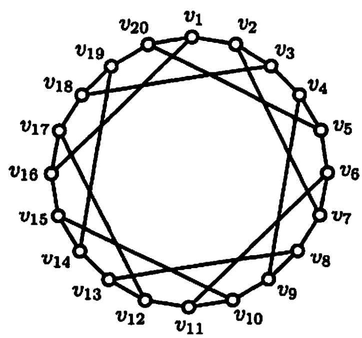

Figure 5: A cubic Hamiltonian graph of order 20

Proof. Since the 6-path $(v_1, v_2, v_3, v_4, v_5, v_{20})$ cannot be extended to any Hamiltonian cycle in G, it suffices to show that every path of order at most 5 lies on a Hamiltonian cycle in G. Every cubic Hamiltonian graph is 3-edge-colorable. Consider the 3-edge-coloring of G shown in Figure 6. There are

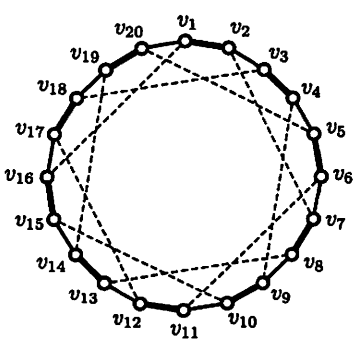

Figure 6: A 3-edge-coloring of G

exactly twelve types of 5-paths, according to how the four edges of each path is colored. Furthermore, each of these twelve 5-paths lies on one of the four Hamiltonian cycles in G shown in Figure 7. Thus, G is 5-path-Hamiltonian.

If P is an ℓ-path, where $1 \leq \ell \leq 4$, then P can be extended to a 5-path as the girth of G is 6, which then can be extended to a Hamiltonian cycle as we already verified. Consequently, G is ℓ-path-Hamiltonian for $1 \leq \ell \leq 5$ but not 6-path-Hamiltonian and so $\text{hce}(G) = 5$. ∎

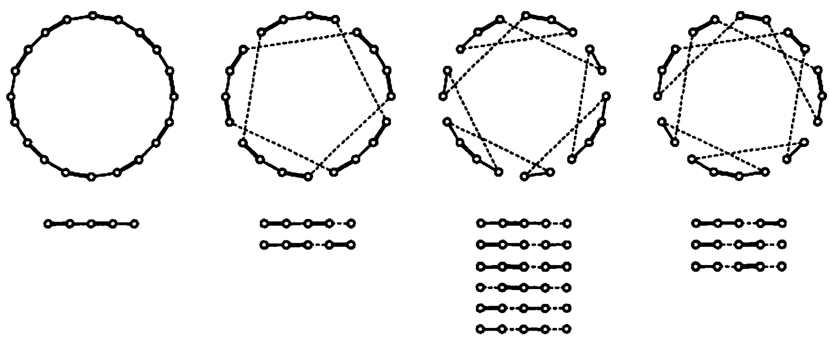

Figure 7: Four Hamiltonian cycles and twelve types of 5-paths in G

There are also cubic Hamiltonian graphs of order 20 that are 6-path-Hamiltonian. For example, the Desargues graph (the generalized Petersen graph $G(10,3)$, shown in Figure 8) is ℓ-path-Hamiltonian for $1 \leq \ell \leq 6$ (but not 7-path-Hamiltonian).

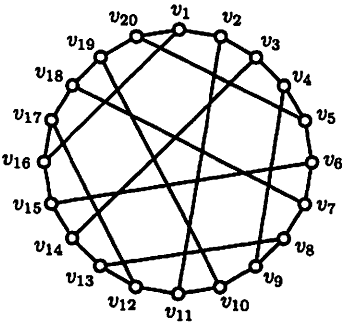

Figure 8: The Desargues graph

3 Two Classes of Cubic Hamiltonian Graphs

In this section, we determine the Hamiltonian cycle extension numbers of two classes of cubic Hamiltonian graphs. We begin with some additional definitions and notation. Let r, g be integers with $r \geq 2$, $g \geq 3$. An (r, g)-*graph* is an r-regular graph having girth g. The order of an (r, g)-graph is bounded below by the *Moore bound* $M(r,g)$ given by

$$M(r,g) = \begin{cases} 1 + \sum_{i=0}^{(g-3)/2} r(r-1)^i & \text{if } g \text{ is odd} \\ 2 \sum_{i=0}^{(g-2)/2} (r-1)^i & \text{if } g \text{ is even.} \end{cases}$$

If G is a $(3,g)$-graph of order 20 (Hamiltonian or not), then $3 \le g \le 6$. We will show later (in Theorem 4.9) that a cubic Hamiltonian graph whose girth g is at most 7 is ℓ-path-Hamiltonian only if $1 \le \ell \le g$. As a result, a cubic Hamiltonian graph of order at most 28 ($= M(3,8) - 2$) and girth g is ℓ-path-Hamiltonian only if $1 \le \ell \le g$. Therefore, every cubic Hamiltonian graph of order 20 is ℓ-path-Hamiltonian only if $1 \le \ell \le 6$.

3.1 The Heawood Graph and Related (3,6)-Graphs

The graph G of Figure 5 is obtained from a 20-cycle $(v_1, v_2, \ldots, v_{20}, v_{21} = v_1)$ by adding ten edges $v_{2i}v_{2i+5}$ $(1 \le i \le 10)$, where the subscripts are expressed modulo 20. More generally, for each even integer $n \ge 8$, we are able to obtain a cubic bipartite Hamiltonian graph $H_{(n)}$ from an n-cycle $(v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ by adding $n/2$ edges $v_{2i}v_{2i+5}$ $(1 \le i \le n/2)$. Thus, $G = H_{(20)}$ for the graph G of Figure 5. Furthermore, $H_{(8)}$ is the cube (the graph G_5 in Figure 3) and $H_{(10)}$ is the Möbius ladder M_{10}. The girth of $H_{(n)}$ equals 4 for $n \in \{8, 10, 12\}$ and is 6 for $n \ge 14$. Also, $H_{(14)}$ is the Heawood graph, which is the unique $(3,6)$-graph of order $M(3,6) = 14$. The graph $H_{(16)}$ is known as the Möbius-Kantor graph (the generalized Petersen graph $G(8,3)$).

While we have seen that $H_{(20)}$ is ℓ-path-Hamiltonian if and only if $1 \le \ell \le 5$, one can verify that the Heawood graph $H_{(14)}$ is ℓ-path-Hamiltonian if and only if $1 \le \ell \le 6$. In fact, the following holds, which can be verified by considering 3-edge-colorings as done in the proof of Theorem 2.3.

Proposition 3.1 *For each even integer $n \ge 8$, the graph $H_{(n)}$ is ℓ-path-Hamiltonian if and only if $1 \le \ell \le \mathrm{hce}(H_{(n)})$, where*

$$\mathrm{hce}(H_{(n)}) = \begin{cases} 4 & \textit{if } n = 10, 12 \textit{ or } n \equiv 0 \pmod 8 \\ 5 & \textit{if } n = 18 \textit{ or } n \equiv 4 \pmod 8 \textit{ except } n = 12 \\ 6 & \textit{if } n \equiv 2 \pmod 4 \textit{ except } n = 10, 18. \end{cases}$$

3.2 Another Class of Cubic Hamiltonian Graphs

There are exactly two $(3,5)$-graphs of order $M(3,5) = 12$. One is the graph in Figure 4, which is 2-path-Hamiltonian but not 3-path-Hamiltonian. The other is shown in Figure 9. The second graph can be obtained from a 9-cycle $C_0 = (u_1, u_2, \ldots, u_9, u_1)$ by adding three new vertices w_1, w_2, w_3 and joining w_i to the three vertices u_i, u_{i+3}, u_{i+6} for $1 \leq i \leq 3$. Here,

$$C_0 = (u_1 = v_2, u_2 = v_3, v_{11}, v_{12}, v_5, v_6, v_7, v_8, v_9 = u_9, v_2)$$

and $w_1 = v_1, w_2 = v_4, w_3 = v_{10}$.

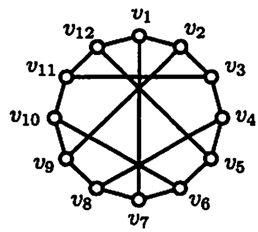

Figure 9: One of the two cubic Hamiltonian graphs of order 12 and girth 5

This suggests another way of constructing cubic Hamiltonian graphs. For each positive integer k, let $G_{(k)}$ be the graph of order $4k$ obtained from a $3k$-cycle $C_0 = (u_1, u_2, \ldots, u_{3k}, u_1)$ by adding k new vertices in the set $W = \{w_1, w_2, \ldots, w_k\}$ and joining w_i to the three vertices u_i, u_{i+k}, u_{i+2k} for $1 \leq i \leq k$. Thus, the graph in Figure 9 is $G_{(3)}$, while $G_{(1)} = K_4$ and $G_{(2)}$ is the cube (the graph G_5 in Figure 3).

Let us first verify that the graph $G_{(k)}$ is indeed Hamiltonian.

Proposition 3.2 *For each positive integer k, the graph $G = G_{(k)}$ is Hamiltonian and its girth $g(G)$ is given by*

$$g(G) = \begin{cases} k+2 & \text{if } 1 \leq k \leq 3 \\ 6 & \text{if } k \geq 4. \end{cases}$$

Proof. Since the result clearly holds for $G_{(1)} = K_4$, we may assume that $k \geq 2$. The girth is straightforward to verify. Construct the graph $G = G_{(k)}$ as described before with $V(G) = U \cup W$, where $U = \{u_1, u_2, \ldots, u_{3k}\}$ and $W = \{w_1, w_2, \ldots, w_k\}$. Let $C_0 = (u_1, u_2, \ldots, u_{3k}, u_1)$ be the $3k$-cycle induced by U. For each integer $\alpha \in \{1, 2, \ldots, k\}$, define the following sequences

s_α : $w_\alpha, u_\alpha, u_{\alpha+1}$

s'_α : $w_\alpha, u_{k+\alpha}, u_{k+\alpha+1}$

s''_α : $w_\alpha, u_{2k+\alpha}, u_{2k+\alpha+1}$,

t'_α : $w_\alpha, u_{k+\alpha}, u_{k+\alpha-1}, u_{k+\alpha-2}, u_{k+\alpha-3}$

t''_α : $w_\alpha, u_{2k+\alpha}, u_{2k+\alpha-1}, u_{2k+\alpha-2}, u_{2k+\alpha-3}$

where the subscripts are expressed modulo $3k$. Now the cycle C given by

- $(s_1, t''_2, s_{k-1}, t''_k, s_{k-3}, t'_{k-2}, s_{k-5}, t''_{k-4}, s_{k-7}, t'_{k-6}, \ldots, s_5, t''_6, s_3, t'_4, w_1)$ if $k \equiv 2 \pmod 4$
- $(s_1, t'_2, s'_{k-1}, t''_k, s'_{k-3}, t'_{k-2}, s'_{k-5}, t''_{k-4}, s'_{k-7}, t'_{k-6}, \ldots, s'_7, t''_8, s'_5, t'_6, s'_3, t''_4, w_1)$ if $k \equiv 0 \pmod 4$
- $(w_1, u_1, u_2, \ldots, u_{k+2}, s''_2, s'_3, s''_4, s'_5, \ldots, s''_{k-1}, s'_k, w_1)$ if k is odd

is a Hamiltonian cycle in G. ∎

Proposition 3.3 *The graph $G_{(k)}$ is ℓ-path-Hamiltonian for $1 \leq \ell \leq 4$.*

Proof. Since the result holds for $G_{(1)} = K_4$, we can suppose once again that $k \geq 2$ and let $G = G_{(k)}$. If $P = (x_1, x_2, x_3, x_4)$ is a 4-path in G, then $0 \leq |V(P) \cap W| \leq 2$. In fact, there are five possibilities, namely (i) $V(P) \cap W = \emptyset$, (ii) $V(P) \cap W = \{x_1, x_4\}$, (iii) $V(P) \cap W = \{x_1\}$, (iv) $V(P) \cap W = \{x_2\}$ and $d_{C_0}(x_1, x_4) = k - 1$ or (v) $V(P) \cap W = \{x_2\}$ and $d_{C_0}(x_1, x_4) = k + 1$. Now suppose that Q is a path of order at most 4 in G. If $k \equiv 2 \pmod 4$, then there exists a path Q' of order 4 and an isomorphism ϕ of G such that $Q \subseteq Q'$ and $\phi(Q')$ is one of the five 4-paths

$$(u_{k+1}, u_{k+2}, u_{k+3}, u_{k+4}),\ (w_1, u_1, u_2, w_2),\ (w_1, u_{k+1}, u_{k+2}, u_{k+3}),$$
$$(u_1, u_2, w_2, u_{2k+2}),\ (u_2, w_2, u_{2k+2}, u_{2k+1}).$$

Since the Hamiltonian cycle C we have obtained in the proof of Proposition 3.2 contains these five 4-paths, it follows that Q can be extended to a Hamiltonian cycle. One can verify that the same holds for other values of k in a similar manner. ∎

For an integer $k \geq 2$ and $G = G_{(k)}$, note that $d(w, w') \geq 3$ for every two distinct vertices $w, w' \in W$. Thus, if $P = (x_1, x_2, x_3, x_4, x_5)$ is a 5-path in G whose central vertex x_3 belongs to W, then $|V(P) \cap W| = 1$. Furthermore, $C_0 - \{x_1x_2, x_4x_5\} = P_{k+i} + P_{2k-i}$, where $i \in \{-1, 0, 1\}$. Let us say that a 5-path P is of *type* 1 (or a *type*-1 *path*) if $i = \pm 1$ and of *type* 2 (or a *type*-2 *path*) otherwise. In particular, for $k \geq 3$, a 5-path P is of type 2 if and only if $d_{C_0}(x_1, x_5) = k$.

For the Hamiltonian cycle C constructed in the proof of Observation 3.2 and for each $w \in W$, let $P(w, C)$ be the 5-subpath of C whose central vertex is w. Then every path $P(w, C)$ is of type 1 if k is odd while no path $P(w, C)$ is of type 1 if k is even. It turns out that this is the case for every Hamiltonian cycle in $G_{(k)}$. We first verify the following.

Proposition 3.4 *Let C be a Hamiltonian cycle in $G_{(k)}$, where $k \geq 2$. If the path $P(w_1, C)$ is of type 1, then so is $P(w_2, C)$.*

Proof. Suppose that $P(w_1, C)$ is of type 1. Without loss of generality, we may assume that $P(w_1, C) = (x, u_1, w_1, u_{k+1}, y)$ and either (i) $(x, y) = (u_{3k}, u_{k+2})$ or (ii) $k \geq 3$ and $(x, y) = (u_2, u_k)$. Since $w_1 u_{2k+1} \notin E(C)$, note that $u_{2k+1} u_{2k+2} \in E(C)$. Also, if (i) occurs, then $u_1 u_2 \notin E(C)$ and so $w_2 u_2, u_2 u_3 \in E(C)$. It then follows that the two end-vertices of $P(w_2, C)$ are u_3 and either u_{k+1} or u_{2k+1}. Similarly, if (ii) occurs, then it can be shown that the end-vertices of $P(w_2, C)$ are u_{k+3} and either u_1 or u_{2k+1}. As a result, $P(w_2, C)$ is of type 1. ∎

Corollary 3.5 *Let C be a Hamiltonian cycle in $G_{(k)}$, where $k \geq 2$. Then the k subpaths $P(w_i, C)$ $(i = 1, 2, \ldots, k)$ are of the same type.*

For this reason, let us say that a Hamiltonian cycle C in $G_{(k)}$ $(k \geq 2)$ is of type 1 if it contains a type-1 path $P(w, C)$ and C is of type 2 otherwise.

If C is a Hamiltonian cycle in $G_{(k)}$ $(k \geq 2)$, then the graph $C - W$ is a union of k nontrivial paths. With this in mind, let us examine Hamiltonian cycles in $G_{(k)}$.

Proposition 3.6 *Let $k \geq 2$ be an integer. A Hamiltonian cycle in $G_{(k)}$ is of type 1 if and only if k is odd.*

Proof. Let $G = G_{(k)}$. First, we show that k must be even if G contains a type-2 Hamiltonian cycle C. To do so, we show that the order of each path in $C - W$ is either 2 or 4. Since this follows readily when $k = 2$, assume that $k \geq 3$.

If $C - W$ contains a path of order greater than 4, say $(u_1, u_2, \ldots, u_5)$ is a path in $C - W$, then the six edges $w_i u_{k+i}, w_i u_{2k+i}$ $(i = 2, 3, 4)$ are in $E(C)$. Since C is of type 2, the path $P(w_3, C)$ is of type 2, say the edges $u_{k+2} u_{k+3}$ and $u_{2k+2} u_{2k+3}$ are on C. However then, $(w_2, u_{k+2}, u_{k+3}, w_3, u_{2k+3}, u_{2k+2}, w_2)$ is a 6-cycle in C, which is impossible.

If $C - W$ contains P_3, say (u_1, u_2, u_3), as a component, then assume, without loss of generality, that $P(w_2, C) = (u_{k+1}, u_{k+2}, w_2, u_{2k+2}, u_{2k+1})$. However then, $P(w_3, C)$ is a $u_2 - u_{k+4}$ path, that is, $P(w_3, C)$ is of type 1. This is impossible.

As a consequence, if C is of type 2, then the order of $C - W$, which equals $3k$, must be even. Thus, if C is of type 2, then k is even.

Next we verify the converse: a Hamiltonian cycle in G is of type 2 if k is even. Assume, to the contrary, that the statement is false. Let k be the smallest positive even integer for which G has a type-1 Hamiltonian cycle. It is straightforward to verify that $k \geq 4$ by examining Hamiltonian cycles in $G_{(2)}$, the cube of order 8.

As before, let us construct G with a $3k$-cycle $(u_1, u_2, \ldots, u_{3k}, u_{3k+1} = u_1)$ and the k vertices in $W = \{w_1, w_2, \ldots, w_k\}$. Let $C = (v_1, v_2, \ldots, v_{4k}, v_{4k+1} = v_1)$ be a type-1 Hamiltonian cycle in G. Then we may assume that

(i) $W = \{w_{i_j} = v_{i'_j} : 1 \le i'_1 < i'_2 < \cdots < i'_k \le 4k\}$ and (ii) if $v_i, v_{i+1} \in U$, then $v_i = u_j$ and $v_{i+1} = u_{j+1}$ for some $j \in \{1, 2, \ldots, 3k\}$. For each $j \in \{1, 2, \ldots, k\}$, let Q_j be the $w_{i_j} - w_{i_{j+1}}$ subpath of C such that $V(Q_j) \cap W = \{w_{i_j}, w_{i_{j+1}}\}$. The order n_j of Q_j satisfies $n_j \equiv i_{j+1} - i_j + 3 \pmod{k}$. Since there must exist an integer j such that i_j and i_{j+1} are of different parity, at least one of the k components of $C - W$ of order $3k$ is of even order. Consequently, at least one component of $C - W$ is P_2. Without loss of generality, suppose that (u_{k-1}, u_k) is a component in $C - W$. Thus, $C = (w_{k-1}, u_{k-1}, u_k, w_k, u_{\alpha_k}, \ldots, u_{\beta_{k-1}}, w_{k-1})$, where $\alpha_k = \beta_{k-1} + 1 \in \{2k, 3k\}$. By the symmetry, we may assume that $\alpha_k = \beta_{k-1} = 2k$. Therefore, C contains each of the paths $Q' = (u_{2k-2}, u_{2k-1}, w_{k-1}, u_{k-1}, u_k, w_k, u_{2k}, u_{2k+1})$ and $Q'' = (u_{3k-2}, u_{3k-1}, u_{3k}, u_1)$ (or its reversal) as a subpath.

We now construct a new graph G' from G by deleting the eight vertices $u_{k-1}, u_k, u_{2k-1}, u_{2k}, u_{3k-1}, u_{3k}, w_{k-1}, w_k$ and adding three edges $u_{k-2}u_{k+1}$, $u_{2k-2}u_{2k+1}, u_{3k-3}u_1$. Then $G' \cong G_{(k-2)}$. Furthermore, replacing Q' and Q'' in C by (u_{2k-2}, u_{2k+1}) and (u_{3k-2}, u_1), respectively, we obtain a type-1 Hamiltonian cycle in G'. This is a contradiction. ■

For $k \ge 2$, the graph $G_{(k)}$ is not ℓ-path-Hamiltonian for $\ell = 5, 6$ as both type-1 5-paths and type-2 5-paths can be easily extended to 6-paths. As mentioned in the previous subsection, a cubic Hamiltonian graph whose girth is at most 7 is ℓ-path-Hamiltonian only if ℓ does not exceed the girth of the graph.

Corollary 3.7 *The graph $G_{(k)}$ of order $4k$, where k is a positive integer, is ℓ-path-Hamiltonian if and only if $1 \le \ell \le 4$.*

4 Cubic Hamiltonian Graphs Having Girth at Most 7

In this section, we show that if G is a cubic Hamiltonian graph with girth g, where $3 \le g \le 7$, then G is ℓ-path-Hamiltonian only if $1 \le \ell \le g$. So we begin with a cubic Hamiltonian graph G whose girth equals g. Let $C = (v_1, v_2, \ldots, v_g, v_1)$ be a g-cycle in G. If $N(v_1) = \{v_2, v_g, u_1\}$ and $N(v_{g-1}) = \{v_{g-2}, v_g, u_{g-1}\}$, then $u_1, u_{g-1} \notin V(C)$. Also, if $g \ge 5$, then $u_1 \ne u_{g-1}$ and so G contains a $(g+1)$-path $P = (u_1, v_1, v_2, \ldots, v_{g-1}, u_{g-1})$. No cycle in G contains both P and v_g simultaneously. The following is an immediate consequence.

Observation 4.1 *If G is a cubic Hamiltonian graph with girth $g \ge 5$, then G is not $(g+1)$-path-Hamiltonian. Hence, $1 \le \mathrm{hce}(G) \le g$.*

Let C be a Hamiltonian cycle in a cubic Hamiltonian graph G of order $n \ge 8$ and girth g. An edge in G that is not on C is called a *chord* of C. If G is a cubic Hamiltonian graph of order $n \geq 4$, then there exists a partition $\{E_1, E_2, E_3\}$ of the edge set of G such that each E_i contains exactly $n/2$ independent edges and the subgraph induced by $E_2 \cup E_3$ is an n-cycle, say C^*. Thus, E_1 is the set of chords of G with respect to C^*.

Let $S_i(G)$ be the set of i-cycles contained in G. If g is the girth of G, then G contains g-cycles. Hence, $S_g(G), S_n(G) \neq \emptyset$. Note that, if $D \in S_i(G)$, then $1 \leq |E(D) \cap E_j| \leq i/2$ for $j \in \{1,2,3\}$. In particular, $1 \leq |E(D) \cap E_1| \leq g/2$ if $D \in S_g(G)$.

Let us write $C^* = (v_1, v_2, \ldots, v_n, v_1)$ and, for each vertex v_i, let $N(v_i) = \{v_{i-1}, v_{i+1}, v_{\alpha_i}\}$, where each subscript is expressed as an integer between 1 and n modulo n. In other words, $E_1 = \{v_i v_{\alpha_i} : 1 \leq i \leq n\}$. Note that $v_{\alpha_i} = v_j$ if and only if $v_{\alpha_j} = v_i$. Also, $2 \leq g - 1 \leq d_{C^*}(v_i, v_{\alpha_i}) = \min\{|\alpha_i - i|, n - |\alpha_i - i|\} \leq n/2$ and

$$|E(D) \cap E_1| = 1 \text{ for some } D \in S_g(G) \text{ if and only if } d_{C^*}(v_i, v_{\alpha_i}) = g - 1 \text{ for some } i. \tag{1}$$

As we mentioned, our goal is to show that G is ℓ-path-Hamiltonian only if $1 \leq \ell \leq g$ for $3 \leq g \leq 7$. By Corollary 1.3(c), it suffices to show that, for each integer ℓ with $g + 1 \leq \ell \leq n - 4$, there exists a path Q of order ℓ and a vertex x not on Q such that no cycle contains both Q and x.

4.1 Cubic Hamiltonian Graphs Having Girth 3, 4, 5

We begin with cubic Hamiltonian graphs having girth 3 or 4.

Proposition 4.2 *If G is a cubic Hamiltonian graph of order $n \geq 8$ and girth 3, then G is ℓ-path-Hamiltonian only if $1 \leq \ell \leq 3$.*

Proof. Let $C^* = (v_1, v_2, \ldots, v_n, v_1)$ be a Hamiltonian cycle in G. Every triangle in G contains exactly one edge in $E_1 = E(G) \backslash E(C^*)$. By (1), suppose that $d_{C^*}(v_1, v_{\alpha_1}) = \alpha_1 - 1 = 2$ and so $(v_1, v_2, v_3, v_1) \in S_3(G)$. Let Q_1 be the $v_n - v_{n-1}$ path in Figure 10, where an edge on C^* is represented by a single line segment while each edge in E_1 is represented by a double line segment. Also, let Q_0 be the $v_n - v_4$ subpath of Q_1. In Figure 10, the four

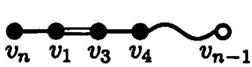

Figure 10: A $v_n - v_{n-1}$ path of order $n - 1$

vertices in Q_0 are shaded. Then for every path Q satisfying $Q_0 \subseteq Q \subseteq Q_1$, there is no cycle in G containing both Q and v_2. Therefore, G is not ℓ-path-Hamiltonian for $4 \leq \ell \leq n - 1$. As a result, G is ℓ-path-Hamiltonian only if $1 \leq \ell \leq 3$. ■

Proposition 4.3 *If G is a cubic Hamiltonian graph of order $n \geq 8$ and girth 4, then G is ℓ-path-Hamiltonian only if $1 \leq \ell \leq 4$.*

Proof. Let $C^* = (v_1, v_2, \ldots, v_n, v_1)$ be a Hamiltonian cycle in G. Note that every 4-cycle in G contains at least one and at most two edges in $E_1 = E(G) \backslash E(C^*)$. We consider the following two cases.

Case 1. $|E(D) \cap E_1| = 1$ *for some* $D \in S_4(G)$. Then we may assume that $D = (v_1, v_2, v_3, v_4, v_1)$, that is, $\alpha_1 = 4$. Note that $6 \leq \alpha_3 \leq n$. If $\alpha_3 = n$, then let Q_1 be the $v_{n-1} - v_{n-2}$ path of order $n-2$ in Figure 11 with the $v_{n-1} - v_5$ subpath Q_0. Similarly, if $\alpha_3 \leq n-1$, then let Q_1 be the $v_{\alpha_3+1} - v_5$ path shown in Figure 12 with the $v_n - v_{\alpha_3}$ subpath Q_0. Again, the vertices belonging to Q_0 are shaded in Figures 11 and 12. In

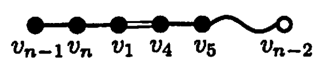

Figure 11: A $v_{n-1} - v_{n-2}$ path of order $n-2$

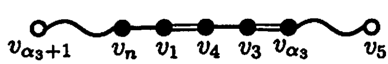

Figure 12: A $v_{\alpha_3+1} - v_5$ path of order $n-1$

either case, consider a path Q satisfying $Q_0 \subseteq Q \subseteq Q_1$. Then no cycle in G contains both Q and v_2, which implies that G is not ℓ-path-Hamiltonian for $5 \leq \ell \leq n-2$.

Case 2. $|E(D) \cap E_1| = 2$ *for every* $D \in S_4(G)$. Then $d_{C^*}(v_i, v_{\alpha_i}) \geq 4$ for $1 \leq i \leq n$. Hence, $i + 4 \leq \alpha_i \leq n + i - 4$ for $i = 1, 2, 3$. We may assume, without loss of generality, that there exists a 4-cycle of the form $(v_1, v_{\alpha_1}, v_{\alpha_2}, v_2, v_1)$, where $v_{\alpha_1} v_{\alpha_2} \in E(C^*)$. Hence, $|\alpha_1 - \alpha_2| = 1$.

Subcase 2.1. $\alpha_2 = \alpha_1 - 1$. Note that $7 \leq \alpha_1 \leq n-3$. In this case, the $v_{\alpha_1+1} - v_3$ path Q_1 in Figure 13 and its $v_n - v_{\alpha_1-2}$ subpath Q_0 show that there is an ℓ-path that cannot belong to a cycle containing v_2 for $5 \leq \ell \leq n-1$.

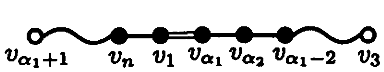

Figure 13: A $v_{\alpha_1+1} - v_3$ path of order $n-1$

Subcase 2.2. $\alpha_2 = \alpha_1 + 1$. Note that $5 \leq \alpha_1 \leq n-3$ and $7 \leq \alpha_3 \leq n-1$. First, if $\alpha_1 + 2 \leq \alpha_3 \leq n-1$, then G is not ℓ-path-Hamiltonian for $5 \leq \ell \leq n-1$ by considering the $v_{\alpha_3+1} - v_{\alpha_1-1}$ path Q_1 of order $n-1$ and its $v_n - v_{\alpha_1+2}$ subpath Q_0 in Figure 14. If $7 \leq \alpha_3 \leq \alpha_1 - 1$, then

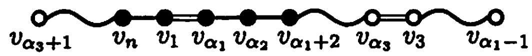

Figure 14: A $v_{\alpha_3+1} - v_{\alpha_1-1}$ path of order $n-1$

let Q_1 and Q_1' be the $v_4 - v_3$ path and $v_{\alpha_2} - v_{\alpha_3-1}$ path in Figure 15, respectively. Also, let Q_0 and Q_0' be the $v_{\alpha_1-1} - v_3$ subpath and $v_n - v_4$ subpath of Q_1 and Q_1', respectively. Then the paths Q_0 and Q_1 show that

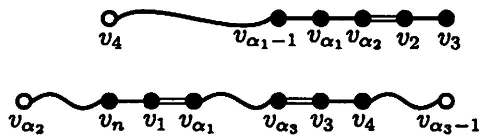

Figure 15: A $v_4 - v_3$ path of order α_1 and a $v_{\alpha_2} - v_{\alpha_3-1}$ path of order $n-1$

G is not ℓ-path-Hamiltonian for $5 \leq \ell \leq \alpha_1$. Similarly, the paths Q_0' and Q_1' show that G is not ℓ-path-Hamiltonian for $\alpha_1 - \alpha_3 + 5 \leq \ell \leq n-1$. This completes the proof. ∎

Case 1 of the proof of Proposition 4.3 suggests the following.

Proposition 4.4 *Let $C^* = (v_1, v_2, \ldots, v_n, v_1)$ be a Hamiltonian cycle in a cubic graph G of order n and girth $g \geq 5$. Then G is ℓ-path-Hamiltonian only if* $1 \leq \ell \leq \min\{d_{C^*}(v_i, v_{\alpha_i}) : 1 \leq i \leq n\} + 1$.

Proof. Without loss of generality, suppose that

$$d_{C^*}(v_1, v_{\alpha_1}) = \alpha_1 - 1 = \min\{d_{C^*}(v_i, v_{\alpha_i}) : 1 \leq i \leq n\}.$$

Hence, $5 \leq g \leq \alpha_1 \leq n/2$ and $\alpha_1 + 2 \leq \alpha_3 \leq n + 4 - \alpha_1$. Then the existence of a path Q in G satisfying $Q_0 \subseteq Q \subseteq Q_1$, where Q_1 is the $v_{\alpha_3+1} - v_{\alpha_1+1}$ path of order $n-1$ shown in Figure 16 with the $v_n - v_{\alpha_3}$ subpath Q_0, shows that G is not ℓ-path-Hamiltonian for $\alpha_1 + 1 \leq \ell \leq n$. ∎

Figure 16: A $v_{\alpha_3+1} - v_{\alpha_1+1}$ path in the graph G

The following is a consequence of Observation 4.4 and (1).

Corollary 4.5 *Let G be a cubic Hamiltonian graph of order n and girth $g \geq 5$. If G contains a g-cycle D such that $|E(D) \cap E_1| = 1$, where E_1 is the set of $n/2$ chords of a Hamiltonian cycle in G, then G is ℓ-path-Hamiltonian only if $1 \leq \ell \leq g$.*

Let us next consider cubic Hamiltonian graphs having girth 5.

Proposition 4.6 *If G is a cubic Hamiltonian graph whose girth equals 5, then G is ℓ-path-Hamiltonian only if $1 \leq \ell \leq 5$.*

Proof. The order n of G is at least $M(3,5) = 10$. (In fact, the Petersen graph, which is not Hamiltonian, is the only $(3,5)$-graph of order 10 and so we may assume that $n \geq 12$.) Let $C^* = (v_1, v_2, \ldots, v_n, v_1)$ be a Hamiltonian cycle in G and $E_1 = E(G) \backslash E(C^*)$. By Corollary 4.5, we may assume that $|E(D) \cap E_1| = 2$ for every $D \in S_5(G)$. (Note that $6 \leq \alpha_1 \leq n-4$, $7 \leq \alpha_2 \leq n-3$ and $8 \leq \alpha_3 \leq n-2$.) Hence, suppose that there exists a 5-cycle of the form $(v_1, v_2, v_3, v_{\alpha_3}, v_{\alpha_1}, v_1)$, where $v_{\alpha_1}v_{\alpha_3} \in E(C^*)$. Thus, $|\alpha_1 - \alpha_3| = 1$. Let $\alpha = \max\{\alpha_1, \alpha_3\}$. (Thus, $8 \leq \alpha \leq n-3$.) By considering the $v_{\alpha+1} - v_{\alpha-2}$ path Q_1 of order $n-1$ and its $v_n - v_4$ subpath Q_0 shown in Figure 17, it follows that G is not ℓ-path-Hamiltonian for $6 \leq \ell \leq n-1$ and the result now follows. ∎

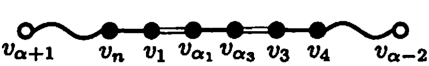

Figure 17: A $v_{\alpha+1} - v_{\alpha-2}$ path of order $n-1$

Let G be a cubic Hamiltonian graph of order n with a Hamiltonian cycle $C^* = (v_1, v_2, \ldots, v_n, v_1)$. If the girth is at least 4, then there exists an integer i $(1 \leq i \leq n)$ such that the vertices v_{i+2} and $v_{\alpha_{i+2}}$ belong to different components in the graph obtained from C^* by deleting v_i and v_{α_i}. Without loss of generality, we may assume that $4 \leq \alpha_1 < \alpha_3 \leq n$. In particular, if the girth g is at least 5, then $\alpha_1 \geq 5$ and $\alpha_3 \leq n-1$. Then G contains the three paths Q_1, Q_2, Q_3 of order $n-1$ shown in Figure 18 and we see that G is not ℓ-path-Hamiltonian for $\min\{\alpha_1 + 1, \alpha_3 - \alpha_1 + 5, n - \alpha_3 + 5\} \leq \ell \leq n$.

Since $(\alpha_1+1)+(\alpha_3-\alpha_1+5)+(n-\alpha_3+5)=n+11$, it follows that G is not ℓ-path-Hamiltonian for $(n+11)/3 \le \ell \le n$.

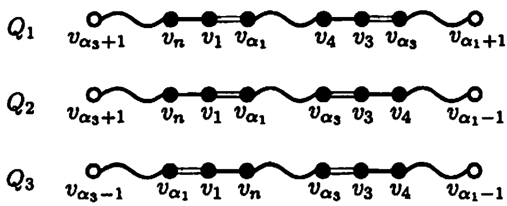

Figure 18: Three paths of order $n-1$

Note that (i) $\alpha_2 < \alpha_1$ or (ii) $\alpha_1 < \alpha_2 < \alpha_3$ or (iii) $\alpha_2 > \alpha_3$. For example, if (i) occurs, then the path Q_1 in Figure 18 contains a $v_n - v_{\gamma_2}$ subpath Q_1' of order $\alpha_1-\alpha_2+4$ and a $v_{\beta_2} - v_{\alpha_3}$ subpath Q_1'' of order α_2 shown in Figure 19. (Here, $\beta_2=\alpha_2+1$ and $\gamma_2=\alpha_2-1$.) Since neither

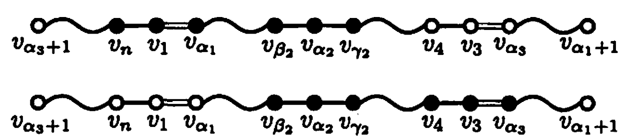

Figure 19: The subpaths Q_1' and Q_1'' of Q_1

Q_1' nor Q_1'' lies on a cycle containing v_2, it follows that G is not ℓ-path-Hamiltonian for $\min\{\alpha_1-\alpha_2+4, \alpha_2, \alpha_3-\alpha_1+5, n-\alpha_3+5\} \le \ell \le n$. In fact, one can verify that G is not ℓ-path-Hamiltonian for $A \le \ell \le n$, where

$$A = \begin{cases} \min\{\alpha_1-\alpha_2+4, \alpha_2, \alpha_3-\alpha_1+5, n-\alpha_3+5\} & \text{if (i) occurs} \\ \min\{\alpha_1+1, \alpha_2-\alpha_1+4, \alpha_3-\alpha_2+4, n-\alpha_3+5\} & \text{if (ii) occurs} \\ \min\{\alpha_1+1, \alpha_3-\alpha_1+5, \alpha_2-\alpha_3+4, n-\alpha_2+4\} & \text{if (iii) occurs.} \end{cases}$$

Combining, we see that G is not ℓ-path-Hamiltonian for $(n+14)/4 \le \ell \le n$.

Corollary 4.7 *A cubic Hamiltonian graph of order n and girth $g \ge 5$ is ℓ-path-Hamiltonian only if $1 \le \ell \le (n+13)/4$.*

4.2 Cubic Hamiltonian Graphs Having Girth 6 or 7

In a cubic Hamiltonian graph G of order n with a Hamiltonian cycle $C^* = (v_1, v_2, \ldots, v_n, v_1)$, recall that $N(v_i) = \{v_{i-1}, v_{i+1}, v_{\alpha_i}\}$ for $1 \le i \le n$, where the subscripts are expressed as one of the n integers $1, 2, \ldots, n$ modulo n. In addition, let us write $N(v_{\alpha_i}) = \{v_i, v_{\beta_i}, v_{\gamma_i}\}$ for each i. Therefore, both $v_{\alpha_i}v_{\beta_i}$ and $v_{\alpha_i}v_{\gamma_i}$ are edges on C^*.

Proposition 4.8 *If G is a cubic Hamiltonian graph having girth 6, then G is ℓ-path-Hamiltonian only if $1 \leq \ell \leq 6$.*

Proof. By Corollary 4.5, we may assume that $|E(D) \cap E_1| \in \{2, 3\}$ for every $D \in S_6(G)$. (Note that $i + 6 \leq \alpha_i \leq n + i - 6$ for $i = 1, 2, 3$.)

Case 1. $|E(D) \cap E_1| = 2$ *for some* $D \in S_6(G)$. We may assume, without loss of generality, that either

(i) $D = (v_1, v_2, v_3, v_{\alpha_3}, v_\beta, v_{\alpha_1}, v_1)$, where $(v_{\alpha_1}, v_\beta, v_{\alpha_3})$ is a subpath of C^* or

(ii) $D = (v_1, v_2, v_{\alpha_2}, v_{\beta_2}, v_{\beta_1}, v_{\alpha_1}, v_1)$, where $(v_{\alpha_2}, v_{\beta_2}, v_{\beta_1}, v_{\alpha_1})$ is a subpath of C^*.

If (i) occurs, then $|\alpha_1 - \alpha_3| = 2$. Let $\alpha = \max\{\alpha_1, \alpha_3\}$. (Thus, $9 \leq \alpha \leq n - 3$. Also, $\beta = \alpha - 1$.) Then the $v_{\alpha+1} - v_{\alpha-3}$ path Q_1 and its $v_n - v_4$ subpath Q_0 shown in Figure 20 show that G is not ℓ-path-Hamiltonian for $7 \leq \ell \leq n - 1$.

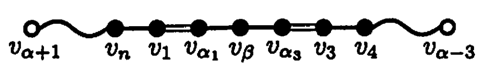

Figure 20: A $v_{\alpha+1} - v_{\alpha-3}$ path of order $n - 1$

Suppose next that (ii) occurs. In this case, $|\alpha_1 - \alpha_2| = 3$. First, if $\alpha_2 = \alpha_1 - 3$, then $\gamma_2 = \alpha_1 - 4$. The $v_{\alpha_1+1} - v_3$ path Q_1 and its $v_n - v_{\gamma_2}$ subpath Q_0 in Figure 21 show that G is not ℓ-path-Hamiltonian for $7 \leq \ell \leq n - 1$.

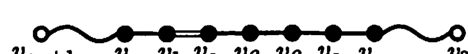

Figure 21: A $v_{\alpha_1+1} - v_3$ path of order $n - 1$

If $\alpha_2 = \alpha_1 + 3$, then $\gamma_2 = \alpha_1 + 4$. Let Q_1 and Q_1' be the $v_n - v_{n-1}$ path and $v_{\beta_1} - v_3$ path in Figure 22, respectively. Also, let Q_0 and Q_0' be the $v_n - v_{\gamma_2}$ subpath and $v_{\beta_2} - v_{\alpha_1}$ subpath of Q_1 and Q_1', respectively. Then the paths Q_0 and Q_1 show that G is not ℓ-path-Hamiltonian for $7 \leq \ell \leq n - \alpha_1 + 2$. Similarly, the paths Q_0' and Q_1' show that G is not ℓ-path-Hamiltonian for $n - \alpha_1 + 1 \leq \ell \leq n - 1$.

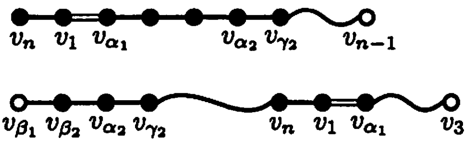

Figure 22: A $v_n - v_{n-1}$ path of order $n - \alpha_1 + 2$ and a $v_{\beta_2} - v_3$ path of order $n - 1$

Case 2. $|E(D) \cap E_1| = 3$ *for every* $D \in S_6(G)$. Without loss of generality, suppose that $D = (v_1, v_2, v_{\alpha_2}, v_{\beta_2}, v_{\beta_1}, v_{\alpha_1}, v_1)$, where $\alpha_{\beta_1} = \beta_2$. (In other words, $v_{\beta_1}v_{\beta_2} \in E_1$.) By symmetry, it suffices to analyze four cases, namely (i) $\alpha_1 > \beta_1 > \beta_2 > \alpha_2$, (ii) $\alpha_1 > \beta_1 > \alpha_2 > \beta_2$, (iii) $\beta_1 > \alpha_1 > \alpha_2 > \beta_2$ and (iv) $\beta_2 > \alpha_2 > \alpha_1 > \beta_1$. Let $\alpha = \max\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$. By considering the paths shown in Figure 23, one can verify that G is not ℓ-path-Hamiltonian for $7 \leq \ell \leq n-1$ in each case. ■

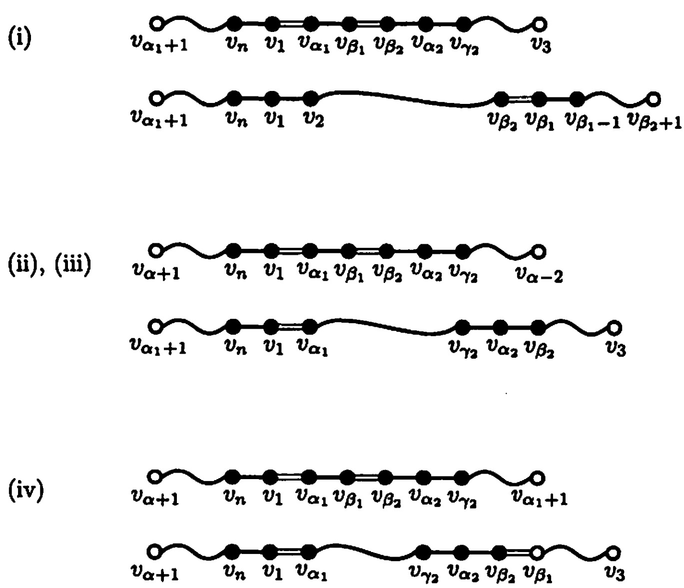

Figure 23: Paths in G

A similar proof verifies that a cubic Hamiltonian graph whose girth equals 7 is ℓ-path-Hamiltonian only if $1 \leq \ell \leq 7$. In summary, we have the following main result of this section.

Theorem 4.9 *If G is a cubic Hamiltonian graph with girth g, where $3 \leq g \leq 7$, then G is ℓ-path-Hamiltonian only if $1 \leq \ell \leq g$.*

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