From Connectivity to Coloring

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November 29, 2015

Abstract

A vertex set $U \subset V$ in a connected graph $G = (V, E)$ is a cutset if $G - U$ is disconnected. If no proper subset of U is also a cutset of G, then U is a minimal cutset. An $\mathcal{MVC}$-partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of the vertex set $V(G)$ of a connected graph G is a partition of $V(G)$

*Research supported in part by the University of Johannesburg.

such that every $V_i \in \pi$ is a minimal cutset of G. For an $\mathcal{MVC}$-partition π of G, the π-graph G_π of G has vertex set π such that $V', V'' \in \pi$ are adjacent in G_π if and only if there exist $v' \in V'$ and $v'' \in V''$ such that $v'v'' \in E(G)$. Graphs that are π-graphs of cycles are characterized. A homomorphic image H of a graph G can be obtained from a partition $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ into independent sets such that $V(H) = \{v_1, v_2, \ldots, v_k\}$, where v_i is adjacent to v_j if and only if some vertex of V_i is adjacent to some vertex of V_j in G. By investigating graphs H that are homomorphic images of the Cartesian product $H \square K_2$, it is shown that for every nontrivial connected graph H and every integer $r \geq 2$, there exists an r-regular graph G such that H is a homomorphic image of G. It is also shown that every nontrivial tree T is a homomorphic image of $T \square K_2$ but that not all graphs H are homomorphic images of $H \square K_2$.

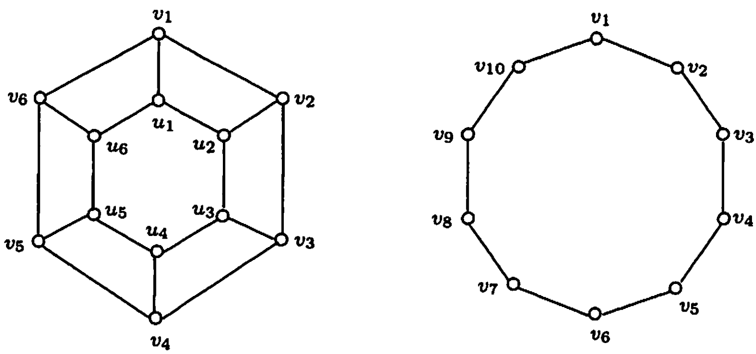
1 Introduction

There are many areas of study in graph theory that involve partitions of the vertex set or edge set of a graph, where each subset in the partition possesses some prescribed property. Another area of study in graph theory concerns the structure of a graph, especially the degree of connectedness of the graph. The most common measure of this is the connectivity of a graph. Combining these two areas leads to a coloring problem that we will discuss here. We refer to the book [1] for graph theory notation and terminology not described in this paper.

A *(vertex) cutset* U of a connected graph G is a subset of the vertex set of G such that $G - U$ is disconnected. If no proper subset of U is also a cutset of G, then U is a *minimal cutset*. A cutset containing a minimum number of vertices is a *minimum cutset* of G. The number of vertices in a minimum cutset of G is the *connectivity* of G and is denoted by $\kappa(G)$. The complete graph K_n of order n contains no cutset but its connectivity is defined as $n - 1$, that is, $\kappa(K_n) = n - 1$.

If a connected graph G contains a collection π of pairwise disjoint vertex cutsets, then π is called a *$\mathcal{VC}$-collection.* If each element of π is a minimal cutset, then π is called an *$\mathcal{MVC}$-collection.* An $\mathcal{MVC}$-collection of the vertex set of a graph G that is a partition of $V(G)$ is called an *$\mathcal{MVC}$-partition.* The prism $G_1 = C_6 \square K_2$ (the Cartesian product of C_6 and K_2) has connectivity 3 and is shown in Figure 1, where the sets $V_1 = \{u_1, v_2, v_6\}$, $V_2 = \{v_1, u_2, u_6\}$, $V_3 = \{u_4, v_3, v_5\}$, $V_4 = \{v_4, u_3, u_5\}$ are minimal cutsets and so $\pi_1 = \{V_1, V_2, V_3, V_4\}$ is an $\mathcal{MVC}$-collection. For $U_1 = \{u_1, v_1, u_4, v_4\}$, $U_2 = \{u_2, v_2, u_5, v_5\}$, $U_3 = \{u_3, v_3, u_6, v_6\}$, the set $\pi' = \{U_1, U_2, U_3\}$ is also an $\mathcal{MVC}$-collection. The 10-cycle $G_2 = C_{10}$ shown in Figure 1 has many

$\mathcal{MVC}$-collections. One such $\mathcal{MVC}$-collection is $\pi_2 = \{V_1, V_2, V_3, V_4, V_5\}$, where $V_1 = \{v_1, v_6\}$, $V_2 = \{v_2, v_{10}\}$, $V_3 = \{v_3, v_8\}$, $V_4 = \{v_4, v_9\}$, $V_5 = \{v_5, v_7\}$. In fact, the $\mathcal{MVC}$-collections π_1 and π_2 above are both $\mathcal{MVC}$-partitions, as is π'. While every noncomplete connected graph contains an $\mathcal{MVC}$-collection, not every such graph has an $\mathcal{MVC}$-partition. For example, no odd cycle has an $\mathcal{MVC}$-partition.



$G_1 = C_6 \square K_2$ $\qquad$ $G_2 = C_{10}$

Figure 1: The prism $G_1 = C_6 \square K_2$ and the 10-cycle $G_2 = C_{10}$

If V_1 is a set consisting of two nonadjacent vertices of the 4-cycle $G_3 = C_4$ and $V_2 = V(G_3) - V_1$, then V_1 and V_2 are minimal cutsets and $\pi_3 = \{V_1, V_2\}$ is an $\mathcal{MVC}$-partition. The fact that $C_4 = K_{2,2}$ and C_4 has an $\mathcal{MVC}$-partition consisting of two minimal cutsets serves to illustrate the proposition below. First, it is necessary to describe a class of graphs constructed from two disconnected graphs.

Let $F = F_1 + F_2 + \cdots + F_a$ and $H = H_1 + H_2 + \cdots + H_b$ be two disconnected graphs with a and b components, respectively, where $a, b \geq 2$. A *component join* of F and H is constructed from F and H by adding edges between F and H such that each vertex of F is adjacent to at least one vertex in each component of H and each vertex of H is adjacent to at least one vertex in each component of F. Thus, a component join of F and H is a connected subgraph of the join $F \wedge H$ of F and H. In particular, $F \wedge H$ itself is a component join of F and H. Therefore, for integers $r, s \geq 2$, the complete bipartite graph $K_{r,s}$ is a component join of $\overline{K}_r$ and $\overline{K}_s$. Also, for $F = K_2 + K_1$ and $H = P_3 + K_2 + K_1$, the graph G of Figure 2 is a component join of F and H.

Proposition 1.1 *A connected graph G of order 4 or more has an $\mathcal{MVC}$-partition consisting of two minimal cutsets, each containing at least two*

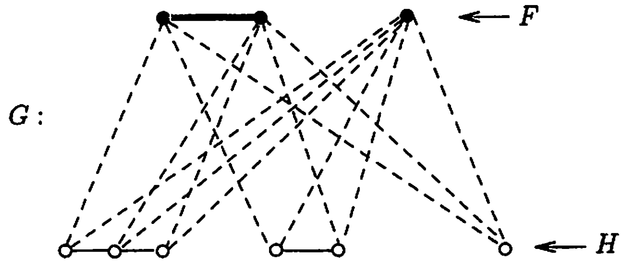


Figure 2: A component join of F and H

vertices, if and only if G is a component join of two disconnected graphs.

Proof. First, suppose that G is a component join of two disconnected graphs G_1 and G_2. Then $\{V(G_1), V(G_2)\}$ is a $\mathcal{VC}$-collection. Since every vertex of G_1 is joined to at least one vertex in each component of G_2, it follows that if U_1 is a proper subset of $V(G_1)$, then $G - U_1$ is connected. Also, if U_2 is a proper subset of $V(G_2)$, then $G - U_2$ is connected. Therefore, $\{V(G_1), V(G_2)\}$ is an $\mathcal{MVC}$-partition.

For the converse, assume that $\{V_1, V_2\}$ is an $\mathcal{MVC}$-partition of G, where $|V_i| \geq 2$ for $i = 1, 2$. This implies that $F = G[V_1]$ and $H = G[V_2]$ are disconnected subgraphs of G. We claim that G is a component join of F and H. Assume, to the contrary, that this is not the case. Then some vertex v of F, say, is not adjacent to any vertex in some component H_1 of H. Then v and H_1 belong to different components in the subgraph $G - (V_1 - \{v\})$ and so $G - (V_1 - \{v\})$ is disconnected, which is a contradiction. ■

2 π-Graphs of Graphs

For any partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of the vertex set of a graph G, the *π-graph* G_π of G is defined as the graph whose vertices correspond 1-to-1 with the sets $V_1, V_2, \ldots, V_k$ and two vertices V_i and V_j are adjacent in G_π if and only if there exist vertices $u \in V_i$ and $v \in V_j$ such that u is adjacent to v in G. In the case where π is an $\mathcal{MVC}$-partition of G, the π-graph G_π represents the structure of π. For the $\mathcal{MVC}$-partitions π_1 and π_2 described in Section 1, the π_1-graph and π_2-graph of the graphs G_1 and G_2 of Figure 1 are shown in Figure 3.

There is a fundamental property that all π-graphs of connected graphs G possess for each $\mathcal{MVC}$-partition π of G. For two sets A and B of vertices of a connected graph G, let

$$d_G(A, B) = \min\{d_G(a, b) : a \in A \text{ and } b \in B\},$$

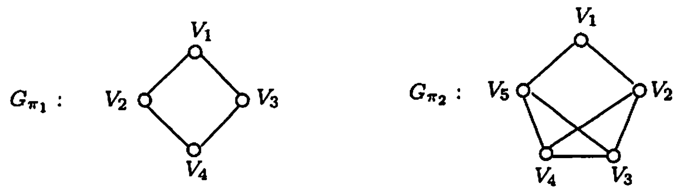


Figure 3: The π-graphs of the graphs of Figure 1

where $d_G(a, b)$ is the distance between two vertices a and b (the length of a shortest $a - b$ path in G).

Proposition 2.1 *If G is a connected graph and π is an $\mathcal{MVC}$-partition of $V(G)$, then the π-graph G_π of G is connected.*

Proof. Assume, to the contrary, that there is a connected graph G and an $\mathcal{MVC}$-partition π of $V(G)$ such that G_π is disconnected. Among all pairs of vertices of G_π that are not connected in G_π, let A and B be a pair such that $d_G(A, B) = k$ is minimum. Let $a \in A$ and $b \in B$ such that $d_G(a, b) = d_G(A, B)$. Let $P = (a = a_0, a_1, \ldots, a_k = b)$ be an $a - b$ geodesic in G. Necessarily, $k \geq 2$. Suppose that a_i is the first vertex on P that does not belong to A, where then $1 \leq i \leq k - 1$. Let $a_i \in C \in \pi$. Hence, $C \neq A$ in π. Since $a_{i-1} \in A$, it follows that A and C are adjacent in G_π and so the vertex C is in the same component as A in G_π. Thus, C and B are not connected in G_π. Since $d_G(C, B) \leq d_G(a_i, b) < k$, this is a contradiction. ■

Since every even cycle has an $\mathcal{MVC}$-partition and each such partition π consists of pairs of nonadjacent vertices, the π-graphs of even cycles are of special interest. The following theorem characterizes all graphs that are π-graphs of even cycles. For two disjoint sets U and W of vertices in a graph G, the set of edges joining U and W in G is denoted by $E[U, W]$. The *underlying graph* of a multigraph M is the graph G for which $V(G) = V(M)$ and $uv \in E(G)$ if u and v are joined by at least one edge in M.

Theorem 2.2 *A connected graph H of order 3 or more is a π-graph of some even cycle if and only if H is the underlying graph of a 4-regular multigraph.*

Proof. First, assume that H is the underlying graph of a 4-regular multigraph M of order $k \geq 3$. Thus, M is an Eulerian multigraph of even size $2k$. Let $C = (v_1, v_2, \ldots, v_{2k}, v_1)$ be an Eulerian circuit in M. Since M is 4-regular, each vertex of M occurs exactly twice as nonconsecutive vertices of C. Next, let $G = C_{2k} = (u_1, u_2, \ldots, u_{2k}, u_1)$ be a $2k$-cycle. Furthermore, let $\pi = \{V_1, V_2, \ldots, V_k\}$ be the $\mathcal{MVC}$-partition of $V(G)$ in which two vertices u_a and u_b of G belong to the same element of π if and only if $v_a = v_b$ on the circuit C. Then two vertices V_i and V_j of G_π are adjacent if and only if some vertex in V_i is adjacent to some vertex of V_j, where $1 \leq i < j \leq k$. Let $V_i = \{u_a, u_b\}$ and $V_j = \{u_c, u_d\}$. Now, V_i and V_j are adjacent if and only if $|E[V_i, V_j]| \geq 1$. Thus, $H \cong G_\pi$.

For the converse, assume that H is a connected graph of order k that is a π-graph of some even cycle G. Necessarily, G has order $2k$. Suppose that $G = C_{2k} = (u_1, u_2, \ldots, u_{2k}, u_1)$ is a $2k$-cycle. Let $\pi = \{V_1, V_2, \ldots, V_{2k}\}$ be an $\mathcal{MVC}$-partition of $V(G)$ such that $G_\pi \cong H$. For each integer i with $1 \leq i \leq k$, let $V_i = \{u_{i_1}, u_{i_2}\}$ where then u_{i_1} and u_{i_2} are two nonadjacent vertices of G. Let M be the multigraph with $V(M) = \pi$, where the number of edges joining V_i and V_j is $|E[V_i, V_j]|$ for $i \neq j$. Thus, H is the underlying graph of M. Since the two vertices in V_i are nonadjacent, $|E[V_i, V(G) - V_i]| = 4$ for $1 \leq i \leq k$, it follows that M is 4-regular. ∎

The following result is then an immediate corollary of Theorem 2.2.

Corollary 2.3 *Every connected 4-regular graph is a π-graph of an even cycle.*

With the aid of Theorem 2.2, those cubic graphs that are π-graphs of an even cycle with an $\mathcal{MVC}$-partition π can be determined. A *1-factor* of a graph G is a 1-regular spanning subgraph of G.

Corollary 2.4 *A connected cubic graph H is a π-graph of an even cycle if and only if H has a 1-factor.*

Proof. Suppose that H is a connected cubic graph having a 1-factor F. By replacing each edge in F by two parallel edges, a 4-regular multigraph M is obtained. Thus, H is the underlying graph of M. It then follows by Theorem 2.2 that H is a π-graph of an even cycle.

For the converse, assume that H is a connected cubic graph that is a π-graph of an even cycle G. Then H has even order, say $2k \geq 4$. By Theorem 2.2, H is the underlying graph of a 4-regular multigraph M. Let v_1 be a vertex of H. Since $\deg_H v_1 = 3$ and $\deg_M v_1 = 4$, it follows that v_1 is incident with exactly one edge e_1 in M that does not belong to H, say $e_1 = v_1 w_1$. Hence, e_1 is the only edge of M that is incident with w_1 that does not belong to H. Continuing in this manner, we obtain k pairwise nonadjacent edges $e_i = u_i w_i$ $(1 \leq i \leq k)$ that belong to M but not to H. Therefore, the edges $u_i w_i$ $(1 \leq i \leq k)$ of H form a 1-factor of H. ∎

Since the cubic graph H_1 of order 10 in Figure 4 has a 1-factor, it follows by Corollary 2.4 that H_1 is a π-graph of the cycle C_{20}. On the other hand,

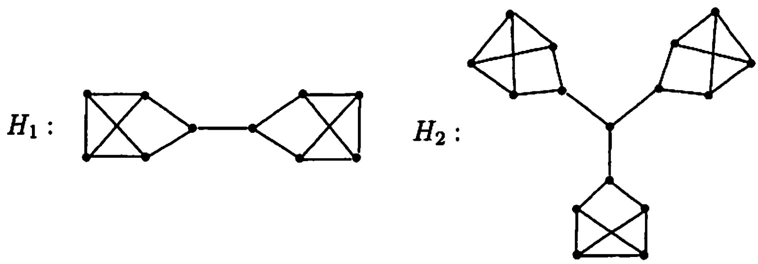


Figure 4: Two cubic graphs H_1 and H_2

the cubic graph H_2 in Figure 4 has no 1-factor and so H_2 is not a π-graph of any even cycle.

Errera [2] proved that if all the bridges of a connected cubic graph G lie on a single path of G, then G has a 1-factor. From this theorem and Corollary 2.4, we have the following.

Corollary 2.5 *Let G be a connected cubic graph. If all the bridges of G lie on a single path of G, then G is a π-graph of an even cycle.*

Corollary 2.4 also implies that (1) the graph $K_{3,3}$ is a π-graph of C_{12}, (2) the Petersen graph is a π-graph of C_{20} and (3) the Mobius ladder M_{2k}, $k \geq 2$, is a π-graph of C_{2k}, where M_{2k} is obtained by joining diametrically opposite vertices of the cycle C_{2k}. In addition, we have the following:

Corollary 2.6 *For each integer $k \geq 3$, the k-cycle C_k is a π-graph of C_{2k}.*

Each graph of order 5 shown in Figure 5 is the underlying graph of a 4-regular multigraph and so it is a π-graph of C_{10} by Theorem 2.2. The corresponding $\mathcal{MVC}$-partition π is given in Figure 5 for each graph, where V_i is the set of vertices labeled i.

For an $\mathcal{MVC}$-partition π of C_{10}, the π-graph G_π of C_{10} has order 5 and each vertex of G_π has degree 2, 3 or 4. With the aid of Theorem 2.2, the following can be verified:

1. The unique graph $K_5 - e$ of order 5 and size 9 is not a π-graph of C_{10}.
2. There is a unique π-graph of C_{10} having size 8.
3. The graph $K_2 \vee \overline{K}_3$ is not a π-graph of C_{10}.
4. The graph $K_1 \vee P_4$ is not a π-graph of C_{10}.
5. The graph $K_{2,3}$ is not a π-graph of C_{10}.

In summary, Figure 5 shows all π-graphs of C_{10}. Incidentally, the only connected graphs of order 4 that are π-graphs of C_8 are C_4 and K_4.

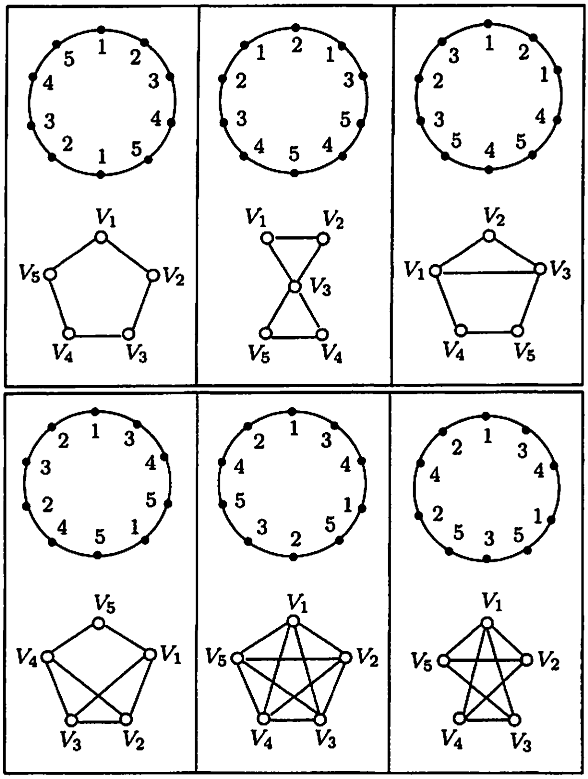


Figure 5: Some $\mathcal{MVC}$-partitions π and corresponding π-graphs G_π of C_{10}

3 Homomorphic Images

We saw that if π is an $\mathcal{MVC}$-partition of an even cycle, then each element of π is an independent set. In fact, if π is a partition of the vertex set of a graph G such that each element of π is an independent set, then both π and G_π concern familiar concepts in graph theory.

A *homomorphism* from a graph G to a graph H' is a function $\phi : V(G) \to V(H')$ that maps adjacent vertices in G to adjacent vertices in H. The subgraph $H = (V, E)$ of H' whose vertex set is $V(H) = \phi(V(G))$ and whose edge set is the set $E(H) = \{\phi(u)\phi(v) : uv \in E(G)\}$ is called the *homomorphic image* of G under ϕ and is denoted by $\phi(G) = H$. A graph H is called a *homomorphic image* of a graph G is there is a homomorphism ϕ of G such that $\phi(G) = H$.

Let $H = (V, E)$ be a homomorphic image of a graph G and let $V(H) = \{v_1, v_2, \ldots, v_k\}$. For any vertex $v \in V(H)$, let $\phi^{-1}(v) = \{u \in V(G) :$

$\phi(u) = v\}$. Since ϕ is a homomorphism, it follows that $\phi^{-1}(v)$ is an independent set in G for every $v \in V(H)$. This defines a *coloring* of the vertices of G, that is, $\pi = \{\phi^{-1}(v_1), \phi^{-1}(v_2), \ldots, \phi^{-1}(v_k)\}$ is a partition of $V(G)$ into independent sets. It also follows that the π-graph G_π is isomorphic to H. In particular, if π is an $\mathcal{MVC}$-partition of a graph G, where each set V_i $(1 \le i \le k)$ is independent, then ϕ represents a k-coloring of G where each vertex of V_i $(1 \le i \le k)$ is colored i and G_π is the homomorphic image of G resulting from this k-coloring.

For a nontrivial connected graph H that is a homomorphic image of a cycle, let $\mu(H)$ be the length of a shortest such cycle and let $e(H)$ be the length of a shortest Eulerian walk in H. (Eulerian walks are discussed in [3].)

Proposition 3.1 *Every nontrivial connected graph H is a homomorphic image of a cycle and so $\mu(H)$ exists. Furthermore, $\mu(H) = e(H)$.*

Proof. Let H be a nontrivial connected graph with $V(H) =\{u_1, u_2, \ldots, u_k\}$. Let $W =(w_1, w_2, \ldots, w_p, w_1)$ be an Eulerian walk of minimum length p in H. Thus, each edge in H occurs in W at least once. (Every edge of H can occur exactly once if and only if H is Eulerian and W is an Eulerian circuit.) Let $C = (v_1, v_2, \ldots, v_p, v_1)$ be a cycle of order p. For each integer i with $1 \le i \le k$, let

$$V_i = \{v_t \in V(C) : \ w_t = u_i \text{ and } 1 \le t \le p\}.$$

Thus, each set V_i is an independent set of vertices of H and a vertex in a set V_i is adjacent to a vertex in V_j in C $(1 \le i \le j$ and $i \ne j)$ if and only if $u_iu_j \in E(H)$. Hence, H is a homomorphic image of C. Therefore, $\mu(H)$ exists and $\mu(H) \le e(H)$.

Next, we show that $e(H) \le \mu(H)$. Let $\mu(H) = \ell$ and let

$$C = (v_1, v_2, \ldots, v_\ell, v_1)$$

be a cycle such that H is a homomorphic image of C. Thus, there is a partition $\mathcal{P} = \{U_1, U_2, \ldots, U_k\}$ of $V(C)$ into independent sets such that $u_iu_j \in E(H)$ if and only if some vertex in U_i is adjacent to some vertex in U_j in C. Since each U_i is an independent set of vertices of C, it follows that every edge in C gives rise to an edge in H (although it is possible that several edges in C produce the same edge in H). Furthermore, for each edge u_iu_j in H, there is at least one edge v_sv_{s+1} in C such that $v_s \in U_i$ and $v_{s+1} \in U_j$. Identifying all vertices in each set U_i, producing a single vertex denoted by u_i for $1 \le i \le k$, and following the ordering of vertices in C, we obtain an Eulerian walk $W = (w_1, w_2, \ldots, w_\ell, w_1)$ of length ℓ in H. Thus, $e(H) \le \mu(H)$ and so $e(H) = \mu(H)$. ∎

By Proposition 3.1, every nontrivial connected graph is a homomorphic image of a connected 2-regular graph. Next, we show that every nontrivial connected graph is, in fact, a homomorphic image of a connected r-regular graph for each integer $r \geq 3$. First, we introduce some notation. For a given integer $n \geq 3$, we denote an n-cycle by C, where $C = (v_1, v_2, \ldots, v_n, v_1)$. Because several n-cycles will be encountered in the graphs to be considered, we denote this first n-cycle by $C(1)$ and write $C(1) = (v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)}, v_1^{(1)})$. The Cartesian product $C \square K_2$ of C and K_2 can also be expressed as $C \square Q_1$. This graph is constructed with the aid of two disjoint n-cycles, namely the n-cycle $C(1)$ and the n-cycle $C(2) = (v_1^{(2)}, v_2^{(2)}, \ldots, v_n^{(2)}, v_1^{(2)})$ together with the edges $v_j^{(1)} v_j^{(2)}$ for $j = 1, 2, \ldots, n$. In the case where C is a 6-cycle, the graph $C \square Q_1$ is shown in Figure 6. For each integer j with $1 \leq j \leq n$, let $V_j = \{v_j^{(1)}, v_{j+1}^{(2)}\}$ be the 2-element independent subset of $V(C \square Q_1)$ where $v_{n+1}^{(2)} = v_1^{(2)}$. Then the n-cycle C is a homomorphic image of $C \square Q_1$ defined by the partition $\mathcal{P} = \{V_1, V_2, \ldots, V_n\}$ of the vertex set of $C \square Q_1$.

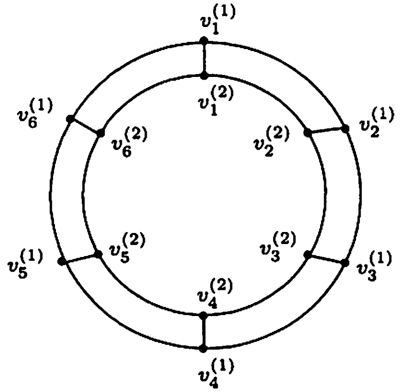


Figure 6: The graph $C_6 \square Q_1$

The graph $C \square Q_2$ is also $(C \square Q_1) \square K_2$. This graph consists of $C \square Q_1$ described above, with another copy of $C \square Q_1$, where the vertex $v_j^{(i)}$ $(1 \leq j \leq n,\ i = 1, 2)$ in the first copy of $C \square Q_1$ corresponds to the vertex $v_j^{(i+2)}$ in the second copy of $C \square Q_1$. In addition to the two copies of $C \square Q_1$ described above, each edge $v_j^{(i)} v_j^{(i+2)}$, $1 \leq j \leq n$, $i = 1, 2$, is added to these two copies. The graph $C \square Q_2$ has therefore $2^2 = 4$ pairwise disjoint n-cycles $C(1), C(2), C(3), C(4)$ where

$$C(i) = (v_1^{(i)}, v_2^{(i)}, \ldots, v_n^{(i)}, v_1^{(i)}) \text{ for } 1 \leq i \leq 2^2.$$

The graph $C \square Q_2$, where C is a 6-cycle, is shown in Figure 7, where an edge in the first copy of $C \square Q_1$ is indicated by a thin line, an edge in the second copy of $C \square Q_1$ is indicated by a dashed line and an edge between these two copies of $C \square Q_1$ is indicated by a bold line. For each integer j with $1 \leq j \leq n$, let $V_{1,j} = \{v_j^{(1)}, v_j^{(4)}\}$ and $V_{2,j} = \{v_j^{(2)}, v_j^{(3)}\}$ be 2-element independent subsets of $V(C \square Q_2)$. Then the graph $C \square Q_1$ is a homomorphic image of $C \square Q_2$ defined by the partition $\mathcal{P} = \{V_{i,j} : i = 1, 2 \text{ and } 1 \leq j \leq n\}$ of the vertex set of $C \square Q_2$.

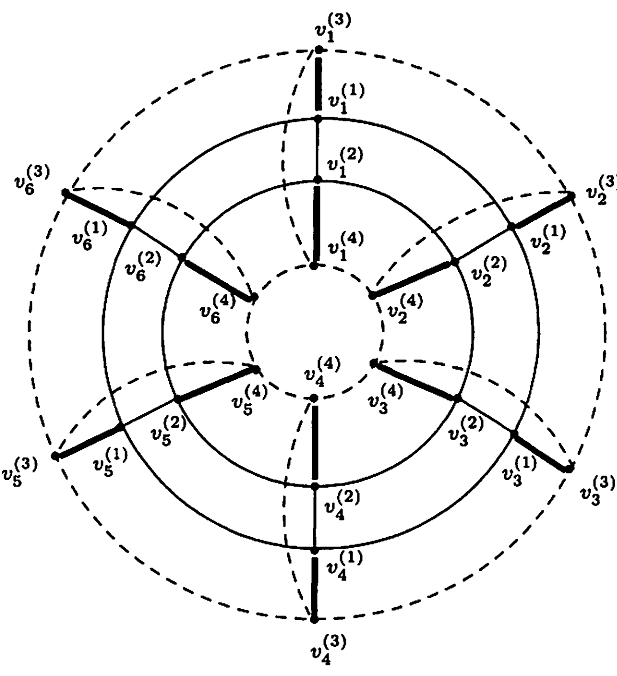


Figure 7: The graph $C_6 \square Q_2$

Suppose now that the graph $C \square Q_k$ has been constructed for an integer $k \geq 2$. We describe the construction of $C \square Q_{k+1}$. By the construction of $C \square Q_k$, it follows that $C \square Q_k$ consists of 2^k pairwise disjoint n-cycles, namely

$$
\begin{aligned}
C(1) &= (v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)}, v_1^{(1)}) \\
C(2) &= (v_1^{(2)}, v_2^{(2)}, \ldots, v_n^{(2)}, v_1^{(2)}) \\
\vdots &= \vdots \\
C(2^k) &= (v_1^{(2^k)}, v_2^{(2^k)}, \ldots, v_n^{(2^k)}, v_1^{(2^k)}).
\end{aligned}
$$

To construct $C \square Q_{k+1}$, we have a second copy of $C \square Q_k$, consisting of the 2^k pairwise disjoint n-cycles

$$\begin{aligned} C^*(1) &= (w_1^{(1)}, w_2^{(1)}, \ldots, w_n^{(1)}, w_1^{(1)}) \\ C^*(2) &= (w_1^{(2)}, w_2^{(2)}, \ldots, w_n^{(2)}, w_1^{(2)}) \\ \vdots &= \vdots \\ C^*(2^k) &= (w_1^{(2^k)}, w_2^{(2^k)}, \ldots, w_n^{(2^k)}, w_1^{(2^k)}), \end{aligned}$$

where the vertex $w_j^{(i)}$, $1 \le j \le n$, $1 \le i \le 2^k$, corresponds to the vertex $v_j^{(i)}$. Then each edge $v_j^{(i)} w_j^{(i)}$ $(1 \le j \le n, 1 \le i \le 2^k)$ is added to these two copies producing the graph $C \square Q_{k+1}$. The graph $C \square Q_{k+1}$ now has 2^{k+1} pairwise disjoint n-cycles $C(1), C(2), \ldots, C(2^{k+1})$ where $C(2^k + p) = C^*(p)$ for $1 \le p \le 2^k$.

For each pair i, j of integers with $1 \le j \le n$ and $1 \le i \le 2^k$, the vertex $v_j^{(i)}$ has degree $k+2$ in $C \square Q_k$. In particular, $v_j^{(i)}$ is adjacent to its two neighbors in the cycle $C(i)$ and is also adjacent to k additional vertices on k of the 2^k disjoint n-cycles in $C \square Q_k$, (exactly one vertex from each of these k cycles).

For $1 \le \ell \le k$, express $i = 2^\ell q_\ell + r_\ell$, where $1 \le r_\ell \le 2^\ell$. Also, let

$$S_\ell = [2^\ell q_\ell + 1,\ 2^\ell q_\ell + 2^\ell] = \{2^\ell q_\ell + 1, 2^\ell q_\ell + 2, \ldots, 2^\ell q_\ell + 2^\ell\}.$$

For each integer i with $1 \le i \le 2^k$ and each j with $1 \le j \le n$, the vertex $v_j^{(i)}$ is adjacent to either $v_j^{(i+2^{\ell-1})}$ or $v_j^{(i-2^{\ell-1})}$ according to which of $i + 2^{\ell-1}$ or $i - 2^{\ell-1}$ belongs to S_ℓ.

For example, in the graph $C \square Q_5$, the vertex $v_j^{(15)}$ on the cycle $C(15)$ is adjacent to its two neighbors in the cycle $C(15)$ and is adjacent to

(1) $v_j^{(16)}$ since $16 = 15 + 1 \in S_1 = [2 \cdot 7 + 1,\ 2 \cdot 7 + 2] = [15, 16]$,

(2) $v_j^{(13)}$ since $13 = 15 - 2 \in S_2 = [4 \cdot 3 + 1,\ 4 \cdot 3 + 4] = [13, 16]$,

(3) $v_j^{(11)}$ since $11 = 15 - 4 \in S_3 = [8 \cdot 1 + 1,\ 8 \cdot 1 + 8] = [9, 16]$,

(4) $v_j^{(7)}$ since $7 = 15 - 8 \in S_4 = [16 \cdot 0 + 1,\ 16 \cdot 0 + 16] = [1, 16]$,

(5) $v_j^{(31)}$ since $31 = 15 + 16 \in S_5 = [32 \cdot 0 + 1,\ 32 \cdot 0 + 32] = [1, 32]$.

As another example, in the graph $C \square Q_5$, the vertex $v_j^{(21)}$ on the cycle $C(21)$ is adjacent to its two neighbors in the cycle $C(21)$ and is adjacent to

(1) $v_j^{(22)}$ since $22 = 21 + 1 \in S_1 = [2 \cdot 10 + 1,\ 2 \cdot 10 + 2] = [21, 22]$,

(2) $v_j^{(23)}$ since $23 = 21 + 2 \in S_2 = [4 \cdot 5 + 1,\ 4 \cdot 5 + 4] = [21, 24]$,

(3) $v_j^{(17)}$ since $17 = 21 - 4 \in S_3 = [8 \cdot 2 + 1,\ 8 \cdot 2 + 8] = [17, 24]$,

(4) $v_j^{(29)}$ since $29 = 21 + 8 \in S_4 = [16 \cdot 1 + 1,\ 16 \cdot 1 + 16] = [17, 32]$,

(5) $v_j^{(5)}$ since $5 = 21 - 16 \in S_5 = [32 \cdot 0 + 1,\ 32 \cdot 0 + 32] = [1, 32]$.

Since the order of $C \,\square\, Q_{k+1}$ is $2^k n$, the order of $C \,\square\, Q_{k+1}$ is $2^{k+1}n$. For each pair i, j of integers where $1 \leq j \leq n$, $1 \leq i \leq 2^k$, let $V_{i,j}$ be the 2-element independent subset of $V(C \,\square\, Q_{k+1})$ defined by

$$V_{i,j} = \begin{cases} \{v_j^{(i)}, w_j^{(i+1)}\} & \text{if } i \text{ is odd} \\ \{v_j^{(i)}, w_j^{(i-1)}\} & \text{if } i \text{ is even.} \end{cases}$$

Thus, the graph $C \,\square\, Q_k$ is the homomorphic image of $C \,\square\, Q_{k+1}$ defined by the partition $\mathcal{P} = \{V_{i,j} : 1 \leq i \leq 2^k \text{ and } 1 \leq j \leq n\}$ of the vertex set of $C \,\square\, Q_{k+1}$. This results in the following theorem.

Theorem 3.2 *For every pair k, n of positive integers where $n \geq 3$, the graph $C_n \,\square\, Q_k$ is a homomorphic image of $C_n \,\square\, Q_{k+1}$.*

The following is then a consequence of Proposition 3.1 and Theorem 3.2.

Corollary 3.3 *Every nontrivial connected graph is a homomorphic image of an r-regular graph for each integer $r \geq 2$.*

Theorem 3.2 also states that H is a homomorphic image of $H \,\square\, K_2$ for $H = C_n \,\square\, Q_k$. This suggests the problem of determining nontrivial connected graphs H having the property that H is a homomorphic image of $H \,\square\, K_2$. For a vertex v in a connected graph G, let $e(v)$ denote the *eccentricity* of v (the largest distance from v to a vertex in G).

Theorem 3.4 *Every nontrivial tree T is a homomorphic image of $T \,\square\, K_2$.*

Proof. Let T be a tree of order $n \geq 2$ and let v_1 be a leaf of T. The tree T may then be considered as a rooted tree with root v_1. Therefore, T can be considered as a directed tree where there is a directed $v_1 - w$ path in T for every vertex w of T. Let $V(T) = \{v_1, v_2, \ldots, v_n\}$ where $d(v_1, v_i) \leq d(v_1, v_j)$ for $1 \leq i \leq j \leq n$. Let $G = T \,\square\, K_2$, where G consists of the tree T (as labeled above), a second copy T' of T with $V(T') = \{u_1, u_2, \ldots, u_n\}$ such that u_i corresponds to v_i and $u_i v_i \in E(G)$ for $1 \leq i \leq n$.

We now show that there is a proper n-coloring c of G using the colors $1, 2, \ldots, n$ resulting in the color classes $V_1, V_2, \ldots, V_n$ such that the homomorphic image resulting from the n color classes $V_1, V_2, \ldots, V_n$ is isomorphic to T. First, color each vertex v_i the color i for $i = 1, 2, \ldots, n$.

The vertex u_2 is the only vertex at distance 1 from u_1 of T'. Assign the color 2 to u_1 and the color 1 to u_2. Next, assign the color 2 to each vertex of T' at distance 1 from u_2. Proceeding recursively, assume that all vertices of T' at distance k from u_1 have been assigned a color where $2 \leq k < e(u_1)$ and let $u_j \in V(T')$ such that $d_{T'}(u_1, u_j) = k + 1$. Let u_i be the unique vertex adjacent to u_j on the $u_1 - u_j$ (directed) path P on T'. We then assign the color $c(v_i)$ to u_j (and so $c(u_j) = c(v_i)$).

Let a and b be distinct colors in $\{1, 2, \ldots, n\}$ such that some vertex in V_a is adjacent to a vertex in V_b. Then $v_a v_b \in E(T)$ where say $a < b$. Thus, the edge u_b is colored a. Also, the two incident vertices of an edge of T' are assigned two distinct colors of $\{1, 2, \ldots, n\}$ if and only if the two incident vertices of some edge of T are also assigned these same two colors. Hence, T is a homomorphic image of G. ■

To illustrate the coloring c described in the proof of Theorem 3.4, consider the tree T with $V(T) = \{v_1, v_2, \ldots, v_{13}\}$ and construct $T \square K_2$ as shown in Figure 8, where one copy of T in $T \square K_2$ is drawn with bold lines, the second copy T' of T is drawn with thin lines and the edges between T and T' are drawn with dashed lines. The color of each vertex of $T \square K_2$ is indicated inside the vertex.

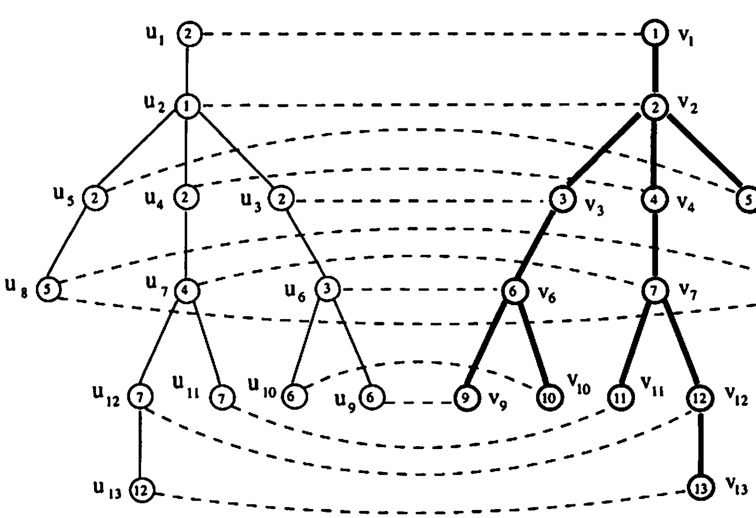


Figure 8: Illustrating the proof of Theorem 3.4

Although every tree T is a homomorphic image of $T \square K_2$, not every nontrivial connected graph G is a homomorphic image of $G \square K_2$.

Proposition 3.5 *The graph H of* Figure 9 *is not a homomorphic image of $H \square K_2$.*

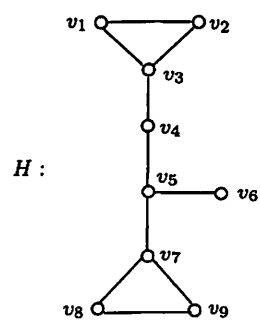


Figure 9: The graph H

Proof. Assume, to the contrary, that H is a homomorphic image of $G = H \square K_2$, shown in Figure 10. Consequently, there is a proper 9-coloring c of G using the colors $1, 2, \ldots, 9$ resulting in the color classes $V_1, V_2, \ldots, V_9$ so that when the vertices of each set V_i $(1 \le i \le 9)$ are identified, producing the vertex v_i, the graph H is obtained.

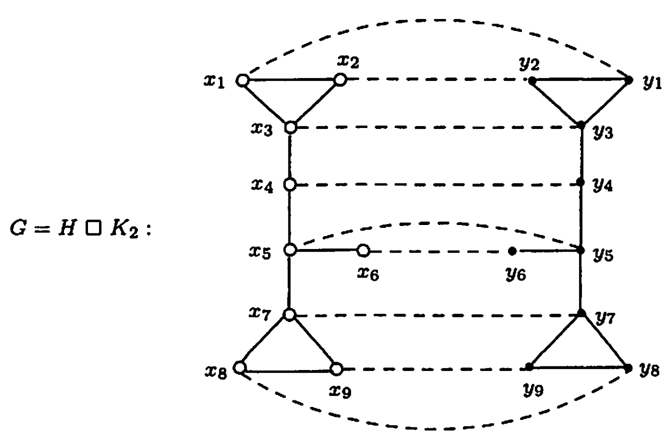


Figure 10: The graph $G = H \square K_2$

Since every vertex in a triangle in $H \square K_2$ belongs to a triangle in H, the vertices of each of the four triangles in G must be colored $1, 2, 3$ or

7, 8, 9. Also, since there are adjacencies between the vertices of triangle x_1, x_2, x_3 and triangle y_1, y_2, y_3, the vertices in these triangles must both be colored 1,2, 3 or both be colored 7, 8, 9. Similarly, the vertices of the triangles x_7, x_8, x_9 and y_7, y_8, y_9 must both be colored 1, 2, 3 or both be colored 7, 8, 9.

Moreover, since any neighbor of a vertex colored 6 must be colored 5, we deduce that none of x_4, x_5, y_4 and y_5 is colored 6. Thus, either x_6 or y_6 is colored 6. Without loss of generality, we may assume that $c(y_6) = 6$, implying that $c(x_6) = c(y_5) = 5$. Then $c(x_5) \in \{4, 7\}$. Furthermore, since y_5 is adjacent to y_7 and y_7 is in a triangle, it follows that $c(y_7) = 7$ and the vertices of triangles y_7, y_8, y_9 and x_7, x_8, x_9 are colored 7, 8, 9. Thus, $c(x_7) \in \{8, 9\}$. Now x_5 is adjacent to x_7 and no vertex colored 4 has a neighbor colored 8 or 9, so $c(x_5) = 7$. But then x_4 is adjacent to a vertex colored 7 and a vertex in triangle colored 1, 2 or 3, and no such vertex exists in H, producing the contradiction. ∎

4 Closing Comments

We close by noting that many problems remain, particularly those dealing with the structure of π-graphs obtained from vertex partitions $\pi = \{V_1, V_2, \ldots, V_k\}$ whose elements have some property of interest. If each set V_i is an independent set, then these partitions give rise to proper colorings of graphs and the corresponding π-graphs are just homomorphic images of a graph. If each set V_i induces a connected subgraph, then the corresponding π-graphs are just *contractions* of a graph. In this case, the famous Hadwiger's conjecture [4] is worth noting; it can be stated as follows:

Conjecture 4.1 (Hadwiger's Conjecture) *For any graph G, if the chromatic number $\chi(G) = k$, then G has a connected vertex partition π whose corresponding π-graph is the complete graph K_k.*

This raises the following question: *Is every graph G having chromatic number k a homomorphic image of a regular graph having chromatic number k?* If each set V_i is an independent set of edges, then the π-graph is a homomorphic image of the line graph of the graph. It is also of interest to study π-graphs arising from vertex partitions π in which each element is a so-called 1-dependent set consisting of a disjoint union of copies of K_1 and K_2. These π-graphs include the homomorphic images of the graph – and more.

It is well known that the chromatic number of any homomorphic image of a graph G is at least the chromatic number of G. The regular graphs constructed in the proof of Theorem 3.2 are bipartite. This raises the question is whether the largest chromatic number k such that a graph G having chromatic number ℓ is a homomorphic image of a regular graph having chromatic number $k \leq \ell$? In particular, is every graph having chromatic number 3 a homomorphic image of a regular graph having chromatic number 3?

Other problems include those of finding other classes of graphs G that are homomorphic images of $G \square K_2$; graphs of the form $G \square K_2$ are often called prisms. So the question becomes: Which graphs are homomorphic images of their own prisms? Theorem 3.4 asserts that every nontrivial tree T is a homomorphic image of $T \square K_2$. Is this also true for every bipartite graph G? Note that the example given in Proposition 3.5 is not bipartite.

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