Small Bipancyclic Graphs

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Abstract

A bipartite graph on $2n$ vertices is called *bipancyclic* if it contains cycles of every length from 4 to $2n$. In this paper we address the question: what is the minimum number of edges in a bipancyclic graph? We present a simple analysis of some small orders using chord patterns.

1 Introduction

For definitions and theorems involving graph theory, the reader is referred to standard texts on the subject, such as [12]. Graphs are finite, simple and undirected.

A graph with v vertices is called *pancyclic* if it contains cycles of every length from 3 to v. Pancyclic graphs were introduced by Bondy [2], although the directed equivalent had been discussed earlier (see [1, 6, 8]), and have been studied by several authors: see, for example, [3, 9, 11, 5]. In particular, some papers have investigated the smallest possible number of edges in such a graph. A pancyclic graph with this number of edges is called *minimal*. In 1978, Sridharan [11] gave constructions that found upper bounds for the number of edges in a minimal pancyclic graph.

In 1982, Schmeichel and Mitchem [10] introduced the concept of *bipancyclic graphs*. A graph is called bipancyclic if it is bipartite, and contains cycles of every even length from 4 up to and including the number of vertices. Any bipartite graph has two disjoint sets of vertices, V_1 and V_2 say, such that every edge has one endpoint in V_1 and the other in V_2. A bipancyclic graph must be Hamiltonian, which implies that the sets V_1 and V_2 must be of equal size (a bipartite graph with this property is called *balanced*), so it will be a subgraph of $K_{n,n}$, where n is the common size of V_1

and V_2.

While minimal bipancyclic graphs can be defined analogously to minimal pancyclic graphs, they have not been investigated extensively. In this paper, we shall discuss some of the smaller cases.

Another idea that has been explored is *uniqueness.* A pancyclic graph is *uniquely pancyclic* if it contains exactly one cycle of every possible length; *uniquely bipancyclic* graphs are defined analogously. While the discussion of uniquely pancyclic graphs has proven difficult, there has been more work on uniquely bipancyclic graphs; two recent papers are [7] and [13].

2 Some basics

For every positive integer $n \geq 2$ there will be an integer $m^*(2n)$ such that any bipancyclic graph with $2n$ vertices must have at least $m^*(2n)$ edges; a bipancyclic graph with $2n$ vertices and $m^*(2n)$ edges is called *minimal.* If a graph G has $e(G)$ edges and $v(G)$ vertices then the difference $e(G) - v(G)$ is called the *excess* of G, so $m^*(2n)$ is the value such that $m^*(2n) - 2n$ is the minimum excess for a bipancyclic graph on $2n$ vertices.

We would conjecture that one cannot decrease $m^*(2n)$ by increasing v; in other words,

$$m^*(2n) \geq m^*(2n-2)$$

There is an easy construction for bipancyclic graphs. Take a $2n$-cycle $(a_1, a_2, \ldots a_{2n}, a_1)$ and add edges from a_1 to a_i for $i = 4, 6, \ldots, n$ if n is even, and up to $n - 1$ if n is odd. The resulting graph is bipancyclic, but the excess is quite large for large n. Another result, proven by Entringer and Schmeichel in 1988, is

Theorem 1 [4] *Suppose G is a balanced bipartite graph on $2n$ vertices. If G has more than $n(n-1)+1$ edges, then G is bipancyclic.*

For convenience we shall denote the vertices of a bipartite graph of order $2n$ by $a_1, a_2, \ldots, a_{2n}$, where $(a_1, a_2, \ldots, a_{2n}, a_1)$ is the Hamilton cycle; the component V_1 consists of the vertices with odd subscripts and V_2 contains the even vertices. The edges not in the Hamilton cycle are called *chords,* so the excess equals the number of chords in the graph.

3 Minimal bipancyclic graphs with excess less than 2

It is obvious that the only bipancyclic graph on four vertices is the 4-cycle.

Suppose a bipancyclic graph has six vertices. Then it must contain cycles of length 4 and 6. The graph will contain a Hamilton cycle, so it will be a 6-cycle with additional edges. It is easy to add a 4-cycle by adding one edge, say from a_1 to a_4. The graph will in fact contain two 4-cycles.

It was shown in [13] that there is a uniquely bipancyclic graph on 8 vertices. This graph and the examples on 4 and 6 vertices are shown in Figure 1.

A graph consisting of a Hamilton cycle and one chord can contain at most three cycles. A bipancyclic graph on 10 or more vertices must contain at least four cycles, so it must contain at least two chords.

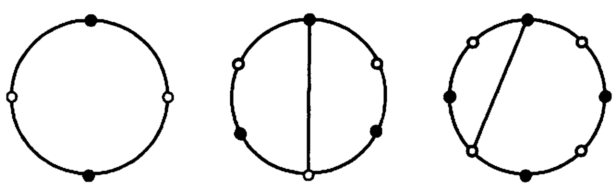


Figure 1: Minimal bipancyclic graphs up to order 8

4 Excess 2

There are three possible patterns for two chords: case A, where the chords share an endpoint; case B, where they do not cross in the standard diagram, and case C, where they cross. The three types are illustrated in Figure 2.

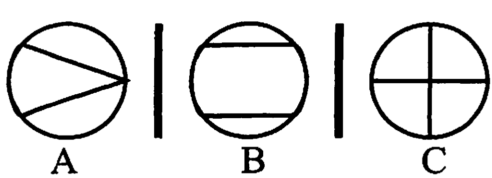


Figure 2: Possible patterns for two chords

Types A and B contain 6 cycles and type C has 7, so the graph can only be bipancyclic if there are 16 or fewer vertices. So we need only consider cases $2n = 10$, 12, 14 and 16.

Figure 3 shows a graph on $2n$ vertices with two chords, X and Y; the circle represents the Hamilton cycle, and the numbers on the cycle show the number of edges in the segment.

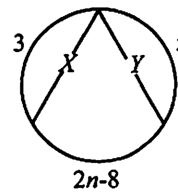


Figure 3: Minimal bipancyclic graphs of orders 10 to 14

The Hamilton cycle is a cycle of length $2n$, the cycles containing X but not Y (the X-cycles) have lengths 4 and $2n-2$, the Y-cycles have lengths 6 and $2n-4$, and the XY-cycle is of length $2n-6$. So the graph is bipancyclic for $2n = 10, 12$ or 14.

In the 16-vertex case, the total number of edges in the two cycles containing a given chord (but not the other one) will total 18, the two cycles that contain both chords will have a total of 20 edges, and the Hamiltonian cycle contains 16. So the seven cycles have a total of $16+18+18+20 = 72$ edges (with repetitions counted multiply). But $4+6+8+10+12+14+16 = 70$. So there are too many edges for exactly one cycle of every length, and not enough if any repeated cycle lengths are allowed.

5 Excess 3

Suppose a Hamiltonian graph contains three chords. It will contain the Hamilton cycle, six one-chord cycles, three to six two-chord cycles, and one or two three-chord cycles, totalling between 11 and 15 cycles. So a three-chord bipancyclic graph could have anywhere up to 32 vertices.

The possible arrangements of three chords were analyzed in [13] and were illustrated in that paper; the types are illustrated in Figure 4, which was taken from that paper.

Figure 5 below shows a graph of type AAC on $2n$ vertices with three chords, X, Y, Z; the circle represents the Hamilton cycle, and the numbers on the cycle show the number of edges in the segment. The Hamilton cycle has length $2n$, the X-cycles have lengths $2n-8$ and 10, the Y-cycles have length 6 and $2n-4$, the Z-cycles have length 4 and $2n-2$, the XY-cycles have length 14 and $2n-10$, the XZ-cycle has length 8, the YZ-cycle has

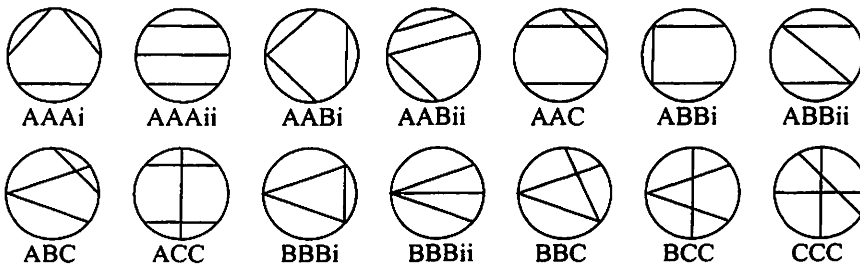


Figure 4: Cases of three chords

length $2n-6$, and the XYZ-cycle has length 12. So the graph is bipancyclic for $10 \leq 2n \leq 26$.

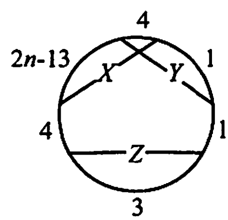
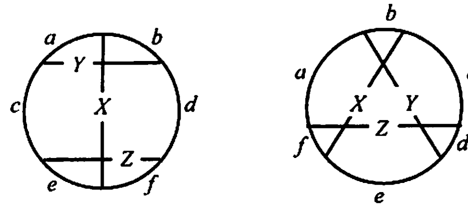


Figure 5: Minimal bipancyclic graphs of orders 16 to 26

A bipancyclic graph on 28 or more vertices must contain at least 13 cycles, so the only possible chord cases with three chords are BCC, ACC or CCC (these patterns are defined in Figure 4). Type BCC has 13 cycles, so a bipancyclic graph of that type on 28 vertices would be uniquely bipancyclic, which was shown in [13] to be impossible. The remaining cases are shown in Figure 6, with the chords named and the lengths of arcs marked.

Type ACC has 14 cycles. In order to achieve 28 or more vertices, we can have at most one pair of cycles of the same length, but this cannot occur. The possible arc lengths to achieve a 4-cycle are $a = 2, b = 1$ or equivalent ($a = 1, b = 2$ or $e = 1, f = 2$ or $e = 2, f = 1$), or $c = d = 1$. If $a = 2, b = 1$, there are an X-cycle and an XY-cycle of length $c+e+3$, and an XZ-cycle and an XYZ-cycle of length $c+f+4$; if $c = d = 1$, the XY-cycles are of lengths $b+e+3$ and $a+f+3$, and the XZ-cycles are of the same two lengths. In either case we have at most 12 distinct cycle lengths. So any bipancyclic graph on more than 26 vertices must have an excess of at least 4.



ACC CCC

Figure 6: Chord patterns ACC and CCC

6 Excess 4

A graph with four chords could have as many as 31 cycles, so it is conceivable, though unlikely, that one might find a 4-chord bipancyclic graph with as many as 64 vertices. Very little work has been done on this topic. A uniquely bipancyclic graph on 44 vertices was found in [7], so we shall look at cases up to 44.

Figure 7 shows a pattern of graph with four chords and $2n$ vertices, where 2n must be at least 28. The numbers on the arcs are the number of edges. We list the lengths of the cycles according to the chords they contain:

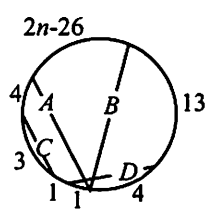


Figure 7: A model for 28 to 44 vertices

No chords, $2n$;	
A only, $10, 2n-8$;	B only, $18, 2n-16$;
C only, $4, 2n-2$;	D only, $6, 2n-4$;
A and $B, 2n-24$;	A and $C, 8$;
A and $D, 14, 2n-10$;	B and $C, 2n-18$;
B and $D, 16, 2n-12$;	C and $D, 2n-6$;
$A, B, C,$ no cycle;	$A, B, D, 24, 2n-18$;
$A, C, D, 12$;	$B, C, D, 2n-14$;
$A, B, C, D, 22$	

So we always have cycles of all even lengths from 4 to 18, 22 and 24, $2n-24$ and all orders from $2n-20$ to $2n$. So the graph is bipancyclic provided $2n \leq 44$. (When $2n = 44$, the graph is the UBPC graph presented in [7].)

We have

Theorem 2 *The minimum excess $m^*(2n)$ of a bipancyclic graph on $2n$ vertices satisfies $m^*(4) = 0$, $m^*(6) = m^*(8) = 1$, $m^*(2n) = 2$ for $10 \leq 2n \leq 16$, $m^*(2n) = 3$ for $18 \leq 2n \leq 26$, $m^*(2n) = 4$ for $28 \leq 2n \leq 44$, and $m^*(2n) \geq 4$ for $2n \geq 46$.*

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