Regular handicap graphs of odd order

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Abstract

A *handicap distance antimagic labeling* of a graph $G = (V, E)$ with n vertices is a bijection $f : V \to \{1, 2, \ldots, n\}$ with the property that $f(x_i) = i$ and the sequence of the weights $w(x_1), w(x_2), \ldots, w(x_n)$ (where $w(x_i) = \sum_{x_j \in N(x_i)} f(x_j)$) forms an increasing arithmetic progression. A graph G is a *handicap distance antimagic graph* if it allows a handicap distance antimagic labeling.

We construct regular handicap distance antimagic graphs for every feasible odd order.

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1 Motivation

The notion of handicap distance antimagic graphs is motivated by a search for round-robin type tournaments in which the weaker teams would have a better chance of winning than in a complete round-robin.

A *complete round robin tournament of n teams* is a tournament in which every team plays the remaining $n - 1$ teams. When the teams are ranked $1, 2, \ldots, n$ according to their standings, that is, the strongest team is ranked 1 and the weakest is ranked n, it is apparent that the sum of rankings of all opponents of the i-th ranked team, denoted $w(i)$, is $w(i) = n(n+1)/2 - i$, and the sequence $w(1), w(2), \ldots, w(n)$ is a decreasing arithmetic progression with difference one. A tournament of n teams in which every team plays precisely r opponents, where $r < n - 1$ and the sequence $w(1), w(2), \ldots, w(n)$ is a decreasing arithmetic progression with difference one is called a *fair incomplete round robin tournament*. This is so because complete round robin tournaments, which obviously satisfy the requirements, are generally considered to be fair.

However, in such a tournament the best team plays the weakest opponents, while the weakest team plays the strongest opponents, which can be undesirable. This property is eliminated in *equalized incomplete round robin tournaments* in which the sum of rankings of all opponents of every team is the same. Some results on existence of fair and equalized incomplete round robin tournaments can be found in [8] and [2].

Even this type of tournament does not give the weaker teams the same chance of winning, simply because if a weak team plays the same opponents as a strong one, it is likely that the total winning record will be worse. Hence, we want the weakest team to play the weakest opponents, while the strongest team should play the strongest ones. That is, the sequence $w(1), w(2), \ldots, w(n)$ should be an increasing arithmetic progression. A tournament in which this condition is satisfied, and every team plays $r < n-1$ games, is called a *handicap incomplete tournament.*

The existence of such tournaments with $n \equiv 0 \pmod 4$ is studied by Froncek and Shepanik in [6] and Kovar and Kovarova [15]. Kovar, Kovarova, and Krajc [14] found such tournaments for $n \equiv 2 \pmod 4$ and $r \leq n - 11$ and proved that they can exist only when $r \equiv 3 \mod 4$ and $r \leq n - 7$. Froncek and Shepanik in [7] completed that result by finding such graphs for $r = n - 7$ with a few exceptions for small n. The exceptional cases for $n \in \{14, 18, 22, 26\}$ were later found computationally by Kovar [13] and for $n = 34, 38$ by Shepanik [18].

For n odd, there was no such classification so far. Constructions based on magic rectangles and magic rectangle sets for non-prime orders were found by the author [3, 4]. This is the first attempt to find such graphs for all odd orders. We present regular handicap graphs with relatively high degrees for all odd orders $n \geq 13, n = 9$ and show that for $n = 3, 5, 7, 11$ no such graphs exist.

2 Definitions

Distance magic labeling was introduced by several sets of authors under at least three different names. Motivated by properties of magic squares, Vilfred [20] introduced the concept of sigma labelings. Miller et al. [17] used the term 1-vertex magic vertex labeling while Sugeng et al. [19] introduced distance magic labeling, which has been most commonly used. A survey on distance magic graphs was published recently [1]. Many newer results can by found in an extensive survey with much wider focus by Gallian [9].

Definition 2.1. A *distance magic labeling* of a graph G of order n is a bijection $f : V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer μ such that $\sum_{y \in N(x)} f(y) = \mu$ for every $x \in V$. The constant μ is called the magic constant of the labeling f. The sum $\sum_{y \in N(x)} f(y)$ is called the weight of the vertex x and is denoted by $w(x)$.

When we identify vertices with their labels (that is, rankings), we can see that a distance magic graph is providing a structure of a fair incomplete tournament described above. It is easy to observe that an incomplete tournament is fair if and only if its complementing tournament is equalized. This type of tournament is related to distance antimagic labeling.

Definition 2.2. A *distance d-antimagic labeling* of a graph $G = (V, E)$ with n vertices is a bijection $f : V \to \{1, 2, \ldots, n\}$ with the property that there exists an ordering of the vertices of G such that the sequence of weights $w(x_1), w(x_2), \ldots, w(x_n)$ forms an arithmetic progression with difference d. When $d = 1$, then f is called just *distance antimagic labeling*. A graph G is a *distance d-antimagic graph* if it allows a distance d-antimagic labeling, and a *distance antimagic graph* when $d = 1$.

The concept of handicap tournaments and handicap distance antimagic labelings was introduced by the author [3] (who called the labeling *ordered distance antimagic*). The term "handicap distance antimagic labeling" was coined by Kovarova [16].

In distance antimagic graphs the weight of a vertex is not required to depend on its own label. We only require that the sequence of weights $w(x_1), w(x_2), \ldots,$ $w(x_n)$ forms an arithmetic progression. To provide models for handicap tournaments, we impose an additional condition on the labeling and require that a vertex with a lower label has a lower weight than a vertex with a higher label.

Definition 2.3. A *handicap distance d-antimagic labeling* of a graph $G = (V, E)$ with n vertices is a bijection $f : V \to \{1, 2, \ldots, n\}$ with the property that $f(x_i) = i$ and the sequence of the weights $w(x_1), w(x_2), \ldots, w(x_n)$ forms an increasing arithmetic progression with difference d. When $d = 1$, the labeling is called just a *handicap distance antimagic labeling* (or a *handicap labeling* for short).

A graph G is a *handicap distance d-antimagic graph* if it allows a handicap distance d-antimagic labeling, and a *handicap distance antimagic graph* or a *handicap graph* when $d = 1$.

Again, if we identify each team in a tournament with its ranking, then an r-regular handicap distance d-antimagic graph is nothing else than a model of a handicap incomplete tournament, since the sum of rankings of opponents of team i is its weight $w(i)$ and the sequence of weights is an increasing arithmetic progression.

Our constructions will be based on the properties of magic rectangles, which are a generalization of magic squares mentioned above.

Definition 2.4. A *magic rectangle* $MR(a, b)$ is an $a \times b$ array whose entries are $1, 2, \ldots, ab$, each appearing once, with all row sums equal to a constant ρ and all column sums equal to a constant σ.

It is easy to observe that a and b must be either both even or both odd. The following existence result was proved by T. Harmuth [11, 12] more than 130 years ago.

Theorem 2.5. [11, 12] *A magic rectangle* $MR(a,b)$ *exists if and only if* $a, b > 1$, $ab > 4$, *and* $a \equiv b \pmod 2$.

However, magic rectangles only allow construction of a relatively narrow class of handicap graphs. For $n = ab$, they only provide graphs with regularity $(a-1)(b-1)$. To construct handicap graphs for a wider spectrum of orders, the author introduced a generalization of magic rectangles, called magic rectangle sets.

Definition 2.6. A *magic rectangle set* $MRS(a, b; c)$ is a collection of c arrays, each of size $a \times b$, whose entries are elements of $\{1, 2, \dots, abc\}$, each appearing once, with all row sums in every rectangle equal to a constant ρ and all column sums in every rectangle equal to a constant σ.

Observe that this generalization is less restrictive than the notion of n-dimensional magic rectangle, which was introduced by Hagedorn in [10]. We present the definition just for a 3-dimensional case, as the higher dimensions are not relevant to our results.

Definition 2.7. An *3-dimensional magic rectangle* 3-$MR(a_1, a_2, a_3)$ is an $a_1 \times a_2 \times a_3$ array with entries d_{i_1,i_2,i_3} which are elements of $\{1, 2, \dots, a_1a_2a_3\}$, each appearing once, such that all sums in the k-th direction are equal to a constant σ_k. That is, we have

$$\sum_{j=1}^{a_1} d_{j,b_2,b_3} = \sigma_1, \sum_{j=1}^{a_2} d_{b_1,j,b_3} = \sigma_2, \sum_{j=1}^{a_3} d_{b_1,b_2,j} = \sigma_3$$

for every selection of indices b_1, b_2, b_3, and $\sigma_k = a_k(a_1a_2a_3 + 1)/2$.

3 Known results

We denote by $H(n, r, d)$ an r-regular handicap distance d-antimagic graph of order n. When $d = 1$, we use just $H(n, r)$. For n even, a complete characterization of such graphs for $d = 1$ follows from results by Kovar, Kovarova, and Krajc [14], Kovar and Kovarova [15], Froncek and Shepanik [6, 7], and Kovar [13].

Theorem 3.1. *Let* $H(n, r)$ *be an* r*-regular handicap graph on* n *vertices. For* $n \equiv 0 \pmod 4$ *an* $H(n, r)$ *exists if and only if* r *is odd and* $3 \le r \le n - 5$. *For* $n \equiv 2 \pmod 4$ *an* $H(n, r)$ *exists if and only if* $r \equiv 3 \pmod 4$ *and* $3 \le r \le n - 7$ *except for* $r = 3$ *and* $n \le 26$.

It is common to denote the Cartesian product of two graphs, H_1 and H_2, by $H_1 \Box H_2$. Also, cH is commonly used to denote the disjoint union of c copies of a given graph H. To avoid confusion, we formally define the disjoint union of uniform Cartesian products of complete graphs. Let $a, b > 1$, $c \geq 1$, then $G = c(K_a \Box K_b)$ with $V(G) = \{v_{i,j}^k \mid 1 \leq i \leq a, 1 \leq j \leq b, 1 \leq k \leq c\}$ and $E(G) = \{v_{i,j}^k v_{i,l}^k \mid 1 \leq i \leq a,\ 1 \leq j < l \leq b,\ 1 \leq k \leq c\} \cup \{v_{i,j}^k v_{l,j}^k \mid 1 \leq i < l \leq a,\ 1 \leq j \leq b,\ 1 \leq k \leq c\}$. Let $\mathcal{M} = \{R^k \mid 1 \leq k \leq c\}$ be a magic rectangle set $MRS(a, b; c)$ with row sums ρ and column sums σ. Then the labeling $f(v_{i,j}^k) = r_{i,j}^k$ is obviously a distance 2-antimagic labeling (except when $a = b = 2$), for when $f(v_{i,j}^k) = r_{i,j}^k = p$, then $w_G(v_{i,j}^k) = \rho + \sigma - 2p$. Hence, the following observation holds. A formal proof can be found in [3].

Observation 3.2. [3] *The graph $G = c(K_a \Box K_b)$ admits a distance 2-antimagic labeling f such that $f(x) = p$ implies $w_G(x) = (a+b)(abc+1)/2 - 2p$ for every $x \in V(G)$ whenever there exists a magic rectangle set $MRS(a, b; c)$.*

In [4] the author made the following observation.

Observation 3.3. [4] *Let G be an r-regular distance 2-antimagic graph with vertices $x_1, x_2, \ldots, x_n$, labeling f and weight function w such that $f(x_i) = i$ and $w(x_i) = k - 2i$ for some constant k. Then $\overline{G}$, the complement of G, is an $(n - r - 1)$-regular handicap graph with labeling f and weight function $\overline{w}$ such that $\overline{w}(x_i) = n(n+1)/2 - k + i$. The converse is obviously also true.*

An obvious consequence for the existence of handicap graphs of odd non-prime orders follows immediately from Theorem 2.5 and Observations 3.2 and 3.3.

Corollary 3.4. *Let n be an odd composite integer, say $n = ab$. Then there exists an $(a-1)(b-1)$-regular handicap graph of order n.*

The following theorem for 3-dimensional magic rectangles of odd order was proved by Hagedorn.

Theorem 3.5. [10] *A 3-dimensional magic rectangle 3-$MR(a_1, a_2, a_3)$ of an odd order $n = a_1 a_2 a_3$ exists whenever $\gcd(a_i, a_j) > 1$ for some $i, j \in \{1, 2, 3\}$.*

This result implies existence of some magic rectangle sets, since slicing a 3-dimensional magic rectangle into single layers in any direction produces magic rectangle sets $MRS(a_i, a_j; a_k)$ for any permutation of $\{i, j, k\} = \{1, 2, 3\}$.

The following is a direct corollary of Theorem 3.5 and Observation 3.2.

Theorem 3.6. *Let a, b, c be positive odd integers such that $1 < a \leq b$, and $q > 1$ divides at least two of a, b, c. Let $n = abc$ and $G = c(K_a \Box K_b)$. Then the complement of G is an $(abc - a - b + 1)$-regular handicap distance antimagic graph with n vertices.*

The author proved the existence of magic rectangle sets for the cases that are not covered by Theorem 3.5 in [5]. Hence, a complete existence characterization of magic rectangle sets of odd order is given.

Theorem 3.7. [5] *Let a, b, c be positive odd integers such that $1 < a \leq b$, and $c \geq 1$. Then a magic rectangle set $MR(a, b; c)$ exists.*

4 New results

Our construction for prime orders will be based on a new generalization of magic rectangle sets. Our sets will consist of rectangles of different sizes, and rather than requiring that in every rectangle the sum of each row is equal to the same constant ρ and the sum of each column equals σ, we require that in each rectangle the sum of any column and any row equals the same constant τ. Obviously, if the rectangles are of the same size, then $\tau = \rho + \sigma$. If they are different, then we still must have within each rectangle the sums of all rows equal, but they may differ if the rectangles have different sizes. The same holds for column sums. We will call such sets *semi-magic rectangle sets.*

An example is shown in Figure 1.

5	10	12
13	6	8
9	11	7

1	16	15	4
17	2	3	14

Figure 1: 6-regular semi-magic rectangle set

Definition 4.1. Let $c > 1$ and (a_i, b_i) be pairs of positive integers for $i = 1, 2, \ldots, c$ such that $1 < a_i \leq b_i$. Set $n = \sum_{i=1}^{c} a_i b_i$ and denote by R^i an $a_i \times b_i$ rectangle with entries $r^i_{j,k}$ for $1 \leq i \leq c, 1 \leq j \leq a_i, 1 \leq k \leq b_i$, where each entry is a distinct element of the set $\{1, 2, \ldots, n\}$. The set of rectangles is called a *semi-magic rectangle set* of order n if in all rectangles the sum of entries in any row and any column is equal to a given constant. More precisely, we have a positive integer τ called a *semi-magic constant* such that

$$\sum_{t=1}^{b_i} r^i_{j,t} + \sum_{s=1}^{a_i} r^i_{s,k} = \tau$$

for every triple (i,j,k). The set is called a *d-regular semi-magic rectangle set* or simply *regular semi-magic rectangle set* if $a_i + b_i = d$ for every $i = 1,2,\ldots,c$ and some positive constant d.

For our purpose of finding handicap graphs of sufficiently large odd orders we need to find regular semi-magic rectangle sets of these orders regardless of the sizes of particular rectangles in the sets. We will show their existence for the cases where one rectangle is of an odd order while the remaining ones are of even orders. As in the case of magic rectangles, the even order rectangles will have both sides even.

First we calculate the semi-magic constant for a given regular semi-magic rectangle set.

Lemma 4.2. *Let $c \geq 1$ and $\mathcal{R} = \{R^i_{a_i\times b_i} \mid i = 1,2,\ldots,c\}$ be a d-regular semi-magic rectangle set of an odd order n with the semi-magic constant τ where $d = 2d'$. Then $\tau = d'(n+1)$.*

Proof. We know that for any $1 \leq i \leq c$, $1 \leq j \leq a_i, 1 \leq k \leq b_i$ we have $\sum_{u=1}^{b_i} r^i_{j,u} + \sum_{v=1}^{a_i} r^i_{v,k} = \tau$. Summing up all such sums for all admissible triples (i,j,k), we count every entry $r^i_{j,k}$ precisely $d = a_i + b_i$ times: Twice in $\sum_{u=1}^{b_i} r^i_{j,u} + \sum_{v=1}^{a_i} r^i_{v,k}$, once in each $\sum_{u=1}^{b_i} r^i_{w,u} + \sum_{v=1}^{a_i} r^i_{v,k}$ for $w \neq j$ (total $a_i - 1$ times), and once in each $\sum_{u=1}^{b_i} r^i_{j,u} + \sum_{v=1}^{a_i} r^i_{v,w}$ for $w \neq k$ (total $b_i - 1$ times). Because the sum of all entries in $\mathcal{R}$ is $n(n+1)/2$ and every entry was counted d times, and we calculated τ for each of the a_ib_i entries in every $R^i_{a_i\times b_i}$, which is $\sum_{i=1}^{c} a_ib_i = n$ times total, we get

$$\tau n = d\frac{n(n+1)}{2} .$$

Substituting $d = 2d'$ and cancelling, we get

$$\tau = d'(n+1)$$

as claimed. □

Now we show how we can extend a regular semi-magic rectangle set of an odd order by adding an extra rectangle of an even order.

Lemma 4.3. *Let $c \geq 1$ and $\mathcal{R} = \{R^i_{a_i\times b_i} \mid i = 1,2,\ldots,c\}$ be a d-regular semi-magic rectangle set of an odd order. Let $a_0 \equiv 0 \pmod 2$, $b_0 \equiv 0 \pmod 4$ and $a_0 + b_0 = d$. Then there exists a d-regular semi-magic rectangle set $\mathcal{Q} = \{Q^i_{a_i\times b_i} \mid i = 0,1,\ldots,c\}$.*

Proof. For $i = 1,2,\ldots,c$ denote the entries of $R^i_{a_i\times b_i}$ by $r^i_{u,v}$ where $1 \leq u \leq a_i$ and $1 \leq v \leq b_i$. Let $a_0 = 2a$ and $b_0 = 4b$.

1	31	3	29	28	6	26	8
32	2	30	4	5	27	7	25
9	23	11	21	20	14	18	16
24	10	22	12	13	19	15	17

Figure 2: 12-regular semi-magic rectangle $R_{4\times 8}$

First we construct a magic rectangle $P^0_{a_0\times b_0}$ with entries $p^0_{s,t}$ as follows. For $s = 1, 2, \ldots, 2a$ and $t = 1, 2, \ldots, 2b$ we define $r^0_{st} = \lfloor\frac{s-1}{2}\rfloor b_0 + t$ when $s+t$ is even and $r^0_{st} = a_0b_0 + 1 - (\lfloor\frac{s-1}{2}\rfloor b_0 + t)$ when $s+t$ is odd. For $s = 1, 2, \ldots, 2a$ and $t = 2b+1, 2b+2, \ldots, 4b$ we define $r^0_{st} = \lfloor\frac{s-1}{2}\rfloor b_0 + t$ when $s+t$ is odd and $r^0_{st} = a_0b_0 + 1 - (\lfloor\frac{s-1}{2}\rfloor b_0 + t)$ when $s+t$ is even. Observe that $P^0_{a_0\times b_0}$ has the semi-magic constant $\tau^0 = (a+2b)(8ab+1)$. An example is shown in Figure 2.

1	7	6	4
8	2	3	5

1	6	8
9	2	4
5	7	3

Figure 3: 6-regular rectangles $P^0_{2\times 4}$ and $R^1_{3\times 3}$

Then we construct $Q^0_{a_0\times b_0}$ by taking $P^0_{a_0\times b_0}$ and increasing the entries equal to $4ab+1, 4ab+2, \ldots, 8ab$ by $\sum_{i=1}^{c} a_ib_i$ each, while keeping the lower entries equal to $1, 2, \ldots, 4ab$ fixed. More formally, $q^0_{s,t} = p^0_{s,t}$ if $p^0_{s,t} \leq 4ab$ and $q^0_{s,t} = p^0_{s,t} + \sum_{i=1}^{c} a_ib_i$ if $p^0_{s,t} > 4ab$. Then we construct the remaining rectangles by setting $q^i_{u,v} = r^i_{u,v} + 4ab$. An example of initial $P^0_{2\times 4}$ and $R^1_{3\times 3}$ is in Figure 3, and resulting $Q^0_{2\times 4}$ and $Q^1_{3\times 3}$ are shown in Figure 4.

Now we need to check that $\mathcal{Q}$ is really a d-regular semi-magic rectangle set. The regularity is obvious, so we just need to check the semi-magic constant.

1	16	15	4
17	2	3	14

5	10	12
13	6	8
9	11	7

Figure 4: 6-regular rectangles $Q^0_{2\times 4}$ and $Q^1_{3\times 3}$

Denote the semi-magic constant of $\mathcal{R}$ by τ. We want to show that $\mathcal{Q}$ has a semi-magic constant $\tau' = \tau + 4abd$. For $i = 1, 2, \ldots, c$ we have for every entry $q^i_{u,v} = r^i_{u,v} + 4ab$. Recall that $\tau = \sum_{v=1}^{b_i} r^i_{j,v} + \sum_{u=1}^{a_i} r^i_{u,k}$ and $d = a_i + b_i$. Then we obtain

$$\begin{aligned}\tau' = \sum_{v=1}^{b_i} q^i_{j,v} + \sum_{u=1}^{a_i} q^i_{u,k} &= \sum_{v=1}^{b_i}(r^i_{j,v} + 4ab) + \sum_{u=1}^{a_i}(r^i_{u,k} + 4ab) \\ &= \sum_{v=1}^{b_i} r^i_{j,v} + \sum_{u=1}^{a_i} r^i_{u,k} + (b_i + a_i)4ab \\ &= \tau + 4abd\end{aligned}$$

Then we examine the row and column sums in $Q^0_{a_0\times b_0}$. First we notice that in each row and each column of $Q^0_{a_0\times b_0}$ exactly half of the entries remained the same as in $P^0_{a_0\times b_0}$, while the other half increased by $\sum_{i=1}^{c} a_i b_i$ each. We know that d must be even, say $d = 2d'$, since have we $a_i \equiv b_i \pmod 2$ for every $i = 0, 1, \ldots, c$. Hence, we have

$$\sum_{t=1}^{b_0} q^0_{j,t} + \sum_{s=1}^{a_0} q^0_{s,k} = \tau^0 + d' \sum_{i=1}^{c} a_i b_i = (a + 2b)(8ab + 1) + d' \sum_{i=1}^{c} a_i b_i \,. \qquad (1)$$

It remains to show that $(a + 2b)(8ab + 1) + d' \sum_{i=1}^{c} a_i b_i = \tau'$. We proved in Lemma 4.2 that

$$\tau = d'(n + 1).$$

It follows that

$$d'n = \tau - d'. \qquad (2)$$

Now substituting (2) into the right-hand side of (1) and recalling that $d = 2d' = 2a + 4b$ and hence $d' = a + 2b$, we obtain

$$\begin{aligned}(a+2b)(8ab+1) + d'n &= d'(8ab+1) + \tau - d' \\ &= 8abd' + \tau \\ &= 4abd + \tau = \tau',\end{aligned}$$

which is precisely what we wanted to show. □

The following theorem is then a straightforward consequence of Lemma 4.3 and can be stated without proof.

Theorem 4.4. *Let $c \geq 1$ and $\mathcal{R} = \{R^i_{a_i \times b_i} \mid i = 1, 2, \ldots, c\}$ be a d-regular semi-magic rectangle set of an odd order. If there exist a and b such that $2a + 4b = d$, then for any $c' > c$ there exists a d-regular semi-magic rectangle set $\mathcal{Q} = \{Q^i_{a'_i \times b'_i} \mid i = 0, 1, \ldots, c'\}$ with $a'_i = a_i$, $b'_i = b_i$ for $i = 1, 2, \ldots, c$ and $a'_i = 2a, b'_i = 4b$ for $i = c+1, c+2, \ldots, c'$.*

Now we can prove an existence result for regular semi-magic rectangle sets of prime orders.

Theorem 4.5. *There exists a regular semi-magic rectangle set of a prime order n if and only if $n = 17$ or $n \geq 31$.*

Proof. First we show that for prime values $n \leq 29$ no such set exists except when $n = 17$. Obviously, as the sets are regular, we cannot have $R^1_{1\times 1}$. Because the order is odd, there must be an odd order rectangle. In particular, it must be one of $R^1_{3\times 3}, R^1_{3\times 5}, R^1_{3\times 7}, R^1_{3\times 9}, R^1_{5\times 5}$. None of them would fit in a set of order 3, 5, or 7. We can also observe that no regular semi-magic rectangle set can contain an $R^i_{2\times 2}$ as the sums of $r^i_{1,1}$ and $r^i_{2,2}$ are different.

For $n = 11$, the odd rectangle would have to be $R^1_{3\times 3}$, which would not leave enough room for another one. For $n = 13$, we would also need $R^1_{3\times 3}$, which would force $R^2_{2\times 2}$, a nonsense. For $n = 19$, $R^1_{3\times 3}$ would imply $d = 6$ and leave us with 10 other entries. As we cannot have $R^2_{2\times 2}$, they would all have to be in one rectangle, namely $R^2_{2\times 5}$, which has $d = 2 + 7 \neq 6$. Also, $R^1_{3\times 5}$ would force $R^2_{2\times 2}$, which is impossible.

For $n = 23$, $R^1_{3\times 3}$ would imply $d = 6$. The only even rectangle with $d = 6$ is $R^2_{2\times 4}$, but we have 14 more entries, which cannot be accommodated. If we have $R^1_{3\times 5}$, then $d = 8$. But there are exactly 8 entries left, which would force $R^2_{2\times 4}$ with $d = 6$, a contradiction. Finally, $R^1_{3\times 7}$ is clearly impossible, as there are only two additional entries.

For $n = 29$, $R^1_{3\times 3}$ forces $d = 6$. The only even rectangle with $d = 6$ is $R^2_{2\times 4}$ and we have 20 more entries, which cannot be accommodated. If we have $R^1_{3\times 5}$, then $d = 8$. We have 14 additional entries, so $R^2_{4\times 4}$ is not a valid option, and we would need $R^2_{2\times 6}$ with 12 entries, which does not use all 14 that need to be used.

Choosing $R^1_{3\times 7}$ gives $d = 10$, but we have only 8 vertices left, so this is impossible. Similarly, $R^1_{5\times 5}$ gives $d = 10$ with only 4 vertices left. Finally, $R^1_{3\times 9}$ or $R^i_{3\times 3}$ for $i = 1, 2, 3$ leaves out only 2 vertices. Therefore, no such set is possible.

The constructive part is divided into cases based on congruence classes modulo 8. Although the theorem is stated only for primes, we construct the sets for all orders, as it is easier to follow.

Case 1: $n \equiv 1 \pmod 8$
We choose $R^1_{3\times 3}$ and $R^i_{2\times 4}$ for $i \geq 2$.

Case 2: $n \equiv 3 \pmod 8$
The cases $n = 11$ and $n = 19$ are impossible as shown above.
For $n \geq 27$, choose $R^i_{3\times 3}$ for $i = 1, 2, 3$ and $R^i_{2\times 4}$ for $i \geq 4$.

Case 3: $n \equiv 5 \pmod 8$
The cases $n = 5$ and $n = 13$ are impossible as shown above.
For $n \equiv 5 \pmod{16}$, choose $R^1_{3\times 7}$ and $R^i_{2\times 8}$ for $i \geq 2$.
For $n \equiv 13 \pmod{16}$, $n = 29$ is impossible as shown above.
When $n \geq 45$, we choose $R^1_{3\times 7}$, $R^2_{4\times 6}$ and $R^i_{2\times 8}$ for $i \geq 3$.

Case 4: $n \equiv 7 \pmod 8$
Subcase 4.1: $n \equiv 7 \pmod{16}$
The case $n = 23$ is impossible as shown above.
For $n = 39$, we use $R^1_{3\times 13}$.
For $n = 55$, we use $R^1_{5\times 11}$.
For $n = 71$, we use $R^i_{3\times 3}$ for $i = 1, 2, \ldots, 7$ and $R^8_{2\times 4}$.
For $n = 87$, we use $R^1_{3\times 29}$.
For $n = 103$, we use $R^1_{3\times 13}$ and $R^2_{8\times 8}$.
For $n = 119$, we use $R^i_{3\times 3}$ for $i = 1, 2, \ldots, 7$ and $R^i_{2\times 4}$ for $i = 8, 9, \ldots, 14$.
For $n \geq 135$, we have $R^i_{3\times 5}$ for $i = 1, 2, \ldots, 9$ and $R^i_{4\times 4}$ for $i \geq 10$.
Subcase 4.2: $n \equiv 15 \pmod{16}$
When $n \equiv 15 \pmod{16}$, we have $R^1_{3\times 5}$ and $R^i_{4\times 4}$ for $i \geq 2$. □

We now need to prove an equivalent of Observation 3.2 for regular semi-magic rectangle sets.

Theorem 4.6. *Let $c \geq 1$ and $\mathcal{R} = \{R^i_{a_i\times b_i} \mid i = 1, 2, \ldots, c\}$ be a d-regular semi-magic rectangle set of an odd order n with the semi-magic constant τ. Let $G = \bigcup_{i=1}^{c} K_{a_i} \square K_{b_i}$ and $\overline{G}$ be the complement of G. Then there exists a distance 2-antimagic labeling f and weight function w of G such that when $f(x) = l$ then $w(x) = \tau - 2l$ and $\overline{G}$ is an $(n - d + 1)$-regular handicap graph with labeling f and weight function $\overline{w}$ such that when $f(x) = l$ then $\overline{w}(x) = n(n+1)/2 - \tau + l$.*

Proof. We denote the vertices in $K_{a_i} \square K_{b_i}$ by $x^i_{j,k}$ for $1 \leq i \leq c$, $1 \leq j \leq a_i$, $1 \leq k \leq b_i$ and define the labeling function as $f(x^i_{j,k}) = r^i_{j,k}$, where $r^i_{j,k}$ is an entry in $R^i_{a_i \times b_i}$. Because the sum of all entries in row j and column k in every rectangle is equal to τ, the sum of labels of vertices in j-th copy of K_{b_i} and k-th copy of K_{a_i} is also τ. The label of the vertex $x^i_{j,k}$, call it l, is counted twice in τ, and hence $w(x^i_{j,k}) = \tau - 2l$.

We know that the sum of all labels in K_n is $n(n+1)/2$, therefore assuming that $f(x^i_{j,k}) = l$ we have

$$\overline{w}(x^i_{j,k}) = n(n+1)/2 - l - (\tau - 2l) = n(n+1)/2 - \tau + l,$$

which we wanted to show. □

The following observation will be useful in our main theorem.

Observation 4.7. *There is no $(n-3)$-regular handicap graph of an odd order n.*

Proof. It is easy to see that in an r-regular handicap graph $H = H(n,r)$ with $f(x_i) = i$ we have $w(x_i) = (r-1)(n+1)/2 + i$. Denote by w' the weight function in $\overline{H}$ under the same labeling. By Observation 3.3, $\overline{H}$ is and $(n-1-r)$-regular distance 2-antimagic graph with $w'(x_i) = (n-r+1)(n+1)/2 - 2i$. For $r = n-3$ we then get $w'(x_i) = 2n+2-2i$ and specifically for x_n it means that $w'(x_n) = 2$. However, this is impossible, as x_n has two neighbors whose labels cannot add up to 2. □

We observe that the above results cover all odd values of n except when $n \in \{11, 13, 19, 23, 29\}$. Kovar [13] used computer assisted search to show that for $n = 11$ no such graph exists and found regular handicap graphs for the remaining values $n = 13, 19, 23, 29$. Based on this, Theorem 4.5 and Theorem 4.6, we can now state our main result.

Theorem 4.8. *There exists a regular handicap graph of an odd order n if and only if $n = 9$ or $n \geq 13$.*

Proof. The existence part follows from [13], Theorems 4.5 and 4.6. Non-existence for $n = 11$ was proved by Kovar [13] and for $n = 3, 5$ was shown in [14]. In the same paper it was shown that no 2-regular handicap graph exists, and by Observation 4.7 an $(n-3)$-regular handicap graph does not exist either. This excludes $n = 7$. □

Recall that for n even, all pairs (n,r) for which there exists an r-regular handicap graph of order n have been characterized. Our result is just a small first step in this direction for odd orders.

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