(r)-Pancyclic, (r)-Bipancyclic and Oddly (r)-Bipancyclic Graphs*

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Abstract

A graph with v vertices is (r)-pancyclic if it contains precisely r cycles of every length from 3 to v. A bipartite graph with even number of vertices v is said to be (r)-bipancyclic if it contains precisely r cycles of each even length from 4 to v. A bipartite graph with odd number of vertices v and minimum degree at least 2 is said to be oddly (r)-bipancyclic if it contains precisely r cycles of each even length from

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4 to $v-1$. In this paper, using a computer search, we classify all (r)-pancyclic and (r)-bipancyclic graphs, $r \geq 2$, with v vertices and at most $v+5$ edges. We also classify all oddly (r)-bipancyclic graphs, $r \geq 1$, with v vertices and at most $v+4$ edges.

1 Definitions

All graphs in this paper are finite, simple and undirected. A *pancyclic* graph, see [1], of order v is a graph that contains cycles of every length from 3 to v. Pancyclic graphs are a generalization of Hamiltonian graphs, which have a cycle containing all the vertices of the graph. A pancyclic graph with exactly one cycle of every possible length is called *uniquely pancyclic* (UPC) (see [4, 3]).

A bipartite graph on v vertices, v even, is called a *uniquely bipancyclic* (UBPC) if it contains precisely one cycle of every even length from 4 to v (see [5]).

The definitions above can be generalized as follows. A graph with v vertices is (r)-*pancyclic* [6] if it contains precisely r cycles of every length from 3 to v. Similarly, a bipartite graph with even number of vertices v is said to be (r)-*bipancyclic* if it contains precisely r cycles of each even length from 4 to v. In the case r = 1, the definitions of (r)-pancyclic and (r)-bipancyclic are precisely the same as that of uniquely pancyclic and uniquely bipancyclic, respectively.

A bipartite graph with odd number of vertices v cannot have a cycle of length v and the largest possible cycle length in such graphs is $v-1$. On the other hand, adding a pendant edge to a graph does not change the cycle structure of that graph. This bring us to the following definition. A bipartite graph with odd number of vertices v and minimum degree at least 2 is said to be *oddly* (r)-*bipancyclic* if it contains precisely r cycles of each even length from 4 to $v-1$.

Every (r)-pancyclic or (r)-bipancyclic graph is Hamiltonian, so we shall represent a v-vertex (r)-pancyclic or (r)-bipancyclic graph as a Hamilton cycle together with some edges, which we shall call *chords*. Similarly, a v-vertex oddly (r)-bipancyclic graph is represented as a cycle C of length $v-1$ together with some edges, which are either chords or are edges at the vertex $u \notin V(C)$, which we shall call *chordettes*.

In this paper, using a computer search, we classify all (r)-pancyclic and (r)-bipancyclic graphs, $r \geq 2$, with v vertices and at most $v+5$ edges. We also classify all oddly (r)-bipancyclic graphs, $r \geq 1$, with v vertices and at most $v+4$ edges.

2 Computer search

The new result presented in the following sections were obtained using different computer programs. The codes for these programs can be found at https://github.com/osawin/Pancyclic. Each of these programs accepts as input a text file containing a positive integer to determine the number of cycles of each length and a list of graph schema. Each schema has a name, and a list of arcs and chords, both represented by a pair of integers denoting their adjacent vertices. The program goes through the schema one by one looking for valid graphs. First, the program determines which edges are arcs, and which are chords or chordettes, based on the formatting of the input. Then, the program finds all cycles within the schema. Next, the program assigns each arc a variable, representing the size of the path it will be replaced with to generate a graph. The program tests all possible combinations of values of these variables to see if it will generate a valid graph. In order to search through graphs faster, the program tests some classes of combinations of arc lengths. For example, the code assigns all possible combinations of even or odd to arcs, and examines which combinations will result in a valid number of even and odd cycle lengths. This allows the code to quickly pass over many non-valid graphs. Once the program has found all valid graphs, it outputs a description of the schema it was checking, whether that layout contains any valid graphs, and the arc lengths that generates valid graphs, if any is found.

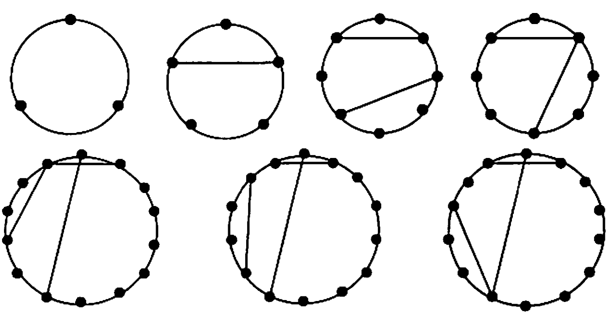


Figure 1: The (1)-pancyclic graphs with three or fewer chords

3 (r)-pancyclic graphs

In [4], Shi classifies all the the UPC graphs with v vertices and $v+m$ edges for $m = 0, 1, 2, 3$ (see Figure 1) and conjectures that:

Conjecture 3.1. *There is no UPC graph with v vertices and $v+m$ edges for $m \geq 4$.*

In [3], using computer programs, Markström reconfirms the results obtained by Shi in [4] and shows that there is no UPC graph with v vertices and $v+5$ edges.

In [6], Zamfirescu studies (2)-pancyclic graphs and finds the two (2)-pancyclic graphs of order 8, which are the smallest (2)-pancyclic graphs. In the same paper, he proves that there exist no (2)-pancyclic graphs on 9 or 10 vertices. For each $v \in \{11, 13, 17, 19\}$ a (2)-pancyclic graph of order v is also presented in [6]. (See Figure 2.)

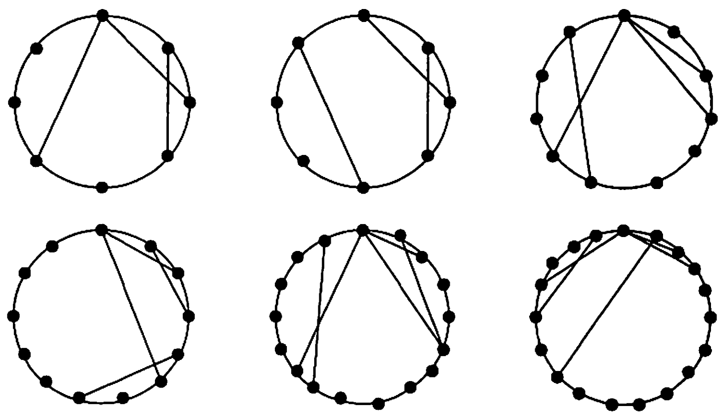


Figure 2: The (2)-pancyclic graphs given in [6], $v = 8, 11, 13, 17, 19$

Computer search shows that:

Theorem 3.2. *Let G be a graph with v vertices and $v+m$ edges, where $0 \leq m \leq 5$.*

1. *The graphs displayed in Figure 3 are the only (2)-pancyclic graphs with v vertices and $v+m$ edges, where $0 \leq m \leq 5$.*

2. *There is no (r)-pancyclic graph, $r \geq 3$, with v vertices and $v+m$ edges, where $0 \leq m \leq 5$.*

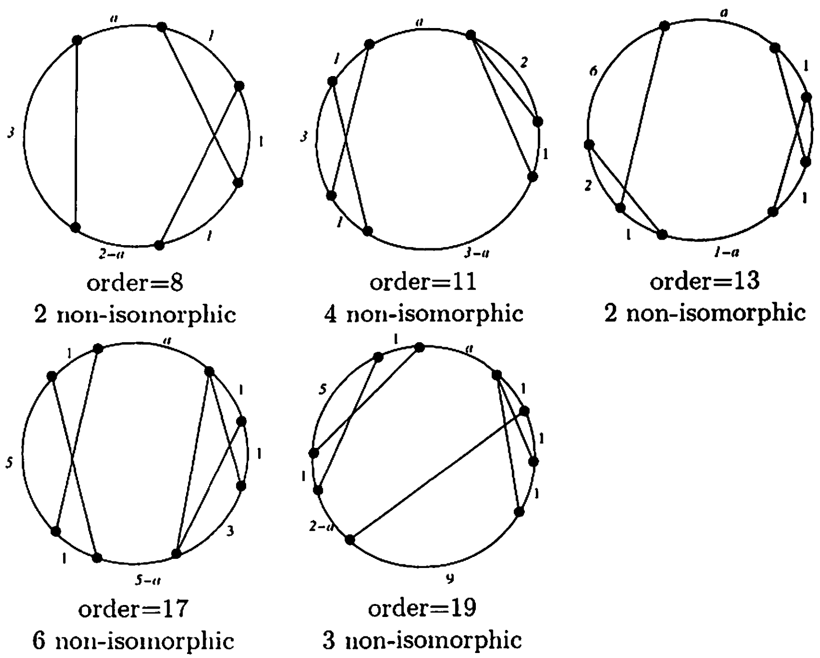


Figure 3: The (2)-pancyclic graphs with five chords or less. The label for an arc indicates the number of edges in that arc.

4 (r)-bipancyclic graphs

Wallis [5] finds all uniquely bipancyclic graphs on at most 30 vertices. These graphs are displayed in Figure 4. In [2], using computer programs, Khodkar et al. show that (see Figure 5):

1. If $32 \leq v \leq 56$, and $v \neq 44$, then there are no UBPC graphs of order v;

2. There are precisely six non-isomorphic UBPC graphs of order 44 (see Figure 5);

3. There are no other UBPC graphs with v vertices and $v + m$ edges for $0 \leq m \leq 5$.

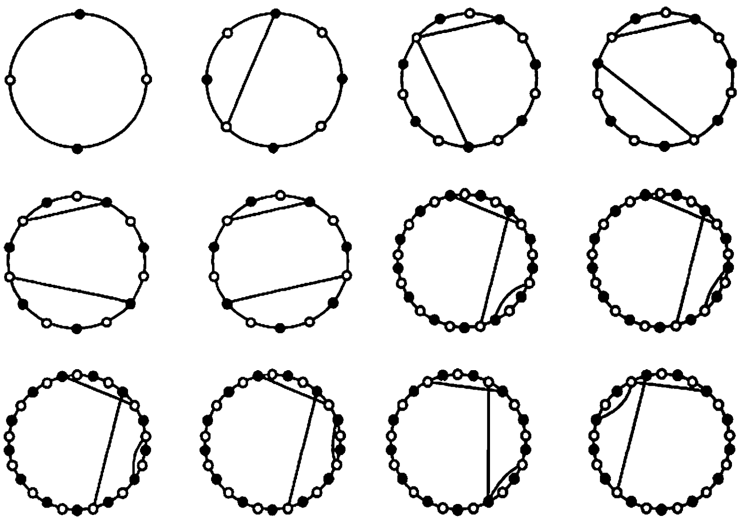


Figure 4: The (1)-bipancyclic graphs of order less than 32

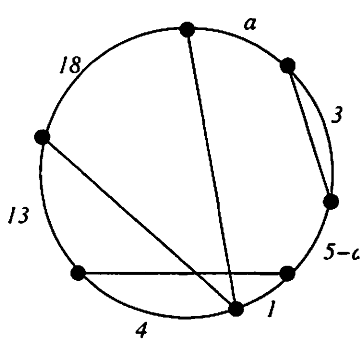
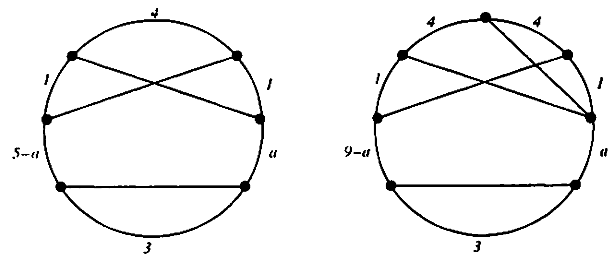


Figure 5: The six non-isomorphic (1)-bipancyclic graphs of order 44

Using a computer search we obtain:

Theorem 4.1. *1. The graphs displayed in Figure 6 are the only (2)-bipancyclic graphs with v vertices and $v+m$ edges, where* $0 \leq m \leq 5$.

2. There is no (r)*-bipancyclic graph,* $r \geq 3$*, with v vertices and $v+m$ edges, where* $0 \leq m \leq 5$.



order=14 | order=22
3 non-isomorphic | 10 non-isomorphic

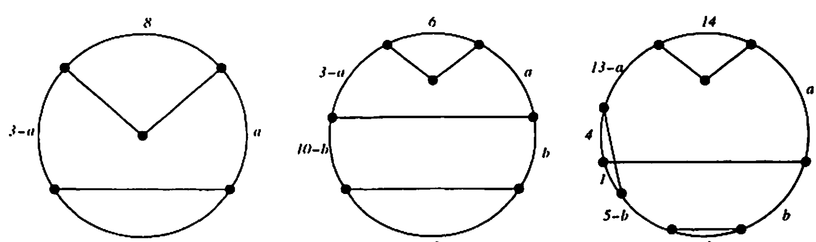
Figure 6: The (2)-bipancyclic graphs with at most four chords

5 Oddly (r)-bipancyclic graphs

In this section we initiate the study of oddly (r)-bipancyclic graphs. These are the bipartite graphs with odd number of vertices v and minimum degree at least 2 which have precisely r cycles of every even length from 4 to $v-1$.

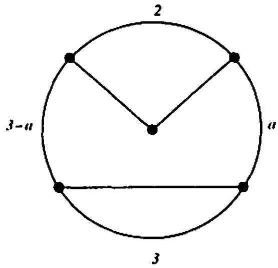
Our computer search shows that:

Theorem 5.1. *The graphs displayed in Figures 7, 8, 9 and 10 are the only oddly (r)-bipancyclic graphs with v vertices and $(v-1)+m$ edges, where $2 \leq m \leq 5$ and $r \geq 1$.*

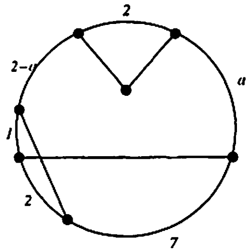


order=15 | order=23 | order=41
2 non-isomorphic | 22 non-isomorphic | 84 non-isomorphic

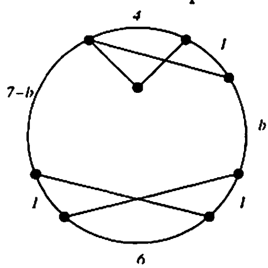
Figure 7: The oddly (1)-bipancyclic graphs with v vertices and $(v-1)+m$ edges, where $m \leq 5$.



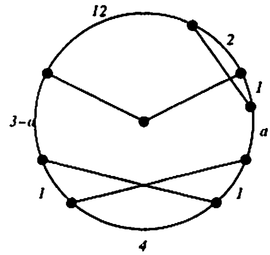
order=9
2 non-isomorphic



order=15
3 non-isomorphic

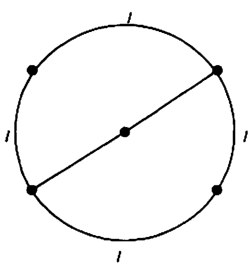


order=21
8 non-isomorphic

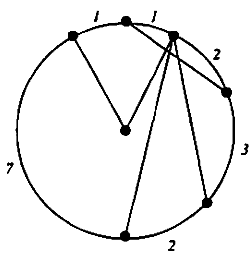


order=25
4 non-isomorphic

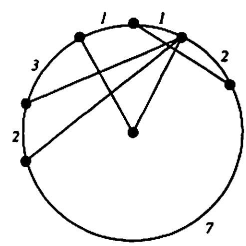
Figure 8: The oddly (2)-bipancyclic graphs with v vertices and $(v-1)+m$ edges, where $m \leq 5$.



order=5

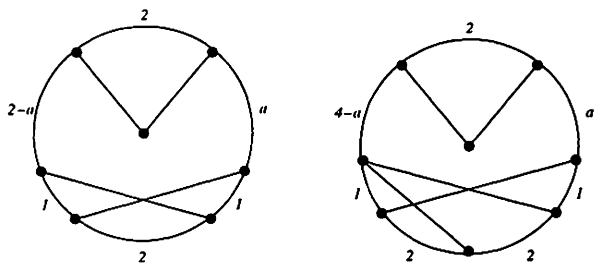


order=17



order=17

Figuer 9: The oddly (3)-bipancyclic graphs with v vertices and $(v-1)+m$ edges, where $m \leq 5$.



order=9
2 non-isomorphic

order=13
5 non-isomorphic

Figure 10: The oddly (4)-bipancyclic graphs with v vertices and $(v-1)+m$ edges, where $m \leq 5$.

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