Super edge-graceful labelings of kites*

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Abstract

A graph G with vertex set V and edge set E is called super edge-graceful if there is a bijection f from E to $\{0, \pm 1, \pm 2, \dots, \pm(|E|-1)/2\}$ when $|E|$ is odd and from E to $\{\pm 1, \pm 2, \dots, \pm |E|/2\}$ when $|E|$ is even such that the induced vertex labeling f^* defined by $f^*(u) = \sum f(uv)$ over all edges uv is a bijection from V to $\{0, \pm 1, \pm 2, \dots, \pm(|V|-1)/2\}$ when $|V|$ is odd and from V to $\{\pm 1, \pm 2, \dots, \pm |V|/2\}$ when $|V|$ is even. A kite is a graph formed by merging a cycle and a path at an endpoint of the path. In this paper, we prove that all kites with $n \geq 5$ vertices, $n \neq 6$, are super edge-graceful.
Keywords: labeling in graphs; edge labeling; super edge-graceful labeling

1 Introduction

In this paper we consider only simple, finite, undirected graphs. A graph of order p and size q is *edge-graceful* [6] if the edges can be labeled by $1, 2, \dots, q$ such that the vertex sums are distinct (mod p). A necessary

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condition for a graph with p vertices and q edges to be edge-graceful is that $q(q+1) \equiv \frac{p(p+1)}{2} \pmod{p}$.

A graph $G=(V,E)$, $|V|=p$ and $|E|=q$, is *super edge-graceful* (SEG) [10] if there is a bijection f from E to $\{0, \pm 1, \pm 2, \ldots, \pm(q-1)/2\}$ when q is odd and from E to $\{\pm 1, \pm 2, \ldots, \pm q/2\}$ when q is even such that the induced vertex labeling f^* defined by $f^*(u) = \sum f(uv)$ over all edges uv is a bijection from V to $\{0, \pm 1, \pm 2, \ldots, \pm(p-1)/2\}$ when p is odd and from V to $\{\pm 1, \pm 2, \ldots, \pm p/2\}$ when p is even.

In [10], the authors showed that super edge-graceful trees are edge-graceful. In particular, they noticed that if a graph G with p vertices and q edges is super-edge graceful and $p|q$, if q is odd, or $p \mid q+1$, if q is even, then G is edge-graceful. Some families of graphs have been shown that to be super edge-graceful. For example, paths of all orders except 2 and 4 and cycles of all orders except 4 and 6 [1] are super edge-graceful, as are complete graphs on $p \geq 5$ vertices [5], complete bipartite graphs except $K_{2,2}$, $K_{2,3}$, and $K_{1,n}$ if n is odd [4], complete tripartite graphs except $K_{1,1,2}$ [3] and trees of odd order with precisely three even vertices [7]. For more information on this topic the reader may also consult [2, 8, 9, 11].

A kite is a graph formed by merging a cycle and a path at an endpoint of the path. In this paper, we prove that all kites with $n \geq 5$ vertices, $n \neq 6$, are super edge-graceful. In addition, we show that a kite with 6 vertices is super edge-graceful if and only if its tail length is 1 or 3. It is easy to see that no kite with 4 vertices is super edge-graceful.

In Section 2, we first present the super edge-graceful labelings (SEGLs) for even cycles given in [1] and will make use of these labelings in Section 5. Then we introduce different SEGLs for even cycles and will employ them in Section 6. In Section 3, we prove that all kites with an odd number of vertices are SEG. In Section 4, we deal with kites with a tail length 1 and prove that such kites with even number of vertices at least 6 are SEG. When the tail length of a kite with even vertices n is at least 2 we transform an n-cycle equipped with a SEGL to the desired kite with a SEGL. For a given kite the existence of such a transformation depends on the SEGL of the n-cycle. In Section 5, we first state our transformation from cycles to kites and then apply it to the SEGLs for even cycles given in [1]. Unfortunately, this does not provide SEGLs for all kites with tail length at least 2. In Section 6, we apply our transformation on n-cycles equipped with the new SEGLs given in Section 2 to settle the remaining kites.

Throughout this paper, we will use the following notation. The vertex set of a graph of order n is $\{1,2,3,\ldots,,n\}$. An n-cycle is written as $C_n = (1,2,3,\ldots,n)$. So the edges are

$$\{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\}.$$

We use the label e_i for the edge $\{i,i+1\}$ and the label v_i for the vertex

i of C_n, where $1 \leq i \leq n$ and addition is modulo n with the residue in $\{1, 2, \ldots, n\}$.

2 Super edge-graceful labelings for even cycles

In this section we first present the construction for the SEGLs of even cycles given in [1]. We make use of these labelings in Section 5.

For $0 \leq i \leq \lfloor \frac{n}{12} \rfloor - 1$, define the 3-tuples a_i^n, b_i^n, c_i^n, d_i^n as follows:

$$\begin{aligned} a_i^n &= (3i+1, \tfrac{-n}{2} + 3i, 3i+2) \\ b_i^n &= (\tfrac{-n}{2} + 3i + 1, 3i+3, \tfrac{-n}{2} + 3i + 2) \\ c_i^n &= (-3i-2, \tfrac{n}{2} - 3i, -3i-1) \\ d_i^n &= (\tfrac{n}{2} - 3i - 2, -3i-3, \tfrac{n}{2} - 3i - 1). \end{aligned}$$

For $n \equiv 0 \pmod{12}$, a SEGL for the cycle C_n is given as follows: the first $n/2$ edges are labeled with the members of a_i^n and b_i^n for $i = 0, 1, 2, \ldots, \frac{n}{12} - 1$. The labels for edges $\{1,2\}, \{2,3\}$ and $\{3,4\}$ are a_0^n, the labels for edges $\{4,5\}, \{5,6\}$ and $\{6,7\}$ are b_0^n and so on. The last $n/2$ edges are labeled with the members of c_i^n and d_i^n for $i = \frac{n}{12} - 1, \frac{n}{12} - 2, \ldots, 0$. The labels for the edges $\{m+1, m+2\}, \{m+2, m+3\}$ and $\{m+3, m+4\}$ are $c_{\frac{n}{12}-1}^n$, the labels for the edges $\{m+4, m+5\}, \{m+5, m+6\}$ and $\{m+6, m+7\}$, are $d_{\frac{n}{12}-1}^n$, where $m = n/2$, and so on.

For $n \equiv 2, 4, 6, 8, 10 \pmod{12}$ define k as follows: $k = 7$ for $n \equiv 2 \pmod{12}$, $k = 8$ for $n \equiv 4 \pmod{12}$, $k = 9$ for $n \equiv 6 \pmod{12}$, $k = 4$ for $n \equiv 8 \pmod{12}$, and $k = 5$ for $n \equiv 10 \pmod{12}$.

The labels for the first $\frac{n}{2} - k$ edges are the members of a_i^n and b_i^n for $i = 0, 1, 2, \ldots, \frac{n-2k}{12} - 1$. The labels for edges $\{m+1, m+2\}, \{m+2, m+3\}, \ldots, \{n-1, n\}$, where $m = \frac{n}{2} + k$, are the members of c_i^n and d_i^n for $i = \frac{n-2k}{12} - 1, \frac{n-2k}{12} - 2, \ldots, 0$. The remaining edge labels are:
for $n \equiv 2 \pmod{12}$,

$$\begin{aligned} (e_{\frac{n}{2}-k+1}, e_{\frac{n}{2}-k+2}, \ldots, e_{\frac{n}{2}+k}) = (&\frac{n-10}{4}, \frac{-(n+14)}{4}, \frac{n-6}{4}, \\ &\frac{-(n+10)}{4}, \frac{n-2}{4}, \frac{-(n+6)}{4}, \frac{n+2}{4}, \frac{-(n-6)}{4}, \frac{n+14}{4}, \\ &\frac{-(n-10)}{4}, \frac{n+6}{4}, \frac{-(n+2)}{4}, \frac{-(n-2)}{4}, \frac{n+10}{4}); \end{aligned}$$

for $n \equiv 4 \pmod{12}$,

$$(e_{\frac{n}{2}-k+1}, e_{\frac{n}{2}-k+2}, \ldots, e_{\frac{n}{2}+k}) = (\frac{n-12}{4}, \frac{-(n+16)}{4}, \frac{n-8}{4}, \frac{-(n+12)}{4}, \frac{n-4}{4}, \frac{-(n+8)}{4}, \frac{n}{4}, \frac{-(n+4)}{4}, \frac{-(n-4)}{4}, \frac{n+8}{4}, \frac{-n}{4}, \frac{n+4}{4}, \frac{-(n-12)}{4}, \frac{n+16}{4}, \frac{-(n-8)}{4}, \frac{n+12}{4});$$

for $n \equiv 6 \pmod{12}$,

$$(e_{\frac{n}{2}-k+1}, e_{\frac{n}{2}-k+2}, \ldots, e_{\frac{n}{2}+k}) = (\frac{n-14}{4}, \frac{-(n+18)}{4}, \frac{n-10}{4}, \frac{-(n+14)}{4}, \frac{n-6}{4}, \frac{-(n+10)}{4}, \frac{n-2}{4}, \frac{-(n+6)}{4}, \frac{n+2}{4}, \frac{-(n-6)}{4}, \frac{n+10}{4}, \frac{-(n-2)}{4}, \frac{-(n+2)}{4}, \frac{n+6}{4}, \frac{-(n-14)}{4}, \frac{n+18}{4}, \frac{-(n-10)}{4}, \frac{n+14}{4});$$

for $n \equiv 8 \pmod{12}$,

$$(e_{\frac{n}{2}-k+1}, e_{\frac{n}{2}-k+2}, \ldots e_{\frac{n}{2}+k}) = (\frac{n-4}{4}, \frac{-(n+8)}{4}, \frac{n}{4}, \frac{-(n+4)}{4}, \frac{-(n-4)}{4}, \frac{n+8}{4}, \frac{-n}{4}, \frac{n+4}{4});$$

and for $n \equiv 10 \pmod{12}$,

$$(e_{\frac{n}{2}-k+1}, e_{\frac{n}{2}-k+2}, \ldots e_{\frac{n}{2}+k}) = (\frac{n-6}{4}, \frac{-(n+10)}{4}, \frac{n-2}{4}, \frac{-(n+6)}{4}, \frac{n+2}{4}, \frac{-(n-6)}{4}, \frac{n+10}{4}, \frac{-(n-2)}{4}, \frac{-(n+2)}{4}, \frac{n+6}{4}).$$

Next we introduce new SEGLs for even n-cycles, $n \geq 18$. We will employ these labelings in Section 6. In what follows the addition is modulo n with the residue in $\{1, 2, 3, \ldots, n\}$.

Construction 2.1. $n \equiv 0 \pmod{12}$, $n \geq 24$.
First label the vertices $\{1, 2, \ldots n\}$ with

$$\begin{array}{lll} v_i = \frac{-n}{2} + i - 1 & \text{for} & 2 \leq i \leq n/2 \\ v_{(n/2)+1} = \frac{-n}{2} & & \\ v_{(n/2)+3i+2} = 3i + 2 & \text{for} & 0 \leq i \leq (n-6)/6 \\ v_{(n/2)+3i+3} = 3i + 1 & \text{for} & 0 \leq i \leq (n-6)/6 \\ v_{(n/2)+3i+4} = 3i + 3 & \text{for} & 0 \leq i \leq (n-6)/6. \end{array}$$

Second, label the edge $\{1,2\}$ with $e_1 = 1$ and the edge $\{i, i+1\}$ with $e_i = v_i - e_{i-1}$ for $2 \le i \le n$. The resulting labeling is a SEGL of an n-cycle, $n \equiv 0 \pmod{12}$.

Construction 2.2. Let $n \equiv 2 \pmod{12}$, $n \ge 26$.
First label the vertices $\{1, 2, \ldots, n\}$ with

$$
\begin{array}{lll}
v_i = \frac{-n}{2} + i - 1 & \text{for} & 2 \le i \le n/2 \\
v_{(n/2)+1} = 2 & & \\
v_{(n/2)+2} = 3 & & \\
v_{(n/2)+3} = 1 & & \\
v_{(n/2)+4} = \frac{-n}{2} & & \\
v_{(n/2)+3i+2} = 3i+1 & \text{for} & 1 \le i \le (n-2)/6 \\
v_{(n/2)+3i+3} = 3i+3 & \text{for} & 1 \le i \le (n-8)/6 \\
v_{(n/2)+3i+4} = 3i+2 & \text{for} & 1 \le i \le (n-8)/6.
\end{array}
$$

Second, label the edge $\{1,2\}$ with $e_1 = 1$ and the edge $\{i, i+1\}$ with $e_i = v_i - e_{i-1}$ for $2 \le i \le n$. The resulting labeling is a SEGL of an n-cycle, $n \equiv 2 \pmod{12}$.

Construction 2.3. $n \equiv 4 \pmod{12}$, $n \ge 28$.
Label the vertices $\{1, 2, \ldots, n\}$ with

$$
\begin{array}{lll}
v_i = \frac{-n}{2} + i - 1 & \text{for} & 2 \le i \le n/2 \\
v_{(n/2)+1} = \frac{-n}{2} & & \\
v_{(n/2)+2} = 3 & & \\
v_{(n/2)+3} = 2 & & \\
v_{(n/2)+4} = 1 & & \\
v_{(n/2)+5} = 4 & & \\
v_{(n/2)+6} = 7 & & \\
v_{(n/2)+7} = 6 & & \\
v_{(n/2)+8} = 5 & & \\
v_{(n/2)+9} = 8 & & \\
v_{(n/2)+3i+1} = 3i+1 & \text{for} & 3 \le i \le (n-4)/6 \\
v_{(n/2)+3i+2} = 3i & \text{for} & 3 \le i \le (n-4)/6 \\
v_{(n/2)+3i+3} = 3i+2 & \text{for} & 3 \le i \le (n-4)/6.
\end{array}
$$

Second, label the edge $\{1,2\}$ with $e_1 = 1$ and the edge $\{i, i+1\}$ with $e_i = v_i - e_{i-1}$ for $2 \le i \le n$. The resulting labeling is a SEGL of an n-cycle, $n \equiv 4 \pmod{12}$.

Construction 2.4. Let $n \equiv 6 \pmod{12}$, $n \ge 18$.
First label the vertices $\{1, 2, \ldots, n\}$ with

$$
\begin{aligned}
v_i &= \tfrac{-n}{2} + i - 1 && \text{for } 2 \le i \le n/2 \\
v_{(n/2)+1} &= 1 \\
v_{(n/2)+2} &= \tfrac{-n}{2} \\
v_{(n/2)+3} &= 2 \\
v_{(n/2)+4} &= 5 \\
v_{(n/2)+5} &= 4 \\
v_{(n/2)+6} &= 3 \\
v_{(n/2)+3i+1} &= 3i && \text{for } 2 \le i \le n/6 \\
v_{(n/2)+3i+2} &= 3i+2 && \text{for } 2 \le i \le (n-6)/6 \\
v_{(n/2)+3i+3} &= 3i+1 && \text{for } 2 \le i \le (n-6)/6.
\end{aligned}
$$

Second, label the edge $\{1,2\}$ with $e_1 = 1$ and the edge $\{i, i+1\}$ with $e_i = v_i - e_{i-1}$ for $2 \le i \le n$. The resulting labeling is a SEGL of an n-cycle, $n \equiv 6 \pmod{12}$, $n \ge 18$.

When $n \equiv 8 \pmod{12}$, $n \ge 20$, we present two different labelings for an n-cycle.

Construction 2.5. Let $n \equiv 8 \pmod{12}$, $n \ge 20$.
(A). First label the vertices $\{1, 2, \ldots, n\}$ with

$$
\begin{aligned}
v_i &= \tfrac{-n}{2} + i - 1 && \text{for } 2 \le i \le n/2 \\
v_{(n/2)+1} &= \tfrac{-n}{2} \\
v_{(n/2)+2} &= 2 \\
v_{(n/2)+3} &= 1 \\
v_{(n/2)+4} &= 3 \\
v_{(n/2)+5} &= 6 \\
v_{(n/2)+6} &= 5 \\
v_{(n/2)+7} &= 4 \\
v_{(n/2)+8} &= 7 \\
v_{(n/2)+3i+3} &= 3i+3 && \text{for } 2 \le i \le (n-8)/6 \\
v_{(n/2)+3i+4} &= 3i+2 && \text{for } 2 \le i \le (n-8)/6 \\
v_{(n/2)+3i+5} &= 3i+4 && \text{for } 2 \le i \le (n-8)/6.
\end{aligned}
$$

Label the edge $\{1,2\}$ with $e_1 = 1$ and the edge $\{i, i+1\}$ with $e_i = v_i - e_{i-1}$ for $2 \le i \le n$. The resulting labeling is a SEGL of an n-cycle, $n \equiv 8 \pmod{12}$, $n \ge 20$.

(B): First label the vertices $\{1, 2, \ldots, n\}$ with

$$
\begin{array}{lll}
v_i = \frac{-n}{2} + i - 1 & \text{for} & 2 \le i \le n/2 \\
v_{(n/2)+1} = \frac{-n}{2} & & \\
v_{(n/2)+3i+2} = 3i + 2 & \text{for} & 0 \le i \le (n-14)/6 \\
v_{(n/2)+3i+3} = 3i + 1 & \text{for} & 0 \le i \le (n-14)/6 \\
v_{(n/2)+3i+4} = 3i + 3 & \text{for} & 0 \le i \le (n-14)/6. \\
v_{n-2} = (n-2)/2 & & \\
v_{n-1} = (n-4)/2 & & \\
v_n = (n-6)/2 & & \\
v_1 = n/2 & &
\end{array}
$$

As before, label the edge $\{1,2\}$ with $e_1 = 1$ and the edge $\{i, i+1\}$ with $e_i = v_i - e_{i-1}$ for $2 \le i \le n$. The resulting labeling is a SEGL of an n-cycle, $n \equiv 8 \pmod{12}$.

Construction 2.6. $n \equiv 10 \pmod{12}$, $n \ge 22$.
First label the vertices $\{1, 2, \ldots, n\}$ with

$$
\begin{array}{lll}
v_i = \frac{-n}{2} + i - 1 & \text{for} & 2 \le i \le n/2 \\
v_{(n/2)+1} = 1 & & \\
v_{(n/2)+2} = \frac{-n}{2} & & \\
v_{(n/2)+3i+3} = 3i + 2 & \text{for} & 0 \le i \le (n-4)/6 \\
v_{(n/2)+3i+4} = 3i + 4 & \text{for} & 0 \le i \le (n-10)/6 \\
v_{(n/2)+3i+5} = 3i + 3 & \text{for} & 0 \le i \le (n-10)/6.
\end{array}
$$

Second, label the edge $\{1,2\}$ with $e_1 = 1$ and the edge $\{i, i+1\}$ with $e_i = v_i - e_{i-1}$ for $2 \le i \le n$. The resulting labeling is a SEGL of an n-cycle, $n \equiv 10 \pmod{12}$.

3 Kites with an odd number of vertices

In [1] it was shown that:

Theorem 3.1. *The path P_n is super edge-graceful for all $n \ge 3$, $n \ne 4$.*

Theorem 3.2. *All kites with an odd number of vertices are super-edge graceful.*

Proof. Let G be a kite with an odd number of vertices $n \ge 5$. Let $e \in G$ be one of the edges on the cycle adjacent to the vertex of degree 3. Then $G \setminus \{e\}$ is a path with n vertices, therefore it has a SEGL by Theorem 3.1. Label the edge e by 0 to obtain a SEGL for G. See Figure 1. □

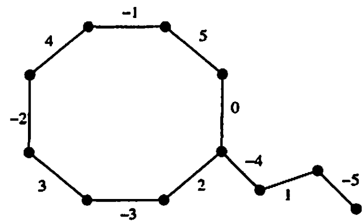

Figure 1: A SEGL for a kite with 11 vertices and a tail length 3

4 Kites with a tail length 1

In this section we present SEGLs for kites with an even number of vertices $n \geq 6$ and a tail length 1.

Theorem 4.1. *Any kite with an even number of vertices $n \geq 6$ and a tail length 1 is super edge-graceful.*

Proof. A kite with n vertices and a tail length 1 consists of a cycle $(1, 2, \ldots, n-1)$ and a pendant edge $\{1, n\}$. We consider four cases.

Case 1: $n \equiv 2 \pmod 4$. First label the vertices $\{1, 2, \ldots, n\}$ with

$$\begin{array}{lll} v_1 = \frac{n-2}{2} & & \\ v_i = \frac{-n}{2} + i - 1 & \text{for} & 2 \leq i \leq n/2 \\ v_{(n/2)+1} = \frac{-n}{2} & & \\ v_{(n/2)+i+1} = i & \text{for} & 1 \leq i \leq (n-4)/2 \\ v_n = \frac{n}{2} & & \end{array}$$

Second, label the edge $\{1, 2\}$ with $e_1 = -n/2$, the edge $\{i, i+1\}$ with $e_i = v_i - e_{i-1}$ for $2 \leq i \leq n-2$, the edge $\{1, n-1\}$ with $e_{n-1} = v_{n-1} - e_{n-2}$ and the pendant edge $\{1, n\}$ with $e_n = n/2$. The resulting labeling is a SEGL of a kite with n vertices and a tail length 1.

Case 2: $n \equiv 0 \pmod{12}$. Figure 2 displays a SEGL for a kite with 12 vertices and a tail length 1.

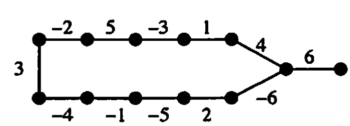

Figure 2: A SEGL for a kite with 12 vertices and a tail length 1

Now let $n \geq 24$. First label the vertices $\{1, 2, \ldots, n\}$ with

$$
\begin{aligned}
v_1 &= \tfrac{n-2}{2} \\
v_i &= \tfrac{-n}{2} + i - 1 && \text{for} \quad 2 \le i \le n/2 \\
v_{(n/2)+1} &= \tfrac{-n}{2} \\
v_{(n/2)+i} &= 5 - i && \text{for} \quad 2 \le i \le 4 \\
v_{(n/2)+5} &= 4 \\
v_{(n/2)+6} &= 6 \\
v_{(n/2)+7} &= 5 \\
v_{(n/2)+3i+2} &= 3i + 1 && \text{for} \quad 2 \le i \le (n-18)/6 \\
v_{(n/2)+3i+3} &= 3i + 3 && \text{for} \quad 2 \le i \le (n-18)/6 \\
v_{(n/2)+3i+4} &= 3i + 2 && \text{for} \quad 2 \le i \le (n-18)/6 \\
v_{(n-4)} &= \tfrac{n-10}{2} \\
v_{(n-3)} &= \tfrac{n-4}{2} \\
v_{(n-2)} &= \tfrac{n-6}{2} \\
v_{(n-1)} &= \tfrac{n-8}{2} \\
v_n &= \tfrac{n}{2}
\end{aligned}
$$

Second, label the edge $\{1,2\}$ with $e_1 = 1$, the edge $\{i, i+1\}$ with $e_i = v_i - e_{i-1}$ for $2 \le i \le n-2$, the edge $\{1, n-1\}$ with $e_{n-1} = v_{n-1} - e_{n-2}$ and the pendant edge $\{1, n\}$ with $e_n = n/2$. The resulting labeling is a SEGL of a kite with n vertices and a tail length 1.

Case 3: $n \equiv 4 \pmod{12}$. First label the vertices $\{1, 2, \ldots, n\}$ with

$$
\begin{aligned}
v_1 &= \tfrac{n-2}{2} \\
v_i &= \tfrac{-n}{2} + i - 1 && \text{for} \quad 2 \le i \le n/2 \\
v_{(n/2)+1} &= \tfrac{-n}{2} \\
v_{(n/2)+2} &= 3 \\
v_{(n/2)+3} &= 2 \\
v_{(n/2)+4} &= 1 \\
v_{(n/2)+3i+2} &= 3i + 1 && \text{for} \quad 1 \le i \le (n-10)/6 \\
v_{(n/2)+3i+3} &= 3i + 3 && \text{for} \quad 1 \le i \le (n-10)/6 \\
v_{(n/2)+3i+4} &= 3i + 2 && \text{for} \quad 1 \le i \le (n-10)/6 \\
v_n &= \tfrac{n}{2}
\end{aligned}
$$

Second, label the edge $\{1,2\}$ with $e_1 = 1$, the edge $\{i, i+1\}$ with $e_i = v_i - e_{i-1}$ for $2 \le i \le n-2$, the edge $\{1, n-1\}$ with $e_{n-1} = v_{n-1} - e_{n-2}$ and the pendant edge $\{1, n\}$ with $e_n = n/2$. The resulting labeling is a SEGL of a kite with n vertices and a tail length 1.

Case 4: $n \equiv 8 \pmod{12}$. First label the vertices $\{1, 2, \ldots, n\}$ with

$$\begin{array}{lll} v_1 = \frac{n-2}{2} & & \\ v_i = \frac{-n}{2} + i - 1 & \text{for} & 2 \le i \le n/2 \\ v_{(n/2)+1} = \frac{-n}{2} & & \\ v_{(n/2)+3i+2} = 3i + 2 & \text{for} & 0 \le i \le (n-8)/6 \\ v_{(n/2)+3i+3} = 3i + 1 & \text{for} & 0 \le i \le (n-8)/6 \\ v_{(n/2)+3i+4} = 3i + 3 & \text{for} & 0 \le i \le (n-14)/6 \\ v_n = \frac{n}{2} & & \end{array}$$

Second, label the edge $\{1,2\}$ with $e_1 = 1$, the edge $\{i, i+1\}$ with $e_i = v_i - e_{i-1}$ for $2 \le i \le n-2$, the edge $\{1, n-1\}$ with $e_{n-1} = v_{n-1} - e_{n-2}$ and the pendant edge $\{1, n\}$ with $e_n = n/2$. The resulting labeling is a SEGL of a kite with n vertices and a tail length 1.

□

5 Kites with a tail length at least 2

In this section we use the SEGLs for even cycles given in [1] (see also Section 2). The following lemma shows that how to transform an n-cycle which is equipped with a SEGL to specific kites with SEGLs.

Lemma 5.1. *Let the n-cycle $(1, 2, \ldots, n)$ have a super edge-graceful labeling. Let e_i be the label of the edge $\{i, i+1\}$ and let v_j be the label of the vertex j such that $e_i = v_j$ for some i and j. Then there exist SEGLs for kites with n vertices and the tail lengths*

1. *$i - j + 1$ and $n - (i - j)$ if $i > j$;*
2. *$j - i$ and $n - (j - i) + 1$ if $j > i$.*

Proof. We only prove Part 1. The proof of Part 2 is similar. Detach the edges $\{i, i+1\}$ and $\{i+1, i+2\}$ at vertex $i+1$ to obtain a path on $n+1$ vertices with a new vertex $(i+1)'$. Identify vertices $i+1$ and j to obtain a kite with a tail length $i - j + 1$. The vertex $(i+1)'$ and edge $\{i, (i+1)'\}$ both have label e_i and the vertex j has label $v_j + e_{i+1}$. The other labels are unchanged. So we have a SEGL for this kite. If we detach the edges $\{i-1, i\}$ and $\{i, i+1\}$ at vertex i and then identify vertices i and j, we will have a SEGL for a kite with a tail length $n - (i - j)$. See Figure 3.

□

Lemma 5.2. *Let $n \equiv 0 \pmod{12}$. Any kite with n vertices and a tail of length t, where $t \notin \{2, \frac{n}{4} + 2, \frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, \frac{3n}{4} - 1\}$, is super edge-graceful.*

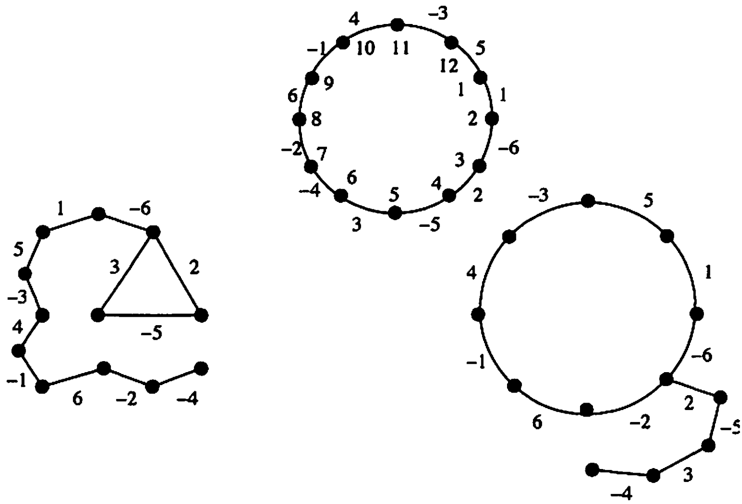

Figure 3: SEGLs for kites with 12 vertices and tail lengths 4 and 9. Note that $e_6 = v_3 = -4$. The number at each vertex is not the induced label for that vertex.

Proof. For a tail of length 1 apply Theorem 4.1. For a tail length of at least 2 we apply Lemma 5.1 on n-cycles, $n \equiv 0 \pmod{12}$, labeled with the SEGLs given in [1] (see also Section 2).

First we note that $v_i = \frac{-n}{2} - 1 + i$ for $2 \leq i \leq \frac{n}{2}$. One can now observe that $v_i = e_{2i}$ for $2 \leq i \leq \frac{n}{4}$. Hence, by Lemma 5.1, a kite with n vertices, where $n \equiv 0 \pmod{12}$, and a tail length $i+1$ or $n-i$ is SEGL, where $2 \leq i \leq \frac{n}{4}$.

Now let $\frac{n}{4} + 1 \leq i \leq \frac{n}{2}$. If $i \equiv 0, 2 \pmod 3$, then $v_i = e_{2i-3}$. So by Lemma 5.1, a kite with n vertices, where $n \equiv 0 \pmod{12}$, and a tail length $i-2$ or $n-i+3$ is SEGL. If $i \equiv 1 \pmod 3$, then $v_i = e_{2i+3}$. So by Lemma 5.1, a kite with n vertices, where $n \equiv 0 \pmod{12}$, and a tail length $i+4$ or $n-i-3$ is SEGL.

Since $i + 4 = (i+6) - 2$, we have generated SEGLs for all kites with $n \equiv 0 \pmod{12}$ vertices and a tail of length $\frac{n}{4} \leq t \leq \frac{n}{2} - 2$ and $t \neq \frac{n}{4} + 2$. Similarly, since $n - i - 3 = n - (i+6) + 3$, we have SEGLs for kites with $n \equiv 0 \pmod{12}$ vertices and a tail of length $\frac{n}{2} + 3 \leq t \leq \frac{3n}{4} + 1$ and $t \neq \frac{3n}{4} - 1$. This completes the proof. □

Recall that for $n \equiv 2, 4, 6, 8, 10 \pmod{12}$ we defined k as follows: $k = 7$ for $n \equiv 2 \pmod{12}$, $k = 8$ for $n \equiv 4 \pmod{12}$, $k = 9$ for $n \equiv 6 \pmod{12}$, $k = 4$ for $n \equiv 8 \pmod{12}$, and $k = 5$ for $n \equiv 10 \pmod{12}$.

Lemma 5.3. *For $n \geq 20$ and $n \equiv 2, 4, 6, 8, 10 \pmod{12}$, a kite with n vertices and a tail of length $3 \leq t \leq \frac{n}{4} - \frac{k}{2} + 1$ or $\frac{3n}{4} + \frac{k}{2} \leq t \leq n - 3$ is super edge-graceful.*

Proof. A closer look at the labelings given in [1] (see also Section 2) for $n \equiv 2, 4, 6, 8, 10 \pmod{12}$ reveals that in all cases, $v_i = \frac{-n}{2} - 1 + i$ for $2 \leq i \leq \frac{n}{2}$. We observe that $v_i = e_{2i}$ for $2 \leq i \leq \frac{n-2k}{4}$. Hence, by Lemma 5.1, a kite with n vertices, where n is even and $n \not\equiv 0 \pmod{12}$, and a tail length $i + 1$ or $n - i$ is SEGL, where $2 \leq i \leq \frac{n-2k}{4}$. So there exists a SEGL for a kite with n vertices and a tail of length $3 \leq t \leq \frac{n-2k}{4} + 1$ or $\frac{3n+2k}{4} \leq t \leq n - 2$. □

Lemma 5.4. *For $n \geq 20$ and $n \equiv 2, 4, 6, 8, 10 \pmod{12}$, a kite with n vertices and a tail of length $\frac{n}{4} + \frac{k}{2} - 1 \leq t \leq \frac{n}{2} - 2$, $t \neq \frac{n}{4} + \frac{k}{2} + 2$, or $\frac{n}{2} + 3 \leq t \leq \frac{3n}{4} - \frac{k}{2} + 2$, $t \neq \frac{3n}{4} - \frac{k}{2} - 1$, is super edge-graceful.*

Proof. Consider the labelings given in [1] (see also Section 2) for n-cycles, $n \equiv 2, 4, 6, 8, 10 \pmod{12}$. Let $\frac{n}{4} + \frac{k}{2} + 1 \leq i \leq \frac{n}{2}$. If $i \equiv k, k + 2 \pmod{3}$, then $v_i = e_{2i-3}$. So by Lemma 5.1, a kite with n vertices, where $n \equiv 2, 4, 6, 8, 10 \pmod{12}$, and a tail length $i - 2$ or $n - i + 3$ is SEGL. If $i \equiv k + 1 \pmod{3}$, then $v_i = e_{2i+3}$. So by Lemma 5.1, a kite with n vertices, where $n \equiv 2, 4, 6, 8, 10 \pmod{12}$, and a tail length $i+4$ or $n-i-3$ is SEGL.

Since $i + 4 = (i + 6) - 2$, we obtain SEGLs for kites with n vertices and a tail of length $\frac{n}{4} + \frac{k}{2} \leq t \leq \frac{n}{2} - 2$, $t \neq \frac{n}{4} + \frac{k}{2} + 2$. Similarly, since $n - j - 3 = n - (j + 6) + 3$, we obtain SEGLs for kites with n vertices and a tail of length $\frac{n}{2} + 3 \leq t \leq \frac{3n}{4} - \frac{k}{2} + 1$, $t \neq \frac{3n}{4} - \frac{k}{2} - 1$.

When $t = n/4 + k/2 - 1$ or $t = 3n/4 - k/2 + 2$ we proceed as follows:
If $n \equiv 2 \pmod{12}$, $e_{n/2+2} = v_{(3n+18)/4} = (n + 14)/4$, which gives tail lengths $(n + 10)/4$ and $(3n - 6)/4$.
If $n \equiv 4 \pmod{12}$, $e_{n/2+3} = v_{n/4+1} = -n/4$, which gives tail lengths $(n + 12)/4$ and $(3n - 8)/4$.
If $n \equiv 6 \pmod{12}$ and $n > 30$, $e_{n/2-4} = v_{(3n-2)/4} = (n - 6)/4$, which gives tail lengths $(n + 14)/4$ and $(3n - 10)/4$.
If $n \equiv 8 \pmod{12}$, $e_{n/2} = v_{n/4} = -(n + 4)/4$, which gives tail lengths $(n + 4)/4$ and $3n/4$.
If $n \equiv 10 \pmod{12}$, $e_{n/2+2} = v_{(3n+14)/4} = (n + 10)/4$, which gives tail lengths $(n + 6)/4$ and $(3n - 2)/4$.

Finally, when $n = 30$ we make use of the SEGL for C_{30} given in Construction 2.4. One can observe that $e_{11} = v_{22} = 6$, which gives tail lengths 11 and 20. This completes the proof. □

It is straightforward to see that kites with 4 vertices and a tail length 1 and kites with 6 vertices and a tail length 2 are not SEG. Figure 4 displays

SEGLs for kites with 6 vertices and tail lengths 1 and 3.

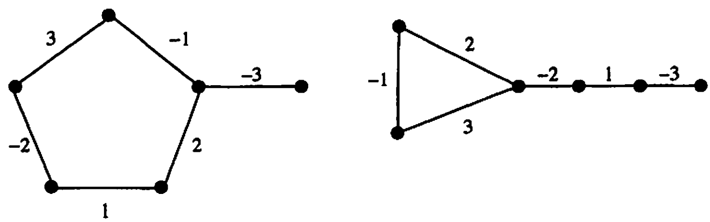

Figure 4: SEGLs for kites with 6 vertices and tail lengths 1, 3

Lemma 5.5. *1. No kite with 4 vertices is super edge-graceful.*

2. A kite with 6 vertices is super edge graceful if and only if its tail length is 1 or 3.

Lemma 5.6. *There are SEGLs for the following kites with n vertices.*

1. $n \equiv 2 \pmod{12}$, *tail lengths* $\frac{n+22}{4}$ *and* $\frac{3n-18}{4}$;
2. $n \equiv 4 \pmod{12}$, *tail lengths* $\frac{n+24}{4}$ *and* $\frac{3n-20}{4}$;
3. $n \equiv 6 \pmod{12}$, *tail lengths* $\frac{n+26}{4}$, $\frac{3n-22}{4}$, $\frac{n-2}{2}$ *and* $\frac{n+4}{2}$;
4. $n \equiv 10 \pmod{12}$, *tail lengths* $\frac{n+18}{4}$, $\frac{3n-14}{4}$, $\frac{n-2}{2}$ *and* $\frac{n+4}{2}$.

Proof. We use the SEGLs given for even cycles in Section [1] (see also Section 2) and apply Lemma 5.1 with the following ingredients.
Proof of 1: Apply Lemma 5.1 with $e_{(n+10)/2} = v_{(n+2)/4} = -\frac{n+2}{4}$ to obtain the tail lengths $\frac{n+22}{4}$ and $\frac{3n-18}{4}$.

Proof of 2: For $n = 16, 28$ and 52, we see that $e_{11} = v_5 = -4$, $e_{27} = v_{12} = -3$ and $e_{13} = v_{32} = 7$, which provide tail lengths of $\{7, 10\}$, $\{16, 13\}$ and $\{34, 19\}$, respectively.

Now let $n \geq 76$ and $n \equiv 4 \pmod{24}$. Then $e_{((n-30)/2)} = v_{(3n-36)/4} = \frac{n-28}{4}$, which gives tail lengths of $(n+24)/4$ and $(3n-20)/4$.

If $n \geq 40$ and $n \equiv 16 \pmod{24}$, then $e_{((n-6)/2)} = v_{((3n+12)/4)} = \frac{n-4}{4}$, which gives tail lengths of $(n+24)/4$ and $(3n-20)/4$.

Proof of 3: First we settle tail lengths $\frac{n+26}{4}$ and $\frac{3n-22}{4}$. We observe that $e_2 = v_{13} = -9$ when $n = 18$ and $e_2 = v_{19} = -15$ when $n = 30$, which provide tail lengths of $\{8, 11\}$ and $\{14, 17\}$ by Lemma 5.1, respectively.

The following table displays a SEGL for a 42-cycle. Since $e_{11} = v_{28} = 6$ we obtain tail lengths 17 and 26 by Lemma 5.1.

i	1	2	3	4	5	6	7	8	9	10	11
e_i	1	-21	2	-20	3	-19	4	-18	5	-17	6
v_i	21	-20	-19	-18	-17	-16	-15	-14	-13	-12	-11
i	12	13	14	15	16	17	18	19	20	21	22
e_i	-16	7	-15	8	-14	9	-13	10	-12	11	-10
v_i	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	1
i	23	24	25	26	27	28	29	30	31	32	33
e_i	-11	13	-8	12	-9	15	-7	14	-5	16	-6
v_i	-21	2	5	4	3	6	8	7	9	11	10
i	34	35	36	37	38	39	40	41	42		
e_i	18	-4	17	-2	19	-3	21	-1	20		
v_i	12	14	13	15	17	16	18	20	19		

Now let $n \geq 54$ and $n \equiv 6 \pmod{12}$. Apply Lemma 5.1 with $e_{(n-20)/2} = v_{(3n-14)/4} = \frac{n-18}{4}$ for tail lengths $\frac{n+26}{4}$ and $\frac{3n-22}{4}$.

For tail lengths $\frac{n-2}{2}$ and $\frac{n+4}{2}$ we have $e_2 = v_{(n+8)/2} = -\frac{n}{2}$. Now the result follows by Lemma 5.1.

Proof of 4: Apply Lemma 5.1 with $e_{(n+8)/2} = v_{(n+2)/4} = -\frac{n+2}{4}$ for tail lengths $\frac{n+18}{4}$, $\frac{3n-14}{4}$ and with $e_2 = v_{(n+8)/2} = -\frac{n}{2}$ for tail lengths $\frac{n-2}{2}$ and $\frac{n+4}{2}$.

□

We are ready to state the main result of this section.

Theorem 5.7. *All kites with $n \geq 8$ vertices are super edge-graceful except for possibly the following cases:*

1. $n \equiv 0 \pmod{12}$ *and the tail lengths*

$$t \in \{2, \frac{n+8}{4}, \frac{n-2}{2}, \frac{n}{2}, \frac{n+2}{2}, \frac{n+4}{2}, \frac{3n-4}{4}\}.$$

2. $n \equiv 2 \pmod{12}$ *and the tail lengths*

$$t \in \{2, \frac{n-6}{4}, \frac{n-2}{4}, \frac{n+2}{4}, \frac{n+6}{4}, \frac{n-2}{2}, \frac{n}{2}, \frac{n+2}{2}, \frac{n+4}{2}, \frac{3n-2}{4}, \frac{3n+2}{4}, \frac{3n+6}{4}, \frac{3n+10}{4}\}.$$

3. $n \equiv 4 \pmod{12}$ *and the tail lengths*

$$t \in \{2, \frac{n-8}{4}, \frac{n-4}{4}, \frac{n}{4}, \frac{n+4}{4}, \frac{n+8}{4}, \frac{n-2}{2}, \frac{n}{2}, \frac{n+2}{2}, \frac{n+4}{2}, \frac{3n-4}{4}, \frac{3n}{4}, \frac{3n+4}{4}, \frac{3n+8}{4}, \frac{3n+12}{4}\}.$$

4. $n \equiv 6 \pmod{12}$ *and the tail lengths*

$$t \in \{2, \frac{n-10}{4}, \frac{n-6}{4}, \frac{n-2}{4}, \frac{n+2}{4}, \frac{n+6}{4}, \frac{n+10}{4}, \frac{n}{2}, \frac{n+2}{2}, \frac{3n-6}{4}, \frac{3n-2}{4}, \frac{3n+2}{4}, \frac{3n+6}{4}, \frac{3n+10}{4}, \frac{3n+14}{4}\}.$$

5. $n \equiv 8 \pmod{12}$ *and the tail lengths*

$$t \in \{2, \frac{n}{4}, \frac{n+16}{4}, \frac{n-2}{2}, \frac{n}{2}, \frac{n+2}{2}, \frac{n+4}{2}, \frac{3n-12}{4}, \frac{3n+4}{4}\}.$$

6. $n \equiv 10 \pmod{12}$ *and the tail lengths*

$$t \in \{2, \frac{n-2}{4}, \frac{n+2}{4}, \frac{n}{2}, \frac{n+2}{2}, \frac{3n+2}{4}, \frac{3n+6}{4}\}.$$

6 SEGLs for the remaining kites

In this section we apply Lemma 5.1 on even cycles equipped with the SEGLs given in Constructions 2.1-2.6 to find SEGLs for the remaining kites.

Lemma 6.1. *Let* $n \equiv 0 \pmod{12}$, $n \geq 24$. *There is a SEGL for a kite with tail length*

$$t \in \{2, \frac{n+8}{4}, \frac{n-2}{2}, \frac{n}{2}, \frac{n+2}{2}, \frac{n+4}{2}, \frac{3n-4}{4}\}.$$

Proof. Consider the SEGL given in Construction 2.1 and apply Lemma 5.1 with $e_n = v_{n-1} = \frac{n-2}{2}$ for a tail length 2, $e_{n/2-3} = v_{3n/4-1} = \frac{n-4}{4}$ for tail lengths $\frac{n}{4}+2$ and $\frac{3n}{4}-1$, $e_2 = v_{n/2+1} = -\frac{n}{2}$ for tail lengths $\frac{n}{2}-1$ and $\frac{n}{2}+2$, and $e_{n-3} = v_{n/2-2} = -3$ for tail lengths $\frac{n}{2}$ and $\frac{n}{2}+1$. This completes the proof. □

Lemma 6.2. *Let* $n \equiv 2 \pmod{12}$, $n \geq 26$. *There is a SEGL for a kite with tail length*

$$t \in \{2, \frac{n-6}{4}, \frac{n-2}{4}, \frac{n+2}{4}, \frac{n+6}{4}, \frac{n-2}{2}, \frac{n}{2}, \frac{n+2}{2}, \frac{n+4}{2}, \frac{3n-2}{4}, \frac{3n+2}{4}, \frac{3n+6}{4}, \frac{3n+10}{4}\}.$$

Proof. Consider the SEGL given in Construction 2.2 and apply Lemma 5.1 with $e_n = v_{n-1} = (n-2)/2$ for a tail length 2, $e_{(n-10)/2} = v_{(n-10)/4} = -\frac{n+14}{4}$ for tail lengths $\frac{n-6}{4}$ and $\frac{3n+10}{4}$, $e_{(n-6)/2} = v_{(n-6)/4} = -\frac{n+10}{4}$ for tail lengths $\frac{n-2}{4}$ and $\frac{3n+6}{4}$, $e_{(n-2)/2} = v_{(n-2)/4} = -\frac{n+6}{4}$ for tail lengths $\frac{n+2}{4}$ and $\frac{3n+2}{4}$, $e_{n/2} = v_{(3n+6)/4} = \frac{n+2}{4}$ for tail lengths $\frac{n+6}{4}$ and $\frac{3n-2}{4}$, $e_1 = v_{(n+6)/2} = 1$ for tail lengths $\frac{n-2}{2}$ and $\frac{n+4}{2}$, and $e_{n-3} = v_{(n-4)/2} = -3$ for tail lengths $\frac{n}{2}$ and $\frac{n+2}{2}$. □

Lemma 6.3. *Let $n \equiv 4 \pmod{12}$, $n \geq 28$. There is a SEGL for a kite with tail length*

$$t \in \{2, \frac{n-8}{4}, \frac{n-4}{4}, \frac{n}{4}, \frac{n+4}{4}, \frac{n+8}{4}, \frac{n-2}{2}, \frac{n}{2}, \frac{n+2}{2}, \frac{n+4}{2}, \frac{3n-4}{4}, \frac{3n}{4}, \frac{3n+4}{4}, \frac{3n+8}{4}, \frac{3n+12}{4}\}.$$

Proof. Consider the SEGL given in Construction 2.3 and apply Lemma 5.1 with $e_n = v_{n-1} = \frac{n-2}{2}$ for a tail length 2, $e_{(n-12)/2} = v_{(n-12)/4} = -\frac{n+16}{4}$ for tail lengths $\frac{n-8}{4}$ and $\frac{3n+12}{4}$, $e_{(n-8)/2} = v_{(n-8)/4} = -\frac{n+12}{4}$ for tail lengths $\frac{n-4}{4}$ and $\frac{3n+8}{4}$, $e_{(n-4)/2} = v_{(n-4)/4} = -\frac{n+8}{4}$ for tail lengths $\frac{n}{4}$ and $\frac{3n+4}{4}$, $e_{n/2} = v_{n/4} = -\frac{n+4}{4}$ for tail lengths $\frac{n+4}{4}$ and $\frac{3n}{4}$, $e_{(n+4)/2} = v_{(3n+16)/4} = \frac{n+8}{4}$ for tail lengths $\frac{n+8}{4}$ and $\frac{3n-4}{4}$, $e_2 = v_{(n+2)/2} = -\frac{n}{2}$ for tail lengths $\frac{n-2}{2}$ and $\frac{n+4}{2}$, and $e_3 = v_{(n+6)/2} = 2$ for tail lengths $\frac{n}{2}$ and $\frac{n+2}{2}$. □

Lemma 6.4. *Let $n \equiv 6 \pmod{12}$, $n \geq 30$. There is a SEGL for a kite with tail length*

$$t \in \{2, \frac{n-10}{4}, \frac{n-6}{4}, \frac{n-2}{4}, \frac{n+2}{4}, \frac{n+6}{4}, \frac{n+10}{4}, \frac{n}{2}, \frac{n+2}{2}, \frac{3n-6}{4}, \frac{3n-2}{4}, \frac{3n+2}{4}, \frac{3n+6}{4}, \frac{3n+10}{4}, \frac{3n+14}{4}\}.$$

Proof. Consider the SEGL given in Construction 2.4 and apply Lemma 5.1 with $e_n = v_{n-1} = \frac{n-2}{2}$ for a tail length 2, $e_{(n+10)/2} = v_{(3n+10)/4} = \frac{n+6}{4}$ for tail lengths $\frac{n-10}{4}$ and $\frac{3n+14}{4}$, $e_{(n-10)/2} = v_{(n-10)/4} = -\frac{n+14}{4}$ for tail lengths $\frac{n-6}{4}$ and $\frac{3n+10}{4}$, $e_{(n-6)/2} = v_{(n-6)/4} = -\frac{n+10}{4}$ for tail lengths $\frac{n-2}{4}$ and $\frac{3n+6}{4}$, $e_{(n-2)/2} = v_{(n-2)/4} = -\frac{n+6}{4}$ for tail lengths $\frac{n+2}{4}$ and $\frac{3n+2}{4}$, $e_{(n+6)/2} = v_{(3n+18)/4} = \frac{n+10}{4}$ for tail lengths $\frac{n+6}{4}$ and $\frac{3n-2}{4}$, $e_{(n+4)/2} = v_{(n+2)/4} = -\frac{n+2}{4}$ for tail lengths $\frac{n+10}{4}$ and $\frac{3n-6}{4}$ and $e_1 = v_{(n+2)/2} = 1$ for tail lengths $\frac{n}{2}$ and $\frac{n+2}{2}$.

□

Lemma 6.5. *Let $n \equiv 8 \pmod{12}$, $n \geq 20$. There is a SEGL for a kite with tail length*

$$t \in \{2, \frac{n}{4}, \frac{n+16}{4}, \frac{n-2}{2}, \frac{n}{2}, \frac{n+2}{2}, \frac{n+4}{2}, \frac{3n-12}{4}, \frac{3n+4}{4}\}.$$

Proof. Consider the SEGL given in Construction 2.5 (A) and apply Lemma 5.1 with $e_n = v_{n-1} = \frac{n-2}{2}$ for a tail length 2, $e_{(n-4)/2} = v_{(n-4)/4} = -\frac{n+8}{4}$ for tail lengths $\frac{n}{4}$ and $\frac{3n+4}{4}$, $e_1 = v_{(n+6)/2} = 1$ for tail lengths $\frac{n-2}{2}$ and $\frac{n+4}{2}$ and $e_7 = v_{(n+14)/2} = 4$ for tail lengths $\frac{n}{2}$ and $\frac{n+2}{2}$.

For tail lengths $\frac{n+16}{4}$ and $\frac{3n-12}{4}$ we use the SEGL given in Construction 2.5 (B) and apply Lemma 5.1 with $e_{(n-14)/2} = v_{(3n-12)/4} = \frac{n-12}{4}$ for tail lengths $\frac{n+16}{4}$ and $\frac{3n-12}{4}$. □

Lemma 6.6. *Let $n \equiv 10 \pmod{12}$, $n \geq 22$. There is a SEGL for a kite with tail length*

$$t \in \{2, \frac{n-2}{4}, \frac{n+2}{4}, \frac{n}{2}, \frac{n+2}{2}, \frac{3n+2}{4}, \frac{3n+6}{4}\}.$$

Proof. We use the SEGL given in Construction 2.6 and apply Lemma 5.1 with $e_n = v_{n-1} = \frac{n-2}{2}$ for a tail length 2, $e_{(n-6)/2} = v_{(n-6)/4} = -\frac{n+10}{4}$ for tail lengths $\frac{n-2}{4}$ and $\frac{3n+6}{4}$, $e_{(n-2)/2} = v_{(n-2)/4} = -\frac{n+6}{4}$ for tail lengths $\frac{n+2}{4}$ and $\frac{3n+2}{4}$ and $e_1 = v_{(n+2)/2} = 1$ for tail lengths $\frac{n}{2}$ and $\frac{n+2}{2}$. □

Lemma 6.7. *Every kite with* 16 *vertices is super edge-graceful.*

Proof. For a tail length 1 apply Theorem 4.1. For the other tail lengths we proceed as follows. Figure 5 displays two different SEGLs for a cycle with 16 vertices. Applying Lemma 5.1 with Graph B and the edge label 7 we obtain a SEGL for a kite with 16 vertices and a tail length 2. For kites with 16 vertices and the other tail lengths apply Lemma 5.1 with Graph A and the edge labels given in the following table.

Edge Label:	−1	−2	−3	−4	−5	7
Tail Lengths:	6, 11	8, 9	4, 13	7, 10	5, 12	3

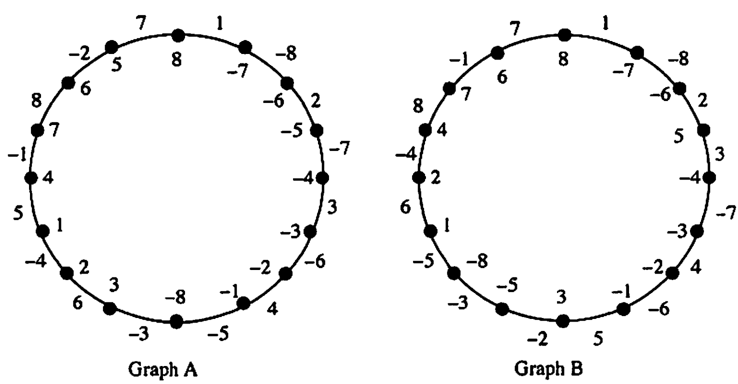

Figure 5: Two different SEGLs for a 16-cycle

□

Lemma 6.8. *The kites on for* $n \in \{8, 10, 12, 14, 18\}$ *vertices are super edge-graceful.*

Proof. For a tail length 1 apply Theorem 4.1. For the other tail lengths we proceed as follows:

For $n = 8$ label the edges of an 8-cycle with 1, -4, 2, -3, -1, 4, -2 and 3 clockwise (or anti clockwise). This is a SEGL for the cycle. Now apply Lemma 5.1 to obtain SEGLs for kites of tail lengths 2, 3, 4, and 5.

For $n = 10$ apply Lemma 5.1 with the SEGL given for a 10-cycle given in [1].

For $n = 12, 14$ apply Lemma 5.1 with the SEGLs given in Constructions 2.1 and 2.2, respectively.

For $n = 18$ apply Lemma 5.1 with the SEGLs given for an 18-cycle in [1] and in Construction 2.4.

□

We summarize our findings with the main theorem of this paper.

Main Theorem 6.9. *All kites on* $n \geq 5$, $n \neq 6$, *vertices are super edge-graceful. Kites on 4 vertices are not super edge-graceful and kites on 6 vertices are super edge-graceful if and only if their tail length is 1 or 3.*

In the future, we wish to consider modified kites with multiple tails and to determine whether they permit a super edge-graceful labeling.

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