GDD($n_1 + n_2, 3; \lambda_1, \lambda_2$) with equal number of blocks of two configurations

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Abstract

A GDD($n_1 + n_2, 3; \lambda_1, \lambda_2$) is a group divisible design with two groups of sizes n_1 and n_2, where $n_1 < n_2$, with block size 3 such that each pair of distinct elements from the same group occurs in λ_1 blocks and each pair of elements from different groups occurs in λ_2 blocks. We prove that the necessary conditions are sufficient for the existence of group divisible designs GDD($n_1+n_2, 3; \lambda_1, \lambda_2$) with equal number of blocks of configuration $(1, 2)$ and $(0, 3)$ for $n_1 + n_2 \leq 20$, $n_1 \neq 2$ and in general for $n_1 = 1, 3, 4, n_2 - 1$, and $n_2 - 2$.

1 Introduction

Definition 1.1. *A group divisible design GDD*$(n, m, k; \lambda_1, \lambda_2)$ *is a collection of k-element subsets, called blocks, of an nm-set X where the elements of X are partitioned into m subsets (called groups) of size n each; pairs of distinct elements within the same group are called first associates of each other and appear together in* λ_1 *blocks while any two elements not in the same group are called second associates and appear together in* λ_2 *blocks.*

Example 1.2. *A GDD*$(3, 2, 4; 3, 2)$ *with two groups* $\{1, 2, 3\}$ *and* $\{4, 5, 6\}$ *is* $\{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3, 6\}, \{4, 5, 6, 1\}, \{4, 5, 6, 2\}, \{4, 5, 6, 3\}\}$.

Note as both the groups are of the same size, every element in the above example occurs in exactly 4 blocks. In fact, in any *GDD* where the groups

*Authors thank the Council for International Exchange of Scholars (CIES) and the United States Department of State's Bureau of Educational and Cultural Affairs for granting Dinesh Sarvate a Fulbright core fellowship and College of Charleston for granting a sabbatical which made this collaboration possible.

are of the same size, every element occurs a fixed number of times. This replication number is usually denoted by r. Also, there are two groups both of the same size and each block has $(1,3)$ configuration, meaning, each block intersects a group in exactly three points or in exactly one point.

The necessary and sufficient conditions for the existence of *GDDs* with block size 3 and $\lambda_1 = 0$ are given below ([1], p.255):

Theorem 1.3. *A GDD$(n, m, 3; \lambda_1 = 0, \lambda_2 = \lambda)$ exists if and only if $m \geq 3$, $\lambda(m-1)n \equiv 0 \pmod 2$, and $\lambda m(m-1)n^2 \equiv 0 \pmod 6$.*

A GDD$(v, k, k; 0, 1)$ is also called a transversal design $TD(k, v)$ and it has v^2 blocks. Group divisible designs are also defined when the group sizes are not equal [10]. We use a simpler notation GDD$(n_1 + n_2, 3; \lambda_1, \lambda_2)$ to represent a group divisible design with block size 3, two groups G_1 and G_2 of distinct sizes n_1 and n_2 where the first associate pairs occur in λ_1 blocks and the second associate pairs occur in λ_2 blocks. It is common to call the blocks of size 3 as triples or triangles. The elements do not have the same replication numbers as the group sizes are different. We denote the replication number for the elements of the first group by r_1 and the replication number of the second group elements by r_2.

Example 1.4. *A GDD$(2+3, 3; 3, 1)$ with two groups of size 2 and 3, say $\{1,2\}$ and $\{3,4,5\}$, is $\{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{3,4,5\},\{3,4,5\},\{3,4,5\}\}$.*

Note here $r_1 = 3$ and $r_2 = 4$. Also, the number of blocks is six where three blocks are of configuration $(0,3)$, meaning these blocks intersect one of the groups in 0 points and the other group in 3 points, and exactly the same number of blocks are of configuration $(1,2)$, meaning these blocks intersect one of the groups in one point and the other group in two points. From now onwards, a GDD with equal number of blocks of configuration (0,3) and (1,2) will be referred to as an EGDD, and blocks with configuration of type (x,y) may be referred to as (x,y)-blocks. GDDs are building blocks for several constructions of other designs including balanced incomplete block designs defined below.

Definition 1.5. *A Balanced Incomplete Block Design, (V, B), is a collection B of b k-subsets (called blocks) of a v-set V, such that each element appears in exactly r of the blocks, every pair of distinct elements of V occurs in λ blocks, and $k < v$. Such a BIBD is usually denoted by BIBD(v, b, r, k, λ) or BIBD(v, k, λ).*

For convenience, $k = v$ is allowed and BIBD(v, v, λ) is used to denote λ copies of the set V.

Definition 1.6. *Suppose (X, A) is a BIBD(v, k, λ). A parallel class in (X, A) is a subset of disjoint blocks from A whose union is X. A parti-*

tion of A into r parallel classes is called a resolution, and (X, A) is called a resolvable BIBD or RBIBD, if A has at least one resolution.

In other words, an RBIBD is a BIBD(v, k, λ) whose blocks can be partitioned into r parallel classes. Observe that a parallel class contains $\frac{v}{k}$ blocks, and therefore a BIBD may have a parallel class only if $v \equiv 0 \pmod{k}$.

A near parallel class is a partial parallel class missing exactly one point. A design is called α-resolvable if its blocks can be partitioned into classes in which each point occurs α times. Similarly one can define resolvability for any block design including GDDs. The following are well known results, these including the definitions from Graph theory are given here for ease of reference only, see [19], [26], [29]:

- A BIBD$(v, 3, 1)$ exists for $v \equiv 1, 3 \pmod{6}$ and has $\frac{v(v-1)}{6}$ blocks.
- A BIBD$(v, 3, 2)$ exists for $v \equiv 0, 1 \pmod{3}$.
- A BIBD$(v, 3, 3)$ exists for all $v \equiv 1 \pmod{2}$.
- A BIBD$(v, 3, 6)$ exists for all $v \geq 3$.

Theorem 1.7. *Necessary conditions for the existence of an α-resolvable BIBD*
(v, k, λ) are
(i) $\lambda(v-1) \equiv 0 \pmod{(k-1)\alpha}$;
(ii) $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$;
(iii) $\alpha v \equiv 0 \pmod{k}$.

Theorem 1.8. *The necessary conditions for the existence of an α-resolvable BIBD$(v, 3, \lambda)$ are sufficient except for $v = 6$, $\alpha = 1$ and $\lambda \equiv 2 \pmod{4}$.*

Corollary 1.9. *(i) A 3-resolvable BIBD$(v, 3, 6)$ exists for all $v \geq 3$, with $(v-1)$ classes.*
(ii) A resolvable BIBD$(v, 3, 1)$ exists for $v \equiv 3 \pmod{6}$.
(iii) A resolvable BIBD$(v, 3, 2)$ exists for $v \equiv 0 \pmod{3}$ except for $v = 6$.

Definition 1.10. *(i) A complete graph K_n is a graph on n vertices where each distinct pair of vertices is connected by an edge.*
(ii) A one-factor is a set of edges from a complete graph in which each point appears only once.
(iii) A one-factorization of a complete graph K_n is the set of one-factors that partitions the edges of the complete graph.
(iv) A 2-factor is a set of edges from a complete graph in which each vertex appears twice.
(v) A 2-factorization of a complete graph K_n is the set of 2-factors that partitions the edges of the complete graph.

Lemma 1.11. *(i) There are* $(n-1)$ *one-factors in a one-factorization of a complete graph* K_n *with* $n \equiv 0 \pmod 2$ *vertices.*
(ii) There are $\frac{n-1}{2}$ *2-factors in a 2-factorization of a complete graph* K_n *with* $n \equiv 1 \pmod 2$ *vertices.*

2 Previous Work

Recently many papers on GDDs with $\lambda_1 \neq 0$ are written. To prove that the necessary conditions are sufficient when the number of groups is less than the block size is especially difficult. In fact, even for the block size 4 where the number of groups $m < 4$ not much has been done. Below we discuss some of the work done recently on this problem:
Clatworthy [2] has listed eleven GDDs with $k = 4$ and $m = 3$ with replication number at most 10. Henson and Sarvate [12] generalized two of these designs. Then, Rodger and Rogers [23] generalized three more designs from the said list. Subsequently in [24], they gave the generalization of another five from that list. Gao and Ge [11] gave general methods of construction of GDDs with $k = 4$ and independently generalized all the eleven designs. Hurd and Sarvate [15] constructed GDDs with $k = 4$ using Bhaskar Rao designs. In this paper they gave necessary and sufficient conditions for these designs for $3 \leq n \leq 8$.
For GDDs with $k = 5$, Hurd, Mishra, and Sarvate [13] have given an explicit construction using MOLS of order n, with $m = 2$ or 3 or 6 groups. These designs are not listed in Clatworthy's table [2] because their parameter range is beyond that of the table. Hurd, Mishra and Sarvate [14] took into account the block-group intersection pattern to construct GDDs with $k = 5$ and $m = 2$ groups. Obviously, there are only three intersection patterns, viz. $(0,5), (1,4)$, and $(2,3)$ with $m = 2$ and $k = 5$, where a block is said to be of type (a,b) or with intersection pattern (a,b) if there are a treatments from one of the groups and b treatments from another.
Keranen and Laffin [17] have constructed GDDs with two groups and block size six. For the block configuration $(s,t) = (3,3)$, they proved that the necessary conditions are sufficient for the existence of GDD$(n, 2, 6; \lambda_1, \lambda_2)$. Further, for GDDs with the configuration $(1,5)$, they gave examples with minimal or near minimal index for group sizes $n \geq 5$, except for $n = 10, 15, 160$ and 190.
Going back to GDDs with block size 4 and two groups, i.e., a GDD$(n, 2, 4; \lambda_1, \lambda_2)$ in which every block intersects each group in exactly two points is called an even GDD while a GDD$(n, 2, 4; \lambda_1, \lambda_2)$ in which each block intersects each group either in one or three points is called an odd GDD. These were first introduced by Hurd and Sarvate [16]. They proved that the necessary conditions for the existence of even and odd GDD$(n, 2, 4; \lambda_1, \lambda_2)$ are sufficient.

Ndungo and Sarvate [20], proved the existence of all but two families of GDDs where number of even blocks is equal to the number of odd blocks. On the other hand, Sarvate and Nanfuka [25] proved that the necessary conditions are sufficient for the existence of two families of GDDs with block size four and two groups where there are equal number of blocks of configurations (2,2), (1,3), and (0,4). They also gave some results when there are three groups and there are equal number of blocks of configurations (2,2) and (1,3). Now a natural question can be raised: What about the existence of GDDs with block size three with two groups BUT unequal cardinality? Recall that for block size 3 where the groups are of the same cardinality can be answered using the results by Fu, Rodger and Sarvate [9] and Fu and Rodger [8] where they completely settle the existence of group divisible designs with $k = 3$. Later another proof was given by Colbourn and Rosa [4]. Hence, in the present paper we concentrate on EGDDs with $k = 3$ and two groups of size n_1 and n_2, where $n_1 < n_2$ and $2 \notin \{n_1, n_2\}$. We must add that several mathematicians, for example, Pabhapote, Punnim, Chaiyasena, Lapchinda, Uiyyasathian [3], [18], [21], [22], [27], [28] and El-Zanati, Punnim, and Rodger [5] have obtained results for block size three with unequal group sizes including the cases where the number of groups is bigger than or equal to the block size 3.

3 Necessary Conditions

We will assume that $n_1 < n_2$. To find the replication number r_1 for elements in G_1, let $x \in G_1$ and suppose it occurs in r_1 triples. Then $\lambda_1(n_1 - 1) + \lambda_2 n_2 = 2r_1$, and $r_1 = \frac{\lambda_1(n_1-1)+\lambda_2 n_2}{2}$. Similarly, if $x \in G_2$ and if it occurs in r_2 triples then we have $\lambda_1(n_2 - 1) + \lambda_2 n_1 = 2r_2$, and $r_2 = \frac{\lambda_1(n_2-1)+\lambda_2 n_1}{2}$.
Hence, necessary conditions include that both $\lambda_1(n_2-1)+\lambda_2 n_1$ and $\lambda_1(n_1-1) + \lambda_2 n_2$ are even and the following holds:
(i) If n_1 is odd and n_2 is even, then λ_1 and λ_2 must be of the same parity.
(ii) If n_1 is even and n_2 is odd, λ_1 and λ_2 are of the same parity.
(iii) If n_1 and n_2 are even, λ_1 is even and
(iv) If n_1 and n_2 are odd, λ_2 is even.

As we want the number of blocks of the configuration $(1,2)$ and $(0,3)$ to be the same, b must be even and max $(n_1, n_2) \geq 3$. To summarize:

Theorem 3.1. *The necessary conditions for the existence of an EGDD($n_1 + n_2, 3; \lambda_1, \lambda_2$) are:*

•

	n_1 *even*	n_1 *odd*
n_2 *even*	λ_1 *even*	λ_1 *and* λ_2 *same parity*
n_2 *odd*	λ_1 *and* λ_2 *same parity*	λ_2 *even*

- *As* $b = n_1 n_2 \lambda_2$ *must be even,* $n_1 n_2 \lambda_2 \equiv 0 \pmod 2$ *and max* $(n_1, n_2) \geq 3$.

We also note that the 2^{nd} associate pairs occur only in the blocks with configuration $(1,2)$ and each such block contains two second associate pairs and one first associate pair. For EGDDs, as there are $n_1 n_2$ second associate pairs and each occurs λ_2 times, $2\frac{b}{2} = b = n_1 n_2 \lambda_2$. Also $3b = n_1 r_1 + n_2 r_2$ and hence $b = \frac{n_1 r_1 + n_2 r_2}{3} = \frac{\lambda_1(n_1(n_1-1)+n_2(n_2-1))+2n_1n_2\lambda_2}{6}$. Therefore, we have $\lambda_1 (n_1(n_1-1) + n_2(n_2-1)) + 2n_1 n_2 \lambda_2 = 6n_1 n_2 \lambda_2$, which implies $\lambda_1 = \frac{4n_1n_2}{n_1(n_1-1)+n_2(n_2-1)}\lambda_2$ or equivalently $\lambda_2 = \frac{(n_1^2+n_2^2-n_1-n_2)}{4n_1n_2}\lambda_1$.

Hence, we want to construct EGDD$(n_1 + n_2, 3; \lambda_1, \frac{(n_1^2+n_2^2-n_1-n_2)}{4n_1n_2}\lambda_1)$ or equivalently EGDD$(n_1 + n_2, 3; \frac{4n_1n_2}{n_1(n_1-1)+n_2(n_2-1)}\lambda_2, \lambda_2)$.

4 General construction

The family for the extreme case of EGDD$(n_1+n_2, 3; \lambda_1, \frac{(n_1^2+n_2^2-n_1-n_2)}{4n_1n_2}\lambda_1)$ is when $\lambda_1 = 4n_1n_2$ and $\lambda_2 = n_1^2 + n_2^2 - n_1 - n_2$. In this section we show that it exists provided $2 \notin \{n_1, n_2\}$. Here after, we assume that no group size is equal to 2.

Theorem 4.1. *A GDD*$(n_1 + n_2, 3; 4n_1n_2, n_1^2 + n_2^2 - n_1 - n_2)$, *for all values of* n_1 *and* n_2, *where* $2 \notin \{n_1, n_2\}$, *exists.*

Proof. The construction of EGDD$(n_1 + n_2, 3; 4n_1n_2, n_1^2 + n_2^2 - n_1 - n_2)$ below will be referred to by "the general construction" in other sections as required.

Let $G_1 = \{1, 2, \cdots, n_1\}$ and $G_2 = \{a_1, a_2, \cdots, a_{n_2}\}$. Use the edges of $n_1 K_{n_1}$ with a_i to construct $(1,2)$ type triples for $i = 1, 2, \cdots, n_2$. Similarly use the edges of $n_2 K_{n_2}$ with $i \in G_1$ to construct $(1,2)$ type triples. Hence the pairs (i, a_j) occur required number of times. The number of $(1,2)$ blocks is $\frac{n_2 n_1^2(n_1-1)}{2} + \frac{n_1 n_2^2(n_2-1)}{2} = \frac{n_1n_2(n_1(n_1-1)+n_2(n_2-1))}{2}$, which is exactly half of the required number of blocks. If BIBD$(n_1, 3, 3n_1n_2)$ and BIBD$(n_2, 3, 3n_1n_2)$ exist we are done as the total number of triples from these BIBDs $\frac{3n_1n_2.n_1(n_1-1)}{6} + \frac{3n_1n_2.n_2(n_2-1)}{6}$ is equal to $\frac{b}{2}$ as required. Now if n_1 or n_2 is even these *BIBDs* exist $3n_1n_2 \equiv 0 \pmod 6$. If n_1 and n_2 both are odd, then also these BIBDs exist, because BIBD$(v, 3, 3)$ exists for odd v's. □

Using Theorem 4.1 and the necessary conditions, we have:

Corollary 4.2. *For the following pairs of* n_1 *and* n_2, *the smallest possible EGDD*$(n_1 + n_2, 3; \lambda_1 = 4n_1n_2, \lambda_2 = n_1(n_1-1) + n_2(n_2-1))$ *exists :*

($n_1 = 3$, $n_2 = 8$), ($n_1 = 3$, $n_2 = 17$), ($n_1 = 4$, $n_2 = 7$), ($n_1 = 4$, $n_2 = 11$), ($n_1 = 5$, $n_2 = 14$), ($n_1 = 7$, $n_2 = 13$), ($n_1 = 8$, $n_2 = 11$).

One of our aims is to construct EGDD$(n_1+n_2, 3; \lambda_1, \frac{(n_1^2+n_2^2-n_1-n_2)}{4n_1n_2}\lambda_1)$ or equivalently EGDD$(n_1+n_2, 3; \frac{4n_1n_2}{n_1(n_1-1)+n_2(n_2-1)}\lambda_2, \lambda_2)$ when $n_1+n_2 \leq 20$, but first we will give some general results.
The following techniques will be used frequently in the next sections: (i) Splitting a block, say $\{x, y, z\}$ with a point w, means creating blocks $\{x, y, w\}$, $\{x, z, w\}$, and $\{y, z, w\}$.
(ii) Using x copies of a graph G with X means creating x copies of triples $e \cup \{i\}$ $\forall e \in E(G)$ and $\forall i \in X$. More specifically, a procedure which will be used often in this paper is "use a set of edges, E, (for example, of a graph or a one-factor) to create triples with an element x," which means create triple $\{i, j, x\}$ for each edge $\{i, j\} \in E$.

5 GDD$(1+n, 3; \lambda_1, \lambda_2)$

For $n_1 = 1$ and $n_2 = n$, the number of blocks $b = n_1n_2\lambda_2 = n\lambda_2$ and as b must be even, we must have $n \equiv 0 \pmod 2$ or $\lambda_2 \equiv 0 \pmod 2$. We also have $\lambda_2 = \frac{(n-1)}{4}\lambda_1$. Let $G_1 = \{a\}$ and $G_2 = \{1, 2, \cdots, n\}$. When n is odd, for $\lambda_1 = 4$ and $\lambda_2 = n-1$, we can easily construct the required EGDDs. The triples $\{i, j, a\}$, $1 \leq i < j \leq n$ (so we are using the edges of K_n with a to get (1,2) triples) and $\frac{n(n-1)}{2}$ blocks of a BIBD$(n, 3, 3)$ together give the required blocks of an EGDD$(1+n, 3; 4, n-1)$.

Example 5.1. *GDD$(1+7, 3; 4, 6)$: We use a K_7 on $G_2 = \{1, 2, \cdots, 7\}$ with $G_1 = \{a\}$, which gives $\lambda_2 = 6$, and the blocks of a BIBD$(7, 3, 3)$, which provides the required number of* (0,3) *type triples.*

Though such a family is easy to construct, it is not enough to prove that the necessary conditions are sufficient. For n odd, $n-1$ is even and the smallest λ_1 need not be 4. For example, if $n = 4t+1$, smallest allowable λ_1 is 1 and corresponding $\lambda_2 = t$, but as 1 and n both are odd, λ_2 has to be even and hence for t odd, EGDD$(1+(4t+1), 3; 1, t)$ does not exist. For t even, such EGDDs exist as we shall see soon.

Example 5.2. *For $t = 2$, an EGDD$(1+9, 3; 1, 2)$ exists as follows. As a resolvable BIBD$(9, 3, 1)$ exists, split the blocks of one of its parallel classes, say $\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}$, to obtain the* (1,2) *triples. These triples together with the remaining* 9 *triples of the RBIBD$(9, 3, 1)$ give the required EGDD.*

If n is odd, then
(i) if $n \equiv 1 \pmod 4$, say $n = 4t+1$, smallest $\lambda_1 = 1$ for t even and smallest

$\lambda_1 = 2$ for t odd.
(ii) if $n \equiv 3 \pmod 4$, then $\frac{(4t+3-1)\lambda_1}{4} = \frac{(4t+2)\lambda_1}{4} = \frac{(2t+1)\lambda_1}{2}$, hence smallest $\lambda_1 = 4$ as λ_2 has to be even as 1 and n both are odd.

Hence for $n = 4t + 3$, smallest such EGDD must be EGDD$(1 + (4t + 3), 3; 4, 2(2t + 1) = 4t + 2)$, which exists by our earlier general construction for any odd n.

If n is even, then, as $\lambda_2 = \frac{(n-1)}{4}\lambda_1$, λ_1 has to be a multiple of 4, but EGDD$(1 + n, 3; 4t, (n - 1)t)$ does not exist for odd t as 1 and n are of opposite parity and $4t$ and $(n - 1)t$ are also of opposite parity. For t even, EGDD$(1 + n, 3; 4t, (n - 1)t)$ exists as EGDD$(1 + n, 3; 8, 2(n - 1))$, can be constructed as follows. A BIBD$(n, 3, 6)$, on G has $n(n - 1)$ blocks. These blocks are of $(0, 3)$ configuration and the $(1, 2)$ configuration triples are obtained by taking 2 copies of $\{i, j, a\}$ $1 \leq i < j \leq n$. It is easy to check that we have the required smallest EGDD, and its multiples give all required EGDDs for n even.

Hence we have following result:

Theorem 5.3. *For n odd and $t \geq 1$, EGDD$(1 + n, 3; 4t, (n - 1)t)$ always exists. Hence, for $n \equiv 3 \pmod 4$, the necessary conditions are sufficient for the existence of EGDD$(1 + n, 3; \lambda_1 = 4t, \lambda_2 = (n - 1)t)$.*
For n even and $t \geq 1$, EGDD$(1+n, 3; 8t, 2(n-1)t)$ always exists and hence the necessary conditions are sufficient. here as well.

Now what remains for $n_1 = 1$ is to construct the required EGDDs for $n = 4t + 1$.

5.0.1 Construction for EGDD$(1 + (4t + 1), 3; 1, t)$ for t even

First we notice the following results: (i) Let $t \equiv 0 \pmod 6$ and $t = 6s$. Then, $4t + 1 = 24s + 1 \equiv 1 \pmod 6 = 6(4s) + 1$. Therefore, there exists a cyclic solution for BIBD$(4t + 1, 3, 1)$. That is, there are $4s$ difference sets which generate BIBD$(4t + 1, 3, 1)$.
(ii) Let $t \equiv 2 \pmod 6$ and $t = 6s + 2$. Then $(4t + 1) = 4(6s + 2) + 1$. Hence a RBIBD$(4t + 1, 3, 1)$ exists with $3(4s + 1) + 1$ parallel classes.
(iii) Let $t \equiv 4 \pmod 6$ and $t = 6s + 4 = 6(4s + 2) + 5$. From Faruqi and Sarvate [6], we know that there are $(4s + 2)$ difference sets covering $12s + 6$ differences modulo $24s + 17$ (out of the possible $12s + 8$ differences). Let $G_1 = \{a\}$ and $G_2 = \{1, 2, \cdots, 4t + 1\}$. Now, if $t \equiv 0 \pmod 6$, then we split the blocks generated by $\frac{t}{3}$ difference sets of BIBD$(4t + 1, 3, 1)$ with a to get the required EGDD. If $t \equiv 2 \pmod 6$, then we split $\frac{t}{2}$ parallel classes of BIBD$(4t + 1, 3, 1)$ with a. If $t \equiv 4 \pmod 6$, we split the blocks generated by $2s$ difference sets with a and also use the edges with the remaining 2 differences not covered by the difference family with a to create $(1, 2)$

type triples. Now for $n = 4t + 1$ when t is odd, we want to construct an EGDD$(1 + (4t + 1), 3; 2, 2t)$.

5.0.2 Construction of EGDD$(1 + (4t + 1), 3; 2, 2t)$ for t odd

If t is odd, the smallest indices λ_1 and λ_2 are 2 and $2t$ respectively. The construction for these EGDDs are exactly the same as in the case for even t.

We observe that if $t = 6s+1$, $4t+1 = 24s+5$, from Faruqi and Sarvate [6] we can get 4s difference sets covering 12s differences, with two differences missing. As $\lambda_1 = 2$, we use two copies of these differences and two copies of the available difference sets. The union of the edges of these two differences with G_1, gives us a contribution of 8 towards λ_2. To get remaining $12s + 2 - 8 = 12s - 6$ value of λ_2, we use $4s - 2$ difference sets, and use each block $\{i, j, k\}$ developed by these difference sets to get three $(1, 2)$ type blocks, viz, $\{i, j, a\}, \{i, k, a\}, \{j, k, a\}$ for the required EGDD, where we recall that $G_1 = \{a\}$. For other cases when $t = 6s + 3$ and $t = 6s + 5$, we have $4t+1 = 6(4s+2)+1$ and $4t+1 = 6(4s+3)+3$ and we can use difference sets for BIBD$(6(4s+2)+1, 3, 2)$ and parallel classes for BIBD$(6(4s+3)+3, 3, 2)$ exactly in the same way as we did for t even case.

Theorem 5.4. *The necessary conditions are sufficient for the existence of EGDD*$(1 + n, 3; \lambda_1, \lambda_2)$ *for all* $n \geq 3$.

6 $n_1 = n_2 - 1$

For an EGDD$((n - 1) + n, 3; \lambda_1, \lambda_2)$ as n_1 and n_2 are of opposite parity, λ_1 and λ_2 must be of the same parity. As we know, $\lambda_2 = \frac{n_1^2+n_2^2-n_1-n_2}{4n_1n_2}\lambda_1$, we have $\lambda_2 = \frac{(n-1)}{2n}\lambda_1$. Necessary conditions imply that when n even, EGDD$((n - 1) + n, 3; 2n, (n - 1))$ can not exist ($2n$ is even and $(n - 1)$ is odd). Hence, the EGDD with the smallest indices is EGDD$((n - 1) + n, 3; 4n, 2(n-1))$. We know that once we construct EGDD with the smallest indies, together with its multiples, we have the result that the necessary conditions are sufficient. For n odd, EGDD$((n - 1) + n, 3; 2n, (n - 1))$ is possible as $(n - 1)$ and $2n$ both are even. Also we need to consider two cases when n is odd as we will see below. We will let $G_1 = \{a_1, \cdots, a_{n-1}\}$ and $G_2 = \{b_1, b_2, \cdots, b_n\}$.

6.1 EGDD$((n - 1) + n; 3; 4n, 2(n - 1))$: even n.

Note, $\lambda_1 - \lambda_2 = 2(n+1)$. Observe first that for $n \equiv 2 \pmod 6$, $2(n+1) \equiv 0 \pmod 6$ and hence a BIBD$(n, 3, 2(n + 1))$ exists. On the other hand, for $n \equiv 0, 4 \pmod 6$, BIBD$(n, 3, 2)$ exists. Hence BIBD$(n, 3, 2(n + 1))$ exists for all even n. Secondly, a BIBD$((n - 1), 3, 4n)$ also exists for all even n as $n - 1$ is odd so $n - 1 \equiv 1, 3, 5 \pmod 6$. Then, if $n - 1 \equiv 1, 3 \pmod 6$, a BIBD$(n - 1, 3, 1)$ exists and if $n - 1 \equiv 5 \pmod 6$, i.e., $n \equiv 0 \pmod 6$, a

BIBD$(n-1, 3, 4n)$ exists.
For the actual construction, we take edges of $2K_n$ with each element of G_1 to create $n(n-1)^2$ triples with configuration $(1, 2)$. Then, the blocks of a BIBD$(n, 3, 2(n+1))$ on G_2 together with the blocks of a BIBD$(n-1, 3, 4n)$ on G_1 give us the required $(0, 3)$ triples.

6.2 EGDD$((n-1)+n; 3; 2n, (n-1))$: odd n.

Note that $2n$ and $n-1$ both are even. Hence this is not a family with smallest indices for all odd n. Indeed, when $n = 2t+1$ and t is odd, the smallest EGDD we need to construct is EGDD$((n-1)+n, 3; n, \frac{(n-1)}{2})$ which will be constructed in the next subsection. Presently, we will prove that EGDD$((n-1)+n, 3; 2n, n-1)$ exists for all odd n.
We observe that when n is odd, BIBD$(n, 3, n+1)$ exists because when $n \equiv 0, 1 \pmod 3$, BIBD$(n, 3, 2)$ exists and when $n \equiv 2 \pmod 3$ i.e., $n \equiv 5 \pmod 6$, $(n+1) \equiv 0 \pmod 6$, a BIBD$(n, 3, n+1)$ exists. Similarly one can check that a BIBD$(n-1, 3, 2n)$ exists for all integers n, because if $n \equiv 1, 2 \pmod 3$, a BIBD$(n-1, 3, 2)$ exists and if $n \equiv 0 \pmod 3$, then $2n \equiv 0 \pmod 6$ and hence BIBD$(n-1, 3, 2n)$ exists. For odd $n > 4$, we use a BIBD$(n, 3, n+1)$ on second group G_2 and a BIBD$(n-1, 3, 2n)$ on the first group G_1 to obtain blocks of configuration $(0, 3)$, and we construct blocks with configuration $(1, 2)$ by using the edges of K_n on G_2 with each element of G_1.
For $n = 3$, the required EGDD is given in the example below:

Example 6.1. *For $n = 3$, the blocks of the required EGDD$(2+3, 3; 6, 2)$ are the 6 blocks of $(1, 2)$-configuration $\{a, b, 1\}, \{a, b, 1\}, \{a, b, 2\}, \{a, b, 2\}, \{a, b, 3\}$ and $\{a, b, 3\}$ together with six copies of the block $\{1, 2, 3\}$ which is of $(0, 3)$-configuration.*

6.3 GDD$((n-1)+n, 3; n, \frac{(n-1)}{2})$, $n = 2t+1$, t odd

Let $n = 4s+3$. We will construct an EGDD$((4s+2)+(4s+3), 3; 4s+3, 2s+1)$.

There are two cases:
Case (a) when $2s+1 \equiv 0, 1 \pmod 3$ and Case (b) when $2s+1 \equiv 2 \pmod 3$.
Construction for Case (a) : It is known ([8], [7]) that a GDD$(2, (2s+1), 3; 0, 1)$, (a GDD with $2s+1$ groups of size 2) for $2s+1 \equiv 0, 1 \pmod 3$ exists. Use 2 copies of the groups of the GDD as one-factors of K_{4s+2} on G_1 as well as the $4s+1$ one-factors of a K_{4s+2} on G_1 with $4s+3$ elements of G_2 and create $(1, 2)$- triples. To complete the requirement for λ_1 for the elements of G_1, we notice that a BIBD$((4s+2), 3, 4s)$ on G_1 exists as (i) $2s+1 \equiv 0 \pmod 3$ implies $(4s+2) \equiv 0 \pmod 6$, and hence a BIBD$((4s+2), 3, 4s)$ exists.
(ii) $2s+1 \equiv 1 \pmod 3$ implies $(4s+2) \equiv 2 \pmod 3$, hence as $4s \equiv 0$

(mod 6) a BIBD($(4s+2), 3, 4s$) exists.
There are $2s(2s+1) = s(4s+2)$ two-factors of a $2sK_{4s+3}$ on G_2. Use s two-factors with each of the elements of G_1. Note λ_2 count for the EGDD has been met. Now, remaining blocks are of a BIBD($4s+3, 3, 2s+3$) on G_2, which exists because $4s+3 \equiv 1, 3 \pmod 6$.

Example 6.2. *An EGDD($(10+11), 3; 11, 5$): Construct a GDD($2, 5, 3; 0, 6$) on G_1. Take six copies of the groups as six one-factors of K_{10} on G_1. Also note there are 27 one-factors of $3K_{10}$. Use three one-factors with each of the elements of G_2. Use 1 two-factor of $2K_{11}$ with each element of G_1 to get remaining $(1,2)$-triples. Hence the count for $\lambda_2 = 3 + 2 = 5$. Use the block of a BIBD($10, 3, 2$) and of the EGDD($2, 5, 3; 0, 6$) on G_1 and of a BIBD($11, 3, 9$) on G_2 as the $(0,3)$-blocks of the required design.*

Now we will look at Case (b). Construction for Case (b): In this case, $2s+1 \equiv 2 \pmod 3$ and hence $s \equiv 2 \pmod 3$. Let $s = 3x+2$. Our aim is to construct an EGDD($(4s+2)+(4s+3), 3; 4s+3, 2s+1$) = EGDD($(12x+10)+(12x+11), 3; 12x+11, 6x+5$).

Note a GDD($2, 2s+1, 3; 0, 6$) exists. Use 6 copies of the groups of the GDD ($2, 2s+1, 3; 0, 6$) on G_1 as one-factors along with $3(4s+1)$ one-factors of $3K_{4s+2}$ on G_1, we have $3(4s+1)+6 = 12s+9$ one-factors, use three with each of the elements of G_2 and create triples with configuration $(1,2)$. We have until now first associate pairs from G_1 occurring 9 times and these blocks contribute 3 towards λ_2. Note that a BIBD($(4s+2), 3, 12x+2$) on G_1 exists, because the necessary conditions for the existence are met as $4s+1 \equiv 3 \pmod 6$. As $2K_{4s+3}$ has $4s+2$ two-factors, $(6x+2)K_{4s+3}$ has $(3x+1)(4s+2)$ two-factors and use $3x+1$ two-factors with each of the elements of G_1. We know that a BIBD($(4s+3), 3, 6x+9$) exists as BIBD($v, 3, 3$) exists for all odd v's. Hence, the remaining blocks are obtained by taking a BIBD($(4s+3), 3, 6x+9$) on G_2.

Hence, we have the result:

Theorem 6.3. *Necessary conditions are sufficient for the existence of EGDD($(n-1)+n, 3; \lambda_1, \lambda_2$) for all integer values of n.*

7 GDD($(n-2)+n, 3; \lambda_1, \lambda_2$) ; $n_2 = n_1 + 2$.

Here, if n is odd, then λ_2 must be even, and if n even then λ_1 must be even. We know that $\lambda_2 = \frac{n_1^2+n_2^2-n_1-n_2}{4n_1n_2}\lambda_1$ therefore, $\lambda_2 = \frac{(n^2-3n+3)}{2n(n-2)}\lambda_1$. Note that (n^2-3n+3) is odd and therefore λ_1 has to be even.
Also note that $\gcd(n^2-3n+3, n(n-2)) = 1$ if $n \equiv 1, 2 \pmod 3$ and $\gcd(n^2-3n+3, n(n-2)) = 3$ if $n \equiv 0 \pmod 3$. (This can be seen easily by using Euclidean algorithm) Therefore, for $n \equiv 1, 2 \pmod 3$ smallest possible λ_1 would be $2n(n-2)$, but n^2-3n+3 is odd and hence for n odd,

we need EGDD$((n-2)+n,3;4n(n-2),2(n^2-3n+3))$ and for even n we need to construct EGDD$((n-2)+n,3;2n(n-2),n^2-3n+3)$. For odd n, the required EGDDs exist by the general construction given in Section 3. Hence, we need to consider only even n.
We will deal with the case of $n \equiv 0 \pmod 3$ where $\gcd(n^2-3n+3,n(n-2))=3$ after the following subsection.

7.1 Construction of EGDD$((n-2)+n,3;2n(n-2),n^2-3n+3)$ n even.

We construct triples with configuration (1,2), by using the edges of $\frac{n-2}{2}K_{n-2}$ with each point from the second group G_2 and the edges of $\frac{n}{2}K_n$ with each point from the first group G_1. The first associates for both groups occur $\frac{n(n-2)}{2}$ times in the process, and second associate pairs occur $\frac{(n-3)(n-2)}{2}+\frac{(n)(n-1)}{2}$ times, which is exactly the value of λ_2. We complete the construction by observing that $\lambda_1-\frac{n(n-2)}{2}=\frac{3n(n-2)}{2}$ and both BIBD$(n,3,\frac{3n(n-2)}{2})$ on G_2 and BIBD$(n-2,3,\frac{3n(n-2)}{2})$ on G_1 exist as for even n as $\frac{3n(n-2)}{2}$ is a multiple of six..

Example 7.1. *For $n=8$, we construct a EGDD$(6+8,3;16\cdot 6=96,64-21=43)$. Use $3K_6$ with each $i\in G_2$, and $4K_8$ with each $i\in G_1$ to create $(1,2)$-triples. For $(0,3)$-triples we use the triples of BIBD$(8,3,96-24)$ $=$ BIBD$(8,3,72)$ and BIBD$(6,3,96-24)$ $=$ BIBD$(6,3,72)$ on G_2 and G_1 respectively. Note that these BIBDs exist.*

7.2 $n \equiv 0 \pmod 3$

We consider two subcases $n \equiv 0 \pmod 6$ and $n \equiv 3 \pmod 6$.

7.2.1 $n \equiv 0 \pmod 6$

We will construct an EGDD$((6t-2)+6t,3;4t(6t-2),12t^2-6t+1)$. Use the edges of K_{6t-2} on G_1 with each point of G_2. This means second associate pairs will occur $(6t-3)$ times. Note required $\lambda_2-(6t-3)=2(6t^2-6t+2)$ is even. Hence, we split the triples of $6t^2-6t+2$ parallel classes of a BIBD$(6t,3,4t(6t-2))$ with each element of G_1. Remaining triples will, as usual, play the role of $(0,3)$-triples along with the triples of BIBD$(6t-2,3,4t(6t-2)-6t)$ on G_1.

7.3 $n \equiv 0 \pmod 3$

Let $n=6t+3$. The GDD with the smallest indices should be EGDD$((6t+1)+(6t+3),3;2(2t+1)(6t+1),12t^2+6t+1)$, but λ_2 has to be even as n is odd. Therefore, the smallest EGDD in this case is EGDD$((6t+1)+(6t+3),3;4(2t+1)(6t+1),2(12t^2+6t+1))$. As λ_2 is even and there are enough parallel classes in a BIBD$(6t+3,3,4(2t+1)(6t+1))$ on G_2, we split $12t^2+6t+1$ parallel classes with each element of G_1. As usual, the remaining triples and the triples of BIBD($6t+1, 3, 4(2t+1)(6t+1)$) are the required $(0,3)$-triples for the EGDD. As we have constructed EGDDs with the smallest possible indices, we have

Theorem 7.2. *Necessary conditions are sufficient for the existence of EGDD($(n-2)+n, 3; \lambda_1, \lambda_2$) with equal number of blocks of both configurations* $(0,3)$ *and* $(1,2)$.

8 EGDD($3+n, 3; \lambda_1, \lambda_2$)

In this section, we assume $G_1 = \{a, b, c\}$ and $G_2 = \{1, ..., n_2 = n\}$. In addition, as in the earlier sections, when we split some of the blocks of a BIBD, it is assumed that the rest of the blocks of the BIBD are used as the $(0,3)$-triples.

Let $n = 12t$ for some integer $t \equiv 0, 1 \pmod 3$. In this case, we need to construct EGDD($3+12t, 3; 48t, 48t^2-4t+2$). We split $24t^2-2t+1$ resolvable classes of an RBIBD($12t, 3, 48t$) with each element of G_1. In addition, we take the blocks of a BIBD($3, 3, 48t$).

Let $n = 36t+24$ for some integer t. In this case, we need to construct EGDD($3+(36t+24), 3; 48t+32, 144t^2+188t+62$). We split $72t^2+94t+31$ resolvable classes of an RBIBD($36t+24, 3, 48t+32$) with each element of G_1. In addition, we take the blocks of a BIBD($3, 3, 48t+32$).

Let $n = 12t+1$ for some integer t. In this case, we need to construct EGDD($3+(12t+1), 3; 48t+4, 48t^2+4t+2$). We split $8t^2$ 3-resolvable classes of 3-RBIBD($12t+1, 3, 48t$) with each element of G_1. We also take $2t$ 2-factors of K_{12t+1} with each element of G_1 and 1 copy of K_3 with each element of G_2. In addition, we take the blocks of a BIBD($3, 3, 36t+3$) on G_1 and the blocks of a BIBD($12t+1, 3, 3$) on G_2.

Let $n = 48t+2$ for some integer t. In this case, we need to construct EGDD($3+(48t+2), 3; 72t+3, 288t^2+18t+1$). We split $48t^2-5t$ 3-resolvable classes of 3-RBIBD($48t+2, 3, 72t$) with each element of G_1. We also take 1 copy of K_{48t+2} with each element of G_1. In addition, we take the blocks of a BIBD($3, 3, 72t+3$).

Let $n = 12t+2$ for some integer $t \equiv 2 \pmod 4$. In this case, we need to construct EGDD($3+(12t+2), 3; 36t+6, 36t^2+9t+2$). We split $\frac{12t^2+3t}{2}$ 3-resolvable classes of a 3-RBIBD($12t+2, 3, 36t+6$) with each element of G_1. We also take one copy of K_3 with each element of G_2. In addition, we take the blocks of a BIBD($3, 3, 24t+4$).

Let $n = 12t+2$ for some integer $t \equiv 1 \pmod 2$. In this case, we need to construct EGDD($3+(12t+2), 3; 72t+12, 72t^2+18t+4$). We split $12t^2+3t$ 3-resolvable classes of a 3-RBIBD($12t+2, 3, 72t+12$) with each element of G_1. We also take 2 copies of K_3 with each element of G_2. In addition, we take the blocks of a BIBD($3, 3, 48t+8$).

Let $n = 12t+3$ for some integer $t \equiv 0, 2 \pmod 6$. In this case, we need to construct EGDD$(3+(12t+3), 3; 24t+6, 24t^2+10t+2)$. We split $12t^2+5t+1$ parallel classes of RBIBD$(12t+3, 3, 24t+6)$ with each element of G_1. In addition, we take the blocks of a BIBD$(3, 3, 24t+6)$.
Let $n = 12t+3$ for some integer $t \equiv 3, 5 \pmod 6$. In this case, we need to construct EGDD$(3+(12t+3), 3; 12t+3, 12t^2+5t+1)$. We split $\frac{12t^2+5t+1}{2}$ parallel classes of RBIBD$(12t+3, 3, 12t+3)$ with each element of G_1. In addition, we take the blocks of a BIBD$(3, 3, 12t+3)$.
Let $n = 36t+15$ for some integer $t \equiv 0 \pmod 2$. In this case, we need to construct EGDD$(3+(36t+15), 3; 12t+5, 36t^2+29t+6)$. We split $\frac{36t^2+29t+6}{2}$ parallel classes of RBIBD$(36t+15, 3, 12t+5)$ with each element of G_1. In addition, we take the blocks of a BIBD$(3, 3, 12t+5)$.
Let $n = 36t+15$ for some integer $t \equiv 1 \pmod 2$. In this case, we need to construct EGDD$(3+(36t+15), 3; 24t+10, 72t^2+58t+12)$. We split $36t^2+29t+6$ parallel classes of RBIBD$(36t+15, 3, 24t+10)$ with each element of G_1. In addition, we take the blocks of a BIBD$(3, 3, 24t+10)$.
Let $n = 12t+4$ for some integer t. In this case, we need to construct EGDD$(3+(12t+4), 3; 48t+16, 48t^2+28t+6)$. We split $8t^2+2t$ 3-resolvable classes of 3-RBIBD$(12t+4, 3, 48t+12)$ with each element of G_1. We also take $16t+4$ one-factors of $4K_{12t+4}$ with each element of G_1 and 1 copy of K_3 with each element of G_2. In addition, we take the blocks of a BIBD$(3, 3, 36t+12)$.
Let $n = 12t+5$ for some integer t. In this case, the required EGDD to construct EGDD$(3+(12t+5), 3; 144t+60, 144t^2+108t+26)$. However, since in this case $\lambda_1 = 4n_1n_2$, such an EGDD has been proven to exist by Theorem 3.1.
Let $n = 72t+6$ for some integer t. In this case, we need to construct EGDD$(3+(72t+6), 3; 48t+4, 288t^2+44t+2)$. We split $144t^2+22t+1$ parallel classes of RBIBD$(72t+6, 3, 48t+4)$ with each element of G_1. In addition, we take the blocks of a BIBD$(3, 3, 48t+4)$.
Let $n = 144t+18$ for some integer t. In this case, we need to construct EGDD$(3+(144t+18), 3; 72t+9, 864t^2+210t+13)$. We split $432t^2+33t-2$ parallel classes of RBIBD$(144t+18, 3, 72t+6)$ with each element of G_1. We also take 1 copy of $K_{144t+18}$ with each element of G_1. In addtion, we take the blocks of a BIBD$(3, 3, 72t+9)$.
For the case of $t = 0$, $n = 18$, we need to construct EGDD$(3+18, 3; 9, 13)$. We split 2 3-resolvable class of 3-RBIBD$(18, 3, 6)$ with each element of G_1. We also take the blocks of a BIBD$(3+18, 3, 1)$. In addition, we take the blocks of a BIBD$(3, 3, 8)$ and the blocks of a BIBD$(18, 3, 2)$.
Let $n = 144t+66$ for some integer t. In this case, we need to construct EGDD$(3+(144t+66), 3; 72t+33, 864t^2+786t+179)$. We split $432t^2+321t+57$ parallel classes of RBIBD$(144t+66, 3, 72t+30)$ with each

element of G_1. We also take 1 copy of $K_{144t+66}$ with each element of G_1 in addition, we take the blocks of a BIBD$(3, 3, 72t + 33)$.
Let $n = 12t+6$ for some integer $t \equiv 7, 11 \pmod{12}$. In this case, we need to construct EGDD$(3+(12t+6), 3; 12t+6, 12t^2+11t+3)$. We split $\frac{12t^2+11t+3}{2}$ parallel classes of RBIBD$(12t + 6, 3, 12t + 6)$ with each element of G_1. In addition, we take the blocks of a BIBD$(3, 3, 12t + 6)$.
Let $n = 12t + 6$ for some integer $t \equiv 2, 4 \pmod{6}$. In this case, we need to construct EGDD$(3 + (12t + 6), 3; 24t + 12, 24t^2 + 22t + 6)$. We split $12t^2 + 11t + 3$ parallel classes of RBIBD$(12t + 6, 3, 24t + 12)$ with each element of G_1. In addition, we take the blocks of a BIBD$(3, 3, 24t + 12)$.
Let $n = 72t + 42$ for some integer $t \equiv 0 \pmod{2}$. In this case, we need to construct EGDD$(3 + (72t + 42), 3; 24t + 14, 144t^2 + 166t + 48)$. We split $72t^2 + 83t + 24$ parallel classes of RBIBD$(72t + 42, 3, 24t + 14)$ with each element of G_1. In addition, we take the blocks of a BIBD$(3, 3, 24t + 14)$.
Let $n = 72t + 42$ for some integer $t \equiv 1 \pmod{2}$. In this case, we need to construct EGDD$(3+(72t+42), 3; 12t+7, 72t^2+83t+24)$. We split $21t+12$ parallel classes of RBIBD$(72t + 42, 3, 9t + 7)$ with each element of G_1. We also take t copies of K_{72t+42} with each element of G_1. In addition, we take the blocks of a BIBD$(3, 3, 12t + 7)$.
Let $n = 12t + 7$ for some integer $t \equiv 0 \pmod{2}$. In this case, we need to construct EGDD$(3 + (12t + 7), 3; 12t + 7, 12t^2 + 13t + 4)$. We split $\frac{4t^2+3t}{2}$ 3-resolvable classes of 3-RBIBD$(12t+7, 3, 12t+6)$ with each element of G_1. We also take $2t + 1$ 2-factors of K_{12t+7} with each element of G_1 and 1 copy of K_3 with each element of G_2.
Let $n = 12t + 7$ for some integer $t \equiv 1 \pmod{2}$. In this case, we need to construct EGDD$(3 + (12t + 7), 3; 24t + 14, 24t^2 + 26t + 8)$. We split $4t^2 + 3t$ 3-resolvable classes of 3-RBIBD$(12t + 7, 3, 24t + 12)$ with each element of G_1. We also take $4t + 2$ 2-factors of $2K_{12t+7}$ with each element of G_1 and 2 copies of K_3 with each element of G_2.
Let $n = 12t + 8$ for some integer t. In this case, the required EGDD to construct is EGDD$(3 + (12t + 8), 3; 144t + 96, 144t^2 + 180t + 62)$. However, since in this case, $\lambda_1 = 4n_1n_2$, such an EGDD has been proven to exists by Theorem 3.1.
Let $n = 12t + 9$ for some integer $t \equiv 0, 1 \pmod{3}$. In this case, we need to construct EGDD$(3 + (12t + 9), 3; 48t + 36, 48t^2 + 68t + 26)$. We split $24t^2 + 34t + 13$ parallel classes of RBIBD$(12t + 9, 3, 48t + 36)$ with each element of G_1. In addition, we take the blocks of a BIBD$(3, 3, 48t + 36)$.
Let $n = 36t + 33$ for some integer t. In this case, we need to construct EGDD$(3+(36t+33), 3; 48t+44, 144t^2+260t+118)$. We split $72t^2+130t+59$ parallel classes of RBIBD$(36t + 33, 3, 48t + 44)$ with each element of G_1. In addition, we take the blocks of a BIBD$(3, 3, 48t + 44)$.
Let $n = 12t + 10$ for some integer $t \equiv 0 \pmod{4}$. In this case, we need to

construct EGDD$(3+(12t+10),3;12t+10,12t^2+19t+8)$. We split $\frac{4t^2+t-2}{2}$ 3-resolvable classes of 3-RBIBD$(12t+10,3,12t+6)$ with each element of G_1. We also take $16t+12$ one-factors of $4K_{12t+10}$ with each element of G_1 and 1 copy of K_3 with each element of G_2. For the case of $t=0$, $n=10$. In this case, we need to construct EGDD$(3+10,3;10,8)$. We split 1 3-resolvable class of 3-RBIBD$(10,3,6)$ with each element of G_1. We also use one copy of K_3 with each element of G_2. In addition, we use the blocks of a BIBD$(10,3,4)$.
Let $n=48t+34$ for some integer t. In this case, we need to construct EGDD$(3+(48t+34),3;24t+17,96t^2+134t+47)$. We split $16t^2+22t+7$ 3-resolvable classes of 3-RBIBD$(48t+34,3,18t+12)$ with each element of G_1. We also take the blocks of a BIBD$(3+(48t+34),3,2t+5)$. In addition, we use the blocks of a BIBD$(3,3,22t+12)$ and the blocks of a BIBD$(48t+34,3,4t)$.
Let $n=12t+10$ for some integer $t\equiv 1 \pmod 2$. In this case, we need to construct EGDD$(3+(12t+10),3;24t+20,24t^2+38t+16)$. We split $4t^2+5t+1$ 3-resolvable classes of 3-RBIBD$(12t+10,3,24t+18)$ with each element of G_1. We also take $8t+6$ one-factors of $2K_{12t+10}$ with each element of G_1 and 2 copies of K_3 with each element of G_2.
Let $n=12t+11$ for some integer $t\equiv 0 \pmod 2$. In this case, we need to construct EGDD$(3+(12t+11),3;72t+66,72t^2+126t+58)$. We split $12t^2+21t+9$ 3-resolvable classes of a 3-RBIBD$(12t+11,3,72t+66)$. We also use 2 copies of K_3 with each element of G_2. In addition, we take the blocks of a BIBD$(3,3,48t+44)$.
Let $n=12t+11$ for some integer $t\equiv 1 \pmod 2$. In this case, we need to construct EGDD$(3+(12t+11),3;36t+33,36t^2+63t+29)$. We split $\frac{12t^2+21t+9}{2}$ 3-resolvable classes of a 3-RBIBD$(12t+11,3,36t+30)$. We also use 1 copy of K_3 with each element of G_2. In addition, we take the blocks of a BIBD$(3,3,24t+22)$ and the blocks of a BIBD$(12t+11,3,3)$.

9 EGDD$(4+n,3;\lambda_1,\lambda_2)$

In this section, we assume $G_1=\{a,b,c,d\}$ and $G_2=\{1,...,n_2=n\}$. In addition, when we split some blocks of a BIBD, it is assumed that the remaining triples of the BIBD are used for $(0,3)$ blocks.

Lemma 9.1. *Necessary conditions are sufficient for an EGDD*$(4+n,3;\lambda_1,\lambda_2)$ *for some number* $n\equiv 0 \pmod 4$.

Proof. We need to consider many cases as the minimum λ_1's depend on modulus 24 arithmetic.
Case $n\equiv 0 \pmod 8$
Let $n=24t$ for some integer t. In this case, we need to construct EGDD$(4+$

$(24t), 3; 32t, 48t^2 - 2t + 1)$. We use $24t$ one-factors of $8tK_4$ on G_1, one with each of the elements of G_2 and $2t$ copies of K_{24t} on G_2 with each element of G_1. In addition, we take the blocks of a BIBD$(24t, 3, 24t)$ and the blocks of a BIBD$(4, 3, 24t)$.

Let $n = 4t$ for some integer $t \equiv 2, 4 \pmod 6$. We need to construct EGDD$(4 + (4t), 3; 16t, 3 + t(4t - 1))$. We take one copy of K_4 on G_1 with each element of G_2 and t copies of K_{4t} on G_2 with each element of G_1. In addition, we take the blocks of a BIBD$(4, 3, 12t)$ and the blocks of a BIBD$(4t, 3, 12t)$.

Case $n \equiv 4 \pmod 8$

Let $n = 48t + 4$ for some integer t. In this case, we need to construct EGDD$(4 + (48t + 4), 3; 8(12t + 1), 288t^2 + 42t + 3)$. We split $48t^2 + 7t$ 3-resolvable classes of a 3-RBIBD$(48t + 4, 3, 96t)$ with each element of G_1. Also, we take one copy of K_4 on G_1 with each element of G_2. In addition, we take the blocks of a BIBD$(48t + 4, 3, 8)$ and the blocks of a BIBD$(4, 3, 48t + 4)$.

Let $n = 96t + 12$ for some integer t. In this case, we need to construct EGDD$(4 + (96t + 12), 3; 4(8t + 1), 192t^2 + 46t + 3)$. We split $96t^2 + 23t + 1$ parallel classes of RBIBD$(96t + 12, 3, 4(8t + 1))$ with each element of G_1. We also use 1 one-factor of $4(8t+1)K_4$ with each element of G_2 to construct the remaining $(1, 2)$-triples.

Let $n = 48t + 20$ for some integer t. In this case, we need to construct EGDD$(4 + (48t + 20), 3; 8(12t + 5), 288t^2 + 234t + 49)$. We split $12t + 5$ 3-resolvable classes of a 3-RBIBD$(48t + 20, 3, 72t + 36)$ with each element of G_1. Also, we take $6t + 1$ copies of K_{48t+20} with each element of G_1. In addition, we take the blocks of a BIBD$(4, 3, 8(12t + 5))$.

Let $n = 96t + 28$ for some integer t. In this case, we need to construct EGDD$(4 + (96t + 28), 3; 2(24t + 7), 288t^2 + 165t + 24)$. We split $14t + 4$ 3-resolvable classes of a 3-RBIBD$(96t + 28, 3, 36t + 6)$ with each element of G_1. Also, we take $3t$ copies of K_{96t+28} with each element of G_1. In addition, we take the blocks of a BIBD$(96t + 28, 3, 8)$ and the blocks of a BIBD$(4, 3, 2(24t + 7))$.

Let $n = 48t + 36$ for some integer t. In this case, we need to construct EGDD$(4 + (48t + 36), 3; 8(4t + 3), 96t^2 + 142t + 53)$. We split $48t^2 + 71t + 26$ parallel classes of RBIBD$(48t + 36, 3, 8(4t + 3))$ with each element of G_1. We also take 1 one-factor of $4(4t + 3)K_4$ with each element of G_2. In addition, we take the blocks of a BIBD$(4, 3, 4(4t + 3))$.

Let $n = 96t + 44$ for some integer t. In this case, we need to construct EGDD$(4 + (96t + 44), 3; 4(24t + 11), 576t^2 + 522t + 119)$. We split $60t + 27$ 3-resolvable classes of a 3-RBIBD$(96t + 44, 3, 4(18t + 12))$ with each element of G_1. Also, we take $6t - 1$ copies of K_{96t+44} with each element of G_1. In addition, we take the blocks of a BIBD$(4, 3, 96t + 44)$. For the case of $t = 0$, $n = 44$. In this case, we need to construct EGDD$(4 + 44, 3; 44, 119)$. We split 5 3-resolvable classes of a 3-RBIBD$(44, 3, 36)$ with each element of G_1. We also take a copy of K_4 with each element of G_2 and 2 copies of K_{44} with each element of G_1.

Let $n = 96t + 60$ for some integer t. In this case, we need to construct EGDD$(4 + (96t + 60), 3; 2(8t + 5), 96t^2 + 119t + 37)$. We split $26t + 16$ 3-resolvable classes of 3-RBIBD$(96t + 60, 3, 12t + 12)$ with each element of G_1. We also take $t - 1$ copies of K_{96t+60} on G_2 with each element of G_1. In addition, we take the blocks of a BIBD$(96t + 60, 3, 2)$ and the blocks of a BIBD$(4, 3, 16t + 10)$. For the case of $t = 0$, $n = 60$. Here, we need to construct EGDD$(4 + 60, 3; 10, 37)$. We split 18 parallel classes of RBIBD$(60, 3, 6)$ with each element of G_1. We take the blocks of a BIBD$(61, 3, 1)$ on $i \cup G_2$ for each element i in G_1. In addition, we take the blocks of a BIBD$(4, 3, 10)$.

Let $n = 96t + 76$ for some integer t. In this case, we need to construct EGDD$(4 + (96t + 76), 3; 4(24t + 19), 576t^2 + 906t + 357)$. We split $96t^2 + 151t + 59$ 3-resolvable classes of a 3-RBIBD$(96t + 76, 3, 4(24t + 18))$ with each element of G_1. Also, we take 1 copy of K_4 with each element of G_2. In addition, we take the blocks of a BIBD$(96t + 76, 3, 4)$.

Let $n = 96t + 92$ for some integer t. In this case, we need to construct EGDD$(4 + (96t + 92), 3; 2(24t + 23), 288t^2 + 549t + 262)$. We split $78t + 74$ 3-resolvable classes of a 3-RBIBD$(96t + 92, 3, 36t + 54)$ with each element of G_1. Also, we take $3t - 2$ copies of K_{96t+92} with each element of G_1. In addition, we take the blocks of a BIBD$(4, 3, 2(24t + 23))$. For the case of $t = 0$, $n = 92$. Here, we need to construct EGDD$(4 + 92, 3; 46, 262)$. We split 43 3-resolvable classes of 3-RBIBD$(92, 3, 42)$. We also take the blocks of a BIBD$(4 + 92, 3, 4)$. In addition, we take the blocks of a BIBD$(4, 3, 42)$. □

Lemma 9.2. *Necessary conditions are sufficient for the existence of an EGDD$(4 + n, 3; \lambda_1, \lambda_2)$ for all integers $n \equiv 1 \pmod 4$.*

Proof. To complete the proof, we need to consider many cases as the minimum λ_1's depend on modulus 24 arithmetic.

Case $n \equiv 1 \pmod 8$

Let $n = 8t + 1$ for some integer $t \equiv 0, 2 \pmod 3$. In this case, we need to construct EGDD$(4 + (8t + 1), 3; 8(8t + 1), 4t(8t + 1) + 6)$. We take 2 copies of K_4 on G_1 with each element of G_2 and $2(8t + 1)$ copies of K_{8t+1} on G_2 with G_1. In addition, we take the blocks of BIBD$(4, 3, 6(8t + 1))$ and the blocks of BIBD$(8t + 1, 3, 6(8t + 1))$.

Let $n = 24t + 9$ for some integer t. In this case, we need to construct EGDD$(4 + (24t + 9), 3; 8(8t + 3), (8t + 3)(12t + 4) + 2)$. We split 1 parallel class of RBIBD$(24t + 9, 3, 4(8t + 3))$ with each element in G_1, and we take $(8t + 3)$ copies of K_{24t+9} with each element of G_1. In addition, we take the blocks of BIBD$(4, 3, 8(8t + 3))$.

Case $n \equiv 5 \pmod 8$

Let $n = 48t + 5$ for some integer $t \equiv 1 \pmod 2$. In this case, we need to construct EGDD$(4+(48t+5), 3; 48t+5, 144t^2+27t+2)$. We split $\frac{48t^2-23t-3}{2}$ classes of 3-RBIBD$(48t+5, 3, 6(8t-1))$ with each element of G_1. Also, we take 2 copies of K_{48t+5} with each element of G_1 and 1 copy of K_4 with each element of G_2. In addition, we take the blocks of a BIBD$(48t+5, 3, 3)$.
Let $n = 48t + 5$ for some integer $t \equiv 0 \pmod 2$. In this case, we need to construct EGDD$(4 + (48t + 5), 3; 2(48t + 5), 2(144t^2 + 27t + 2))$. We split $48t^2 - 23t - 3$ 3-resolvable classes of a 3-RBIBD$(48t + 5, 3, 12(8t - 1))$ with each element of G_1. Also, we take 4 copies of K_{48t+5} with each element of G_1 and 2 copies of K_4 with each element of G_2. In addition, we take the blocks of a BIBD$(48t+5, 3, 6)$. However, for the case of $t = 0$, one can note that $n_2 = n_1 + 1$ and thus an EGDD with $n_2 = 5$ has been proven to exist.
Let $n = 48t + 13$ for some integer t. In this case, we need to construct EGDD$(4+(48t+13), 3; 4(48t+13), 576t^2+300t+42)$. We split $96t^2+50t+7$ 3-resolvable classes of 3-RBIBD$(48t + 13, 3, 48t + 18)$ with each element of G_1. In addition, we take the blocks of a BIBD$(48t + 13, 3, 144t + 34)$ and the blocks of a BIBD$(4, 3, 4(48t + 13))$.
Let $n = 48t + 21$ for some integer $t \equiv 0 \pmod 2$. In this case, we need to construct EGDD$(4 + (48t + 21), 3; 16t + 7, 48t^2 + 41t + 9)$. We split $\frac{21t+8}{2}$ parallel classes of RBIBD$(48t+21, 3, 12t+7)$ with each element of G_1. Also, we take 1 one-factor of $(16t + 7)K_4$ with each element of G_2 and t copies of K_{48t+21} with each element of G_1.
Let $n = 48t + 21$ for some integer $t \equiv 1 \pmod 2$. In this case, we need to construct EGDD$(4 + (48t + 21), 3; 2(16t + 7), 2(48t^2 + 41t + 9))$. We split $21t + 8$ parallel classes of RBIBD$(48t + 21, 3, 2(12t + 7))$ with each element of G_1. Also, we take 2 one-factors of $2(16t + 7)K_4$ with each element of G_2 and $2t$ copies of K_{48t+21} with each element of G_1.
Let $n = 48t + 29$ for some integer t. In this case, we need to construct EGDD$(4 + (48t + 29), 3; 4(48t + 29), 576t^2 + 684t + 206)$. We split $96t^2 + 98t + 25$ 3-resolvable classes of 3-RBIBD$(48t + 29, 3, 48t + 24)$ with each element of G_1. We also take 2 copies of K_{48t+29} on G_2 with each element of G_1. In addition, we take the blocks of a BIBD$(48t+29, 3, 144t+84)$ and the blocks of a BIBD$(4, 3, 4(48t + 29))$.
Let $n = 48t + 37$ for some integer $t \equiv 1 \pmod 2$. In this case, we need to construct EGDD$(4 + (48t + 37), 3; 48t + 37, 144t^2 + 219t + 84)$. We split $\frac{5t+3}{2}$ classes of a 3-RBIBD$(48t + 37, 3, 36t + 24)$ with each element of G_1. Also, we take $3t+2$ copies of K_{48t+37} on G_2 with each element of G_1 and 1 copy of K_4 on G_1 with each element of G_2. In addition, we take the blocks of a BIBD$(48t + 37, 3, 5)$.
Let $n = 48t + 37$ for some integer $t \equiv 0 \pmod 2$. In this case, we need to construct EGDD$(4+(48t+37), 3; 2(48t+37), 2(144t^2+219t+84))$. We split $5t + 3$ 3-resolvable classes of a 3-RBIBD$(48t + 37, 3, 2(36t + 24))$ with each

element of G_1. Also, we take $2(3t+2)$ copies of K_{48t+37} on G_2 with each element of G_1. In addition, we take the blocks of a BIBD$(48t+37, 3, 10)$ on G_2 and a BIBD$(4, 3, 2(48t+37))$.
Let $n = 48t + 45$ for some integer t. In this case, we need to construct EGDD$(4 + (48t+45), 3; 4(16t+15), 2(96t^2+178t+83))$. We split $96t^2 + 178t + 83$ classes of RBIBD$(48t+45, 3, 4(16t+5))$ with each element of G_1. In addition, we take the blocks of a BIBD$(4, 3, 4(16t+15))$.

□

Lemma 9.3. *Necessary conditions are sufficient for the existence of an EGDD*$(4+n, 3; \lambda_1, \lambda_2)$ *for integers* $n \equiv 2 \pmod 4$.

Proof. There are two cases:

(i) $n = 4t + 2$ for $t \equiv 0, 2 \pmod 3$: In this case, we need to construct EGDD$(4+(4t+2), 3; 16(2t+1), (2t+1)(4t+1)+6)$. We take 2 copies of K_4 on G_1 with each element of G_2 and $(2t+1)$ copies of K_{4t+2} on G_2 with each element in G_1. In addition, we take the blocks of BIBD$(4t+2, 3, 12(2t+1))$ and the blocks of BIBD$(4, 3, 12(2t+1))$.

(ii) $n = 12t + 6$ for some integer t: In this case, we need to construct EGDD$(4 + (12t+6), 3; 16(2t+1), (2t+1)(12t+5)+2)$. We split 1 parallel class of RBIBD$(12t+6, 3, 12(2t+1))$ with each element in G_1. We also take $(2t+1)$ copies of K_{12t+6} on G_2 with each element of G_1. In addition, we take the blocks of BIBD$(4, 3, 16(2t+1))$.

□

Lemma 9.4. *Necessary conditions are sufficient for the existence of an EGDD*$(4+n, 3; \lambda_1, \lambda_2)$ *for integers* $n \equiv 3 \pmod 4$.

Proof. There are two cases:

(i) $n = 12t + 3$ for some integer t: In this case, we need to construct EGDD$(4 + (12t+3), 3; 16(4t+1), (4t+1)(12t+2)+4)$. We split 2 parallel classes of RBIBD$(12t+3, 3, 2)$ with each element of G_1. We also take $(4t+1)$ copies of K_{12t+3} on G_2 with each element in G_1. In addition, we take the blocks of BIBD$(4, 3, 16(4t+1))$ and the blocks of BIBD$(12t+3, 3, 48t+10)$.

(ii) $n = 4t + 3$ for $t \equiv 1, 2 \pmod 3$: In this case, we need to construct EGDD$(4+(4t+3), 3; 16(4t+3), (4t+3)(4t+2)+12)$. However, since in this case $\lambda_1 = 4n_1n_2$, this EGDD has been proven to exist.

□

As an aside, we have

Theorem 9.5. *Necessary conditions are sufficient for the existence of a GDD($n_1 + n_1^2, 3; \lambda_1, \lambda_2$) where $\lambda_1 = n_1^2, \lambda_2 = n_1^3 - 1$ for all $n_1 \geq 2$.*

Proof. Suppose we have two groups, $G_1 = \{a_1, a_2, ..., a_{n_1}\}$ and $G_2 = \{1, ..., n_1^2\}$. Then, if we use a K_{n_1} on G_1 with each element of G_2 and n_1 copies of $K_{n_1^2}$ on G_2 with each element of G_1, we get an EGDD($n_1 + n_1^2, 3; n_1^2, n_1^3 - 1$). □

10 EGDD($n_1 + n_2, 3; \lambda_1, \lambda_2$), $n_1 + n_2 \leq 20$

10.1 EGDD($5 + n, 3; \lambda_1, \lambda_2$)

Now, we assume $G_1 = \{a, b, c, d, e\}$ and $G_2 = \{1, ..., n_2 = n\}$.

For EGDD($5 + 8, 3; \lambda_1, \lambda_2$), $\lambda_1 = \frac{40}{19}\lambda_2$. Hence, for some integer $t \geq 1$, $\lambda_1 = 40t$ and $\lambda_2 = 19t$. In order for an EGDD($5 + 8, 3; 40t, 19t$) to exist, $t \equiv 0 \pmod 2$. We use the blocks of BIBD(13, 3, 38), BIBD(5, 3, 42), and BIBD(8, 3, 42).

For EGDD($5 + 9, 3; \lambda_1, \lambda_2$), $\lambda_1 = \frac{45}{23}\lambda_2$. Hence, for some integer $t \geq 1$, $\lambda_1 = 45t$ and $\lambda_2 = 23t$. In order for an EGDD($5 + 9, 3; 45t, 23t$) to exist, $t \equiv 0 \pmod 2$. We use 23 parallel classes of RBIBD(9, 3, 6) with each element of G_1. In addition, we use the blocks of a BIBD (5, 3, 90) on G_1 and the blocks of a BIBD(9, 3, 84) on G_2.

For EGDD($5 + 10, 3; \lambda_1, \lambda_2$), $\lambda_1 = \frac{20}{11}\lambda_2$. Hence, for some integer $t \geq 1$, $\lambda_1 = 20t$ and $\lambda_2 = 11t$. In order for an EGDD($5 + 10, 3; 20t, 11t$) to exist, $t \equiv 0 \pmod 2$. We use 1 copy of K_5 on G_1 with each element of G_2 and 2 copies of K_{10} on G_2 with each element of G_1. In addition, we use the blocks of a BIBD(5, 3, 30) on G_1 and the blocks of a BIBD(10, 3, 30) on G_2.

For EGDD($5 + 11, 3; \lambda_1, \lambda_2$), $\lambda_1 = \frac{22}{13}\lambda_2$. Hence, for some integer $t \geq 1$, $\lambda_1 = 22t$ and $\lambda_2 = 13t$. In order for an EGDD($5 + 11, 3; 22t, 13t$) to exist, $t \equiv 0 \pmod 2$. We use 4 copies of K_5 on G_1 with each element of G_2 and 1 copy of K_{11} on G_2 with each element of G_1. In addition, we use the blocks of a BIBD(11, 3, 39) on G_2.

For EGDD($5 + 12, 3; \lambda_1, \lambda_2$), $\lambda_1 = \frac{30}{19}\lambda_2$. Hence, for some integer $t \geq 1$, $\lambda_1 = 30t$ and $\lambda_2 = 19t$. In order for an EGDD($5 + 12, 3; 30t, 19t$) to exist, $t \equiv 0 \pmod 2$. We use 4 copies of K_5 on G_1 with each element of G_2 and 2 copies of K_{12} on G_2 with each element of G_1. In addition, we use the blocks of a BIBD(5, 3, 12) on G_1 and the blocks of a BIBD(12, 3, 50) on G_2.

For EGDD($5 + 13, 3; \lambda_1, \lambda_2$), $\lambda_1 = \frac{65}{44}\lambda_2$. Hence, for some integer $t \geq 1$, $\lambda_1 = 65t$ and $\lambda_2 = 44t$. We use 2 copies of K_5 on G_1 with each element of G_2 and 3 copies of K_{13} on G_2 with each element of G_1. In addition, we use the blocks of a BIBD(5, 3, 39) on G_1 and the blocks of a BIBD(13, 3, 50) on G_2.

From Corollary 4.2, EGDD($5 + 14, 3; 280t, 202t$) exists.

For EGDD($5 + 15, 3; \lambda_1, \lambda_2$), $\lambda_1 = \frac{30}{23}\lambda_2$. Hence, for some integer $t \geq 1$,

$\lambda_1 = 30t$ and $\lambda_2 = 23t$. In order for an EGDD$(5+15, 3; 30t, 23t)$ to exist, $t \equiv 0 \pmod 2$. We use 1 copy of K_5 on G_1 with each element of G_2 and 3 copies of K_{15} on G_2 with each element of G_1. In addition, we use the blocks of a BIBD$(5, 4, 45)$ on G_1 and the blocks of a BIBD$(15, 4, 45)$ on G_2.

10.2 EGDD$(6+n, 3; \lambda_1, \lambda_2)$

Now, we assume $G_1 = \{a, b, c, d, e, f\}$ and $G_2 = \{1, ..., n_2 = n\}$.
For EGDD$(6+9, 3; \lambda_1, \lambda_2)$, $\lambda_1 = \frac{36}{17}\lambda_2$. Hence, for some integer $t \geq 1$, $\lambda_1 = 36t$ and $\lambda_2 = 17t$. In order for an EGDD$(6+9, 3; 36t, 17t)$ to exist, $t \equiv 0 \pmod 2$. We split 17 parallel classes from the RBIBD$(9, 3, 6)$ with each element of G_1. In addition, we use the blocks of a BIBD$(6, 3, 72)$ on G_1 and the blocks of a BIBD$(9, 3, 66)$ on G_2.
For EGDD$(6+10, 3; \lambda_1, \lambda_2)$, $\lambda_1 = 2\lambda_2$. Hence, for some integer $t \geq 1$, $\lambda_1 = 2t$ and $\lambda_2 = t$. We use t one-factors of $2tK_6$ on G_1 with each element of G_2. In addition, we use a BIBD$(10, 3, 2t)$.
For EGDD$(6+11, 3; \lambda_1, \lambda_2)$, $\lambda_1 = \frac{66}{35}\lambda_2$. Hence, for some integer $t \geq 1$, $\lambda_1 = 66t$ and $\lambda_2 = 35t$. In order for an EGDD$(6+11, 3; 66t, 35t)$ to exist, $t \equiv 0 \pmod 2$. We split 10 3-resolvable classes from the 3-RBIBD$(11, 3, 6)$ with each element of G_1 and use 1 copy of K_{11} on G_2 with each element of G_1. In addition, we use the blocks of a BIBD$(6, 3, 132)$ and the blocks of a BIBD$(11, 3, 120)$ on G_2.
For EGDD$(6+12, 3; \lambda_1, \lambda_2)$, $\lambda_1 = \frac{16}{9}\lambda_2$. Hence, for some integer $t \geq 1$, $\lambda_1 = 16t$ and $\lambda_2 = 9t$. We split 2 parallel classes from RBIBD$(12, 3, 2)$ with each element of G_1 and use 1 copy of K_6 on G_1 with each element of G_2. In addition, we use the blocks of a BIBD$(6, 3, 4)$ on G_1 and the blocks of a BIBD$(12, 3, 14)$ on G_2.
For EGDD$(6+13, 3; \lambda_1, \lambda_2)$, $\lambda_1 = \frac{52}{31}\lambda_2$. Hence, for some integer $t \geq 1$, $\lambda_1 = 52t$ and $\lambda_2 = 31t$. In order for an EGDD$(6+13, 3; 52t, 31t)$ to exist, $t \equiv 0 \pmod 2$. We split 7 3-resolvable classes from the 3-RBIBD$(13, 3, 6)$ with each element of G_1 and use 4 copies of K_6 on G_1 with each element of G_2. In addition, we use the blocks of a BIBD$(13, 3, 98)$ on G_2 and the blocks of a BIBD$(6, 3, 52)$ on G_1.
For EGDD$(6+14, 3; \lambda_1, \lambda_2)$, $\lambda_1 = \frac{84}{53}\lambda_2$. Hence, for some integer $t \geq 1$, $\lambda_1 = 84t$ and $\lambda_2 = 53t$. We split 8 3-resolvable classes from the 3-RBIBD$(14, 3, 6)$ with each element of G_1 and 1 copy of K_6 on G_1 with each element of G_2. In addition, we use the blocks of a BIBD$(6, 3, 70)$ on G_1 and the blocks of a BIBD$(14, 3, 78)$ on G_2.

10.3 EGDD$(7+n, 3; \lambda_1, \lambda_2)$

Now, we assume $G_1 = \{a, b, c, d, e, f, g\}$ and $G_2 = \{1, ..., n_2 = n\}$.
For EGDD$(7+10, 3; \lambda_1, \lambda_2)$, $\lambda_1 = \frac{70}{33}\lambda_2$. Hence, for some integer $t \geq 1$, $\lambda_1 = 70t$ and $\lambda_2 = 33t$. In order for an EGDD$(7+10, 3; 70t, 33t)$ to exist, $t \equiv 0 \pmod 2$. We use 11 copies of K_7 on G_1 with each element of G_2. In addition, we use the blocks of a BIBD$(10, 3, 140)$ on G_2 and the blocks of

a BIBD(7, 3, 30) on G_1.
For EGDD$(7+11, 3; \lambda_1, \lambda_2)$, $\lambda_1 = \frac{77}{38}\lambda_2$. Hence, for some integer $t \geq 1$, $\lambda_1 = 77t$ and $\lambda_2 = 38t$. We use 3 copies of K_7 on G_1 with each element of G_2 and 2 copies of K_{11} on G_2 with each element of G_1. In addition, we use the blocks of a BIBD(7, 3, 44) on G_1 and the blocks of a BIBD(11, 3, 63) on G_2.
For EGDD$(7+12, 3; \lambda_1, \lambda_2)$, $\lambda_1 = \frac{56}{29}\lambda_2$. Hence, for some integer $t \geq 1$, $\lambda_1 = 56t$ and $\lambda_2 = 29t$. In order for an EGDD$(7+12, 3; 56t, 29t)$ to exist, $t \equiv 0 \pmod 2$. We use 6 copies of K_7 on G_1 with each element of G_2 and 2 copies of K_{12} on G_2 with each element of G_1. In addition, we use the blocks of a BIBD(7, 3, 40) on G_1 and the blocks of a BIBD(12, 3, 98) on G_2.
From Corollary 4.2, EGDD$(7+13, 3; 364t, 198t)$ exists.

10.4 EGDD$(8+n, 3; \lambda_1, \lambda_2)$

Now, we assume $G_1 = \{a, b, c, d, e, f, g, h\}$ and $G_2 = \{1, ..., n_2 = n\}$.
From Corollary 4.2, EGDD$(8+11, 3; 352t, 166t)$ exists.
For EGDD$(8+12, 3; \lambda_1, \lambda_2)$, $\lambda_1 = \frac{96}{47}\lambda_2$. Hence, for some integer $t \geq 1$, $\lambda_1 = 96t$ and $\lambda_2 = 47t$. We use 2 copies of K_8 on G_1 with each element of G_2 and 3 copies of K_{12} on G_2 with each element of G_2. In addition, we use the blocks of a BIBD(8, 3, 72) on G_1 and the blocks of a BIBD(12, 3, 72) on G_2.

11 Summary

Theorem 11.1. *The necessary conditions are sufficient for the existence of EGDD$(n_1+n_2, 3; \lambda_1, \lambda_2)$ when $n_1+n_2 \leq 20$.*

We have also proved:

Theorem 11.2. *The necessary conditions are sufficient for the existence of EGDD$(n_1+n_2, 3; \lambda_1, \lambda_2)$ for $n_1 = n_2-1, n_2-2, 1, 3, 4$.*

Many GDDs can be constructed by taking appropriate BIBDs on G_1 and G_2 away from the EGDD constructions. A small sample of such GDDs, which are not EGDDs, are shown in the following table, assuming an integer $t \geq 1$:

n_1	n_2	λ_1	λ_2	Comment
5	8	$38t$	$38t$	Section 10.1
5	9	$6t$	$46t$	Section 10.1
5	10	$10t$	$22t$	Section 10.1
5	12	$48t$	$38t$	Section 10.1
5	13	$26t$	$44t$	Section 10.1
5	15	$15t$	$46t$	Section 10.1
6	9	$6t$	$34t$	Section 10.2
6	11	$12t$	$70t$	Section 10.2
6	12	$12t$	$9t$	Section 10.2
6	13	$52t$	$62t$	Section 10.2
6	14	$18t$	$53t$	Section 10.2
7	10	$110t$	$66t$	Section 10.3
7	11	$35t$	$38t$	Section 10.3
7	12	$72t$	$58t$	Section 10.3
8	12	$24t$	$47t$	Section 10.4

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