CANCELLABLE NUMBERS

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Abstract

A cancellable number (CN) is a fraction in which a decimal digit can be removed ("cancelled") in the numerator and denominator without changing the value of the number; examples include $\frac{16}{64}$ where the 6 can be cancelled and $\frac{49}{98}$ where the 9 can be cancelled. We provide explicit infinite families of CNs with certain properties, subject to the restriction that the numerator and denominator have equal decimal length and the cancellable digits "line up", i.e., all cancellation lines are vertical. The properties in question are that the CNs contain one of the following: a cancellable nine, a cancellable zero, a sequence of adjacent cancellable zeros, sequences of adjacent cancellable zeros and nines of the same length, and sequences of adjacent cancellations for certain other digits.

1 Introduction

In 1933 Morley [1] posed the problem of illegitimate cancellation, identifying $\frac{16}{64} = \frac{1\cancel{6}}{\cancel{6}4}$, $\frac{26}{65} = \frac{2\cancel{6}}{\cancel{6}5}$, $\frac{19}{95} = \frac{1\cancel{9}}{\cancel{9}5}$ and $\frac{49}{98} = \frac{4\cancel{9}}{\cancel{9}8}$ as the only proper fractions with denominators less than one hundred which could be reduced by the indicated illegal cancellation. B·L· Schwartz [2] provided the solutions for three digit denominators in 1961. In this paper several generalizations are provided, a method of counting is identified and explored and some limit theorems are derived. Before pursuing these inquiries a

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few immediate results are noted. The reciprocal of a cancellable number (CN)is a CN. If M has m digits and $m > 1$, $\frac{M}{M}$ is a CN in m ways. It will be sufficient, therefore, to work with CNs of the form $\frac{L}{M}$, $L > M$ and this convention is adopted. Again, if $M = 10M'$ and $N = 10N'$ then $\frac{M}{N}$ is a CN equal to $\frac{M'}{N'}$ after cancellation of the terminal zero. Finally, we note that, from the CN $\frac{64}{16} = \frac{\cancel{6}4}{1\cancel{6}}$, the sets of CNs, $\{\frac{464}{116}, \frac{864}{216}, \frac{1264}{316}, \cdots\}$ and $\{\frac{644}{161}, \frac{648}{162}, \frac{6412}{1603}, \cdots\}$ can be generated, so the set of CNs is infinite.

2 CNs FOR ALL RATIONAL NUMBERS

Although it is possible to generate an infinite set of CNs from each CN, a more interesting problem is to identify those rationals, $\frac{L}{M}$, for which a CN exists. We have,

Theorem 1: If L has l digits and M has m digits $(l \geq m)$ then

$$\frac{L}{M} = \frac{(L-1)10^{l+1} + 9 * 10^{l} + (10^{l} - L)}{(M-1)10^{l+1} + 9 * 10^{l} + (10^{l} - M)} =$$
$$\frac{(L-1)10^{l} + (10^{l} - L)}{(M-1)10^{l} + (10^{l} - M)}$$

is a CN for $\frac{L}{M}$ with cancellable nine.
The proof of the above theorem consists only of reducing the fractions to establish the two identities, produces a CN with cancellable nine (vertical cancellation) for every rational $\frac{L}{M}$. The next theorem is analogous for cancelling zeros. ■

Theorem 2: If L has l digits and M has m digits $(l \geq m)$ then

$$\frac{(10^{l+1} - 2)10^{l+1}L + L}{(10^{l+1} - 2)10^{l+1}M + M}$$

is a CN for $\frac{L}{M}$ with cancellable zero.
To prove Theorem 2 we need merely note the existence of zero for cancellation and that

$\frac{L}{M} = \frac{(10^{l+1}-2)10^{l+1}L+L}{(10^{l+1}-2)10^{l+1}M+M} = \frac{(10^{l+1}-2)10^{l}L+L}{(10^{l+1}-2)10^{l}M+M}$ ■

Theorem 3: If L has l digits and M has m digits ($l \geq m$), then

$$\frac{10^{l+1}L + L}{10^{l+1}M + M}$$

is a CN for $\frac{L}{M}$ with cancellable zero. ■

The results of Theorem 1 and 2 can be generalized in two different ways. One can seek theorems where digits other than zero or nine are cancellable.Alternately, theorems can be sought where it is possible to cancel p consecutive times, each time retaining the equality. For tactical reasons we do the latter for zero cancellation first.

Definition: A p_H CN for $\frac{L}{M}$ is a representative for $\frac{L}{M}$ such that H can be cancelled 1,2,3, …, p times, each cancellation producing a representative of $\frac{L}{M}$.

Theorem 4: If L has l digits and M has m digits ($l \geq m$) then

$$\frac{(10^{l+p} - 2)10^{l+p}L + L}{(10^{l+p} - 2)10^{l+p}M + M}$$

is a p_0 CN for $\frac{L}{M}$ with cancellable zeros in positions $l + 1, l + 2, \ldots, l + p$ counting from the right. ■

By analogy with Theorem 3 we have the extension.

Theorem 5: If L has l digits and M has m digits ($l \geq m$) then

$$\frac{10^{l+p}L + L}{10^{l+p}M + M}$$

is a P_0 CN for $\frac{L}{M}$ with cancellable zeros in positions $l + 1, l + 2, \ldots, l + p$ counting from the right. ■

The expressions given by Theorems 2 and 4 have a property not generally enjoyed by simpler expressions in Theorems 3 and 4.

Theorem 6: If L has l digits and M has m digits $(l \geq m)$ and L and $M \leq 5 * 10^{l-1}$ and $5 * 10^{m-1}$ respectively then

$$\frac{(10^{l+p} - 2)10^{l+p}L + L}{(10^{l+p} - 2)10^{l+p}M + M}$$

is a $2p$-CN for $\frac{L}{M}$ with p cancellable nines and p cancellable zeros.
Proof: It suffices to establish the required number of zeros and nines for the numerator. Since L has l digits, the factors 10^{l+p} in $(10^{l+p} - 2)10^{l+p}L$ assures p zeros. We proceed to show there are p nines.

$$(10^{l+p} - 2)10^{l+p}L + L = L10^{2l+2p} - (2 * 10^{l+p} - 1)L$$

Since L has l digits, $(2 * 10^{l+p} - 1)L$ has

$$\begin{array}{c} 2l + p \text{ digits if } L \leq 5 * 10^{l-1} \\ 2l + p + 1 \text{ digits if } L > 5 * 10^{l-1} \end{array}$$

Since $L * 10^{2l+2p}$ has $2l + 2p$ zeros, the difference, $L * 10^{2l+2p} - (2 * 10^{l+p} - 1)L$ has p nines if $L \leq 5 * 10^{l-1}$.
It is also necessary to show that quotient still represents $\frac{L}{M}$ after j zeros and k nines have been cancelled. First,

$$\frac{(10^{l+p}-2)10^{l+p}L+L}{(10^{l+p}-2)10^{l+p}M+M} = \frac{L10^{2l+2p}-(2*10^{l+p}-1)L}{M10^{2l+2p}-(2*10^{l+p}-1)M}$$

If L and M meet the magnitude constraints of the theorem the result, after cancellation of j zeros and k nines, $j, k \leq p$ is

$$\frac{L10^{2l+2p-j-k}-(2*10^{l+p-j}-1)L}{M10^{2l+2p-j-k}-(2*10^{l+p-j}-1)M}$$

which is clearly $\frac{L}{M}$ as required. ■

From Theorems 1 and 2 it is seen that all rational fractions have representatives with cancellable zeros and cancellable nines. When can other digits $(\frac{28}{19} = \frac{21756}{14763} = \frac{21\cancel{7}56}{14\cancel{7}63} = \frac{2156}{1463} = \frac{28}{19})$ be cancelled? Except as otherwise noted it is assumed that

1. all cancelling lines are vertical
2. if $\frac{L}{M}$ is represented by a P_H-CN the p cancellations of H are adjacent.

Subject to these constraints, the statement that a representation of $\frac{L}{M}$ is a P_H-CN is:

$$\frac{L}{M} = \frac{A_1 10^{k+p} + H 10^k \sum_{i=0}^{p-1} 10^i + A_2}{B_1 10^{k+p} + H 10^k \sum_{i=0}^{p-1} 10^i + B_2} = \tag{2.1}$$

$$\frac{A_1 10^{k+p-q} + H 10^k \sum_{i=0}^{p-1-q} 10^i + A_2}{B_1 10^{k+p-q} + H 10^k \sum_{i=0}^{p-1-q} 10^i + B_2} \tag{2.2}$$

where $q = 1, 2, \ldots, p$,the integer k needs to be specified, and $\sum_{i=0}^{-1} 10^i = 0$. From (2.1-2), it follows that $\frac{L}{M} = (\frac{9A_1+H}{9B_1+H})$, so that $A_1 = \frac{\frac{H}{9}(L-M)+LB_1}{M}$ which has solutions if and only if $9|\frac{H(L-M)}{(L,M)}$. Similarly, solving (2.1-2) for A_2 and B_2, $A_2 = \frac{\frac{H}{9}10^k(M-L)+LB_2}{M}$ which again has solutions if and only if $9|\frac{H(L-M)}{(L,M)}$ giving the following existence theorem.

Theorem 7: $\frac{L}{M}$ has P_H-CN representations for

a $H \in \{0, 9\}$ if $\frac{L-M}{(L,M)} = 9K + t; K, t$ arbitrary integers.

b $H \in \{0, 3, 6, 9\}$ if $\frac{L-M}{(L,M)} = 3J(3K \pm 1); J, K$ arbitrary integers.

c arbitrary H if $\frac{L-M}{(L,M)} = 9K, K$ an integer. ■

Before developing the general theory for representing $\frac{L}{M}$ as a CN which, though constructive, is (by virtue of the need to compute the totient of L) computationally feasible only in the simplest cases, some special situations will be noted.

Theorem 8: If $(L, M) = 1$ and L has l digits and M has m digits and $l \geq m$ then

a $\frac{L}{M}\left(\frac{9\sum_{i=0}^{p+l-1} 10^i}{9\sum_{i=0}^{p+l-1} 10^i}\right)$ is a P_9-CN for $\frac{L}{M}$, for arbitrary L and M

(Theorems 4 and 5 provide similar results for zero cancellation)

b $\frac{L}{M}\left(\frac{3E\sum_{i=0}^{p+l-1} 10^i}{3E\sum_{i=0}^{p+l-1} 10^i}\right)$ is a P_{3E}-CN for $\frac{L}{M}$ $(E = 1, 2, 3)$ for L and M of the form $3F + 1$

c $\frac{L}{M}\left(\frac{H\sum_{i=0}^{p+l-1} 10^i}{H\sum_{i=0}^{p+l-1} 10^i}\right)$ is a P_H-CN for $\frac{L}{M}(H = 1, 2, 3, \ldots, 9)$ for L and M of the form $9F + 1$.

The proof of (c) is illustrative and will be outlined. Since

1. $\frac{L}{M} = \frac{L}{M}\left(\frac{H\sum_{i=0}^{p+l-1} 10^i}{H\sum_{i=0}^{p+l-1} 10^i}\right) = \frac{L}{M}\left(\frac{H\sum_{i=0}^{p+l-j-1} (10^i)}{H\sum_{i=0}^{p+l-j-1} (10^i)}\right)$ it remains only to show

2. $\frac{L}{M}\left(\frac{H\sum_{i=o}^{p+l-1} 10^i}{H\sum_{i=o}^{p+l-1} 10^i}\right)$ has p consecutive H's in the numerator and

denominator. By symmetry, it suffices to examine the numerator.

$$LH\sum_{i=0}^{p+l-1} 10^i = \frac{LH}{9}(10^{p+l}-1) = (9F+1)\frac{H}{9}(10^{p+l}-1) = \frac{H}{9}[10^{p+l}(9F+1)-(9F+1)]$$

since $9F+1 = L$ has l digits, $10^{p+l}(9F+1)$ has $p+2l$ digits/ Further, $\frac{H}{9}[10^{p+l}(9F+1)-(9F+1)] = \frac{H}{9}[9F10^{p+l} + 10^{p+l} - (9F+1)]$ and $10^{p+l} - (9F+1)$ has at least p consecutive 9's. Since, $9|10^m - (9F+1)$, it follows that $\frac{H}{9}[9F10^{p+l} + 10^{p+l} - (9F+1)]$ is integral and contains at least p consecutive H's. ■

Given $\frac{28}{19}$, notice that $H = 2$, $p = 3$ and $\frac{28}{19}(\frac{(2)(11111)}{(2)(11111)}) = \frac{6\cancel{222}16}{4\cancel{222}18}$.

Theorem 9: If $\frac{L-M}{(L,M)} = 9K, L = 9R+t, M = 9S+t$ and L has l digits, then $\frac{L}{M}(\frac{\sum_{i=0}^{p+l-1} 10^i}{\sum_{i=0}^{p+l-1} 10^i})$ is a p_t-CN for $\frac{L}{M}$.

To prove this it is only necessary to note that

$$\frac{L}{M}(\frac{\sum_{i=0}^{p+l-1} 10^i}{\sum_{i=0}^{p+l-1} 10^i}) = \frac{(9R+1)(\sum_{i=0}^{p+l-1} 10^i) + (t-1)(\sum_{i=0}^{p+l-1} 10^i)}{(9S+1)(\sum_{i=0}^{p+l-1} 10^i) + (t-1)(\sum_{i=0}^{p+l-1} 10^i)}$$

It is an easy exercise to show that $(9R+1)(\sum_{i=0}^{p+l-1} 10^i)$ has at least p consecutive ones, that the addition of $(t-1)(\sum_{i=0}^{p+l-1} 10^i)$ converts at least p of the consecutive ones to t's and that the expression is a p_t-CN for $\frac{L}{M}$. ■

For $L = 29, M = 20, p = 3$, the above produces the 3_2-CN $\frac{322219}{222220}$.

From the discussions preceding Theorem 7 and with the assumption of $(L, M) = 1$ for CN $\frac{L}{M}$, we have $LB_1 = A_1M - (\frac{L-M}{9})H$ which yields $A_1 = M^{\phi(L)-1}(\frac{L-M}{9})H(modL)$ where $\phi(L)$ is the Euler totient of L. Similarly, $LB_2 = A_2M - (\frac{10^k(M-L}{9})H$ or $A_2 = M^{\phi(L)-1}(\frac{M-L}{9})10^kH(modL)$. This leads us to the following result.

Theorem 10: If $(L, M) = 1$, $L > M$, L has l digits and 9 divides $(L - M) \times H$, then the following rational number is a 1_H-CN for $\frac{L}{M}$:

$$\frac{X + H \cdot 10^l + Y}{S + H \cdot 10^l + T}$$

where

$X = M^{\phi(L)-1} \cdot (\frac{L-M}{9}) \cdot H(modL) \cdot 10^{l+1}$,

$Y = M^{\phi(L)-1} \cdot (\frac{M-L}{9}) \cdot 10^l \cdot H(modL)$,

$S = \frac{1}{L}\left[M^{\phi(L)} \cdot (\frac{L-M}{9}) \cdot H(modL) - (\frac{L-M}{9}) \cdot H\right] \cdot 10^{l+1}$,

$T = \frac{1}{L}\left[M^{\phi(L)} \cdot (\frac{M-L}{9}) \cdot 10^l \cdot H(modL) - 10^l \cdot (\frac{M-L}{9}) \cdot H\right]$ ■

Let $L = 8$, $l = 1$, $M = 5$, $H = 3$. Notice $\phi(8) = 4$. Now

$$\frac{8}{5} = \frac{500 + 30 + 6}{\frac{1}{8}[25 - 1] \cdot 10^2 + 30 + \frac{1}{8}[5 \cdot 6 + 10]} = \frac{5\cancel{3}6}{\cancel{3}35} = \frac{56}{35} = \frac{8}{5}$$

Let $L = 11$, $l = 1$, $M = 2$, $H = 7$. Notice $\phi(11) = 10$. Now

$$\frac{11}{2} = \frac{9000 + 700 + 2}{1000 + 700 + 64} = \frac{9\cancel{7}02}{1\cancel{7}64} = \frac{902}{164} = \frac{11}{2}$$

The question of P_H cancellable numbers is settled by demonstrating that the situation is substantially the same sa for P_9 cancellable numbers, i.e., H's are concatenated.

Theorem 11: If $(L, M) = 1$, $L > M$, L has l digits and 9 divides $(L - M) \times H$, then the following rational number is a P_H-CN for $\frac{L}{M}$:

$$\frac{X + H \cdot (\sum_{i=0}^{p-1} 10^{l+i}) + Y}{S + H \cdot (\sum_{i=0}^{p-1} 10^{i}) + T}$$

where

$X = M^{\phi(L)-1} \cdot (\frac{L-M}{9}) \cdot H(modL) \cdot 10^{l+p}$,

$Y = M^{\phi(L)-1} \cdot (\frac{M-L}{9}) \cdot 10^{l} \cdot H(modL)$,

$S = \frac{1}{L}\Big[M^{\phi(L)} \cdot (\frac{L-M}{9}) \cdot H(modL) - (\frac{L-M}{9}) \cdot H\Big] \cdot 10^{l+p}$,

$T = \frac{1}{L}\Big[M^{\phi(L)} \cdot (\frac{M-L}{9}) \cdot 10^{l} \cdot H(modL) - 10^{l} \cdot (\frac{M-L}{9}) \cdot H\Big]$ ■

3 A Tribute

This is a personal note from Hossein Shahmohamad: This work is primarily about honoring and resurrecting the great work of one of my undergraduate professors, Dr. Jerome Henry Manheim, in the 1980s. My first attempt at a course in Differential Equations exposed me to the great mathematical talent of Jerry. While he was the main reason why I am writing my first book of puzzles in Farsi language, he successfully managed to intertwine second degree homogenous linear differential equations with mesmerizing and tantalizingly fascinating puzzles throughout his semester course.

Among all the puzzles presented during the first ten minutes of every lecture, the recurring theme of the cancellable Numbers was enigmatic. Each Monday, Professor Manheim would share a bit of some new theorem and on the following Wednesday, I would see a part of another possibly-true conjecture. For some reason, I managed to not only take down as many mental and written notes, but also to save my course notebooks after nearly 28 years.

While I have received great help from the two brilliant graduate students involved in this project, the bulk of the work must be contributed and dedicated to Jerry Manheim who managed not only to teach me Mathematics, but to strongly change my life and to intensely influence and mentor me to become the person who I am today. While many statements were spoken, written or presented to me 3 decades ago, I have done my best to carefully put together, fill holes and present arguments from the pieces of my ancient notes. We declare Dr. Jerome Henry Manheim to be the (main) author of this research endeavor and we humbly pay homage and tribute to him and his family.

References

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[2] Schwartz, B.L., "Illegal Cancellation," proposal 434; Mathematical magazine (September-October, 1961), Vol. 34, No. 6, 367-68