New Sufficient Conditions for Some Hamiltonian Properties of Graphs

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Abstract

For a connected graph $G = (V, E)$, its inverse degree is defined as $\sum_{v \in V} \frac{1}{d(v)}$. Using an upper bound for the inverse degree of a graph obtained by Cioabă in [6], In this note, we present new sufficient conditions for some Hamiltonian properties of graphs.

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow that in [4]. For a graph $G = (V, E)$, we use n and e to denote its order $|V|$ and size $|E|$, respectively. We use $\delta = d_1 \leq d_2 \leq \cdots \leq d_n = \Delta$ to denote the degree sequence of G. If G is connected, its inverse degree is defined as $\sum_{v \in V} \frac{1}{d(v)}$. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path. A graph G is called Hamilton - connected if for each pair of vertices in G there is a Hamiltonian path between them. A graph G is called r - Hamiltonain (where r is a nonnegative integer) if for each subset X of the vertex set of G with $|X| \leq r$ the induced subgraph $G[V(G) - X]$

is Hamiltonain (see [5] and [3]). Obviously, every 0 - Hamiltonian graph is Hamiltonian. A graph G is called r - edge - Hamiltonain (where r is a nonnegative integer) if any collection of vertex - disjoint paths with at most r edges is contained in a Hamiltonian cycle of G (see Page 204 in [2]). Obviously, every 0 - edge - Hamiltonian graph is Hamiltonian. The smallest number of pairwise disjoint paths covering all the vertices of G is denoted by $pc(G)$ (see [3]). A graph G is called r - piece - traceable (where r is a positive integer) if $pc(G) \leq r$, Obviously, every 1 - piece - traceable graph is traceable. A graph G of order n is called pancyclic if G contains cycles of lengths from 3 to n.

Using an upper bound for the inverse degree of a graph obtained by Cioabă in [6], Li presented sufficient conditions for Hamiltonicity, traceability, Hamilton - connectivity, and k - connectivity of graphs in [7]. Using the ideas similar to the ones in [7], we in this note present sufficient conditions for the Hamiltonicity of bipartite graphs, the r - Hamiltonicity, the r - edge - Hamiltonicity, the r - piece - traceability, and the pancycility of graphs. The main results are as follows.

Theorem 1. Let $G = (X, Y; E)$, where $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1, y_2, ..., y_n\}$, and $n \geq 2$, be a connected bipartite graph. If

$$\frac{2n^2}{e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(2n - 1 - \frac{e}{n}\right) < 3,$$

then G is Hamiltonian.

Theorem 2. Let G be a connected graph of order $n \geq 3$ and size e. Suppose r is a nonnegative integer such that $r \leq n - 3$. If

$$\frac{n^2}{2e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n - 1 - \frac{2e}{n}\right) < \frac{1}{1+r} + \frac{1}{n-2} + \frac{1+r}{\Delta},$$

then G is r - Hamiltonian.

Theorem 3. Let G be a connected graph of order $n \geq 3$ and size e. Suppose r is a nonnegative integer such that $r \leq n - 3$. If

$$\frac{n^2}{2e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n - 1 - \frac{2e}{n}\right) < 1 - r + \frac{1}{n-2} + \frac{1+r}{\Delta},$$

then G is r - edge - Hamiltonian.

Theorem 4. Let G be a connected graph of order n and size e. Suppose r is a positive integer such that $r \leq n-3$. If

$$\frac{n^2}{2e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n - 1 - \frac{2e}{n}\right) < \frac{n+r-1}{n-r-1} + \frac{1}{n-3} + \frac{1}{\Delta},$$

then G is r - piece - traceable.

Theorem 5. Let G be a connected graph of order $n \geq 3$ and size e. If

$$\frac{n^2}{2e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n - 1 - \frac{2e}{n}\right) < 1 + \frac{1}{n-2} + \frac{1}{\Delta},$$

then G is pancyclic or bipartite.

Plugging in $r = 0$ in Theorems 2 and 3, they become Theorem 2.1 in [7]. Thus Theorems 2 and 3 are generalizations of Theorem 2.1 in [7]. Plugging in $r = 1$ in Theorem 4, it becomes Theorem 2.2 in [7]. Thus Theorem 4 is a generalization of Theorem 2.2 in [7].

2. Lemmas

In order to prove the theorems above, we need the following results as our lemmas. Lemma 1 below is Corollary 5 on Page 210 in [2].

Lemma 1. Let $G = (X, Y; E)$ be a bipartite graph such that $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1, y_2, ..., y_n\}$, $n \geq 2$, $d(x_1) \leq d(x_2) \leq \cdots \leq d(x_n)$, and $d(y_1) \leq d(y_2) \leq \cdots \leq d(y_n)$. If

$$d(x_k) \leq k < n \Longrightarrow d(y_{n-k}) \geq n - k + 1,$$

then G is Hamiltonian.

Lemma 2 below is Corollary 1.2 on Page 166 in [5].

Lemma 2. Let G be a graph of order $n \geq 3$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Suppose r is a nonnegative integer such that $r \leq n-3$. If for each k with $1 \leq k < (n-r)/2$

$$d_k \leq k + r \Longrightarrow d_{n-k-r} \geq n - k,$$

then G is r - Hamiltonian.

Lemma 3 below is Theorem 8 on Page 204 in [2].

Lemma 3. Let G be a graph of order $n \geq 3$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Suppose r is a nonnegative integer such that $r \leq n-3$. If for each k with $r+1 \leq k < (n+r)/2$

$$d_{k-r} \leq k \Longrightarrow d_{n-k} \geq n-k+r,$$

then G is r - edge - Hamiltonian.

Lemma 4 below is Theorem 5.1 on Pages 119 and 121 in [3].

Lemma 4. Let G be a graph of order n with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Suppose r is a positive integer. If for each k with $1 \leq k < (n-r)/2$

$$d_{k+r} \leq k \Longrightarrow d_{n-k} \geq n-k-r,$$

then G is r - piece - traceable.

Lemma 5 below is Theorem 1 on Pages 112 in [1].

Lemma 5. Let G be a graph of order $n \geq 3$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If for each k with $1 \leq k < n/2$

$$d_k \leq k \Longrightarrow d_{n-k} \geq n-k,$$

then G is pancyclic or bipartite.

Lemma 6 is from Theorem 9 on Page 1963 in [6].

Lemma 6. Let G be a connected graph of order n and size e. Then

$$\sum_{v\in V}\frac{1}{d(v)} \leq \frac{n^2}{2e} + \left(\frac{1}{\delta}-\frac{1}{\Delta}\right)\left(n-1-\frac{2e}{n}\right).$$

3. Proofs

Proof of Theorem 1. Let G be a graph satisfying the conditions in Theorem 1. Suppose that G is not Hamiltonian. Without loss of generality, we assume that $d(x_1) \leq d(x_2) \leq \cdots \leq d(x_n)$ and $d(y_1) \leq d(y_2) \leq \cdots \leq d(y_n)$. Then, from Lemma 1, there exists an integer k such that $d(x_k) \leq k < n$ and $d(y_{n-k}) \leq n-k$. Obviously, $k \geq 1$. Therefore, from Lemma 6, we have that

$$\frac{2n^2}{e} + \left(\frac{1}{\delta}-\frac{1}{\Delta}\right)\left(2n-1-\frac{e}{n}\right)$$
$$= \frac{(2n)^2}{2e} + \left(\frac{1}{\delta}-\frac{1}{\Delta}\right)\left(2n-1-\frac{2e}{2n}\right) \geq \sum_{v\in V}\frac{1}{d(v)}$$

$$= \frac{1}{d(x_1)} + \cdots + \frac{1}{d(x_k)} + \frac{1}{d(x_{k+1})} + \cdots + \frac{1}{d(x_n)}$$
$$+ \frac{1}{d(y_1)} + \cdots + \frac{1}{d(y_{n-k})} + \frac{1}{d(y_{n-k+1})} + \cdots + \frac{1}{d(y_n)}$$
$$\geq \frac{k}{d(x_k)} + \frac{n-k}{d(x_n)} + \frac{n-k}{d(y_{n-k})} + \frac{k}{d(y_n)}$$
$$\geq \frac{k}{k} + \frac{n-k}{n} + \frac{n-k}{n-k} + \frac{k}{n} = 3,$$

a contradiction. This completes the proof of Theorem 1. QED

Proof of Theorem 2. Let G be a graph satisfying the conditions in Theorem 2. Suppose that G is not r - Hamiltonian. Then, from Lemma 2, there exists an integer k such that $1 \leq k < (n-r)/2$, $d_k \leq k+r$, and $d_{n-k-r} \leq n-k-1$. Therefore, from Lemma 6, we have that

$$\frac{n^2}{2e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n - 1 - \frac{2e}{n}\right) \geq \sum_{v \in V} \frac{1}{d(v)}$$

$$= \frac{1}{d_1} + \cdots + \frac{1}{d_k} + \frac{1}{d_{k+1}} + \cdots + \frac{1}{d_{n-k-r}} + \frac{1}{d_{n-k-r+1}} + \cdots + \frac{1}{d_n}$$
$$\geq \frac{k}{d_k} + \frac{n-2k-r}{d_{n-k-r}} + \frac{k+r}{d_n} \geq \frac{k}{k+r} + \frac{n-2k-r}{n-k-1} + \frac{k+r}{\Delta}$$
$$\geq \frac{1}{1+r} + \frac{1}{n-2} + \frac{1+r}{\Delta},$$

a contradiction. This completes the proof of Theorem 2. QED

Proof of Theorem 3. Let G be a graph satisfying the conditions in Theorem 3. Suppose that G is not r - edge - Hamiltonian. Then, from Lemma 3, there exists an integer k such that $r+1 \leq k < (n+r)/2$, $d_{k-r} \leq k$, and $d_{n-k} \leq n-k+r-1$. Therefore, from Lemma 6, we have that

$$\frac{n^2}{2e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n - 1 - \frac{2e}{n}\right) \geq \sum_{v \in V} \frac{1}{d(v)}$$

$$= \frac{1}{d_1} + \cdots + \frac{1}{d_{k-r}} + \frac{1}{d_{k-r+1}} + \cdots + \frac{1}{d_{n-k}} + \frac{1}{d_{n-k+1}} + \cdots + \frac{1}{d_n}$$
$$\geq \frac{k-r}{d_{k-r}} + \frac{n-2k+r}{d_{n-k}} + \frac{k}{d_n} \geq \frac{k-r}{k} + \frac{n-2k+r}{n-k+r-1} + \frac{k}{\Delta}$$
$$\geq 1 - r + \frac{1}{n-2} + \frac{1+r}{\Delta},$$

a contraction. This completes the proof of Theorem 3. QED

Proof of Theorem 4. Let G be a graph satisfying the conditions in Theorem 4. Suppose that G is not r - piece - traceable. Then, from Lemma 4, there exists an integer k such that such that $1 \leq k < (n-r)/2$, $d_{k+r} \leq k$, and $d_{n-k} \leq n-k-r-1$. Therefore, from Lemma 6, we have that

$$\frac{n^2}{2e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n - 1 - \frac{2e}{n}\right) \geq \sum_{v \in V} \frac{1}{d(v)}$$

$$= \frac{1}{d_1} + \cdots + \frac{1}{d_{k+r}} + \frac{1}{d_{k+r+1}} + \cdots + \frac{1}{d_{n-k}} + \frac{1}{d_{n-k+1}} + \cdots + \frac{1}{d_n}$$

$$\geq \frac{k+r}{d_{k+r}} + \frac{n-2k-r}{d_{n-k}} + \frac{k}{d_n} \geq \frac{k+r}{k} + \frac{n-2k-r}{n-k-r-1} + \frac{k}{\Delta}$$

$$\geq 1 + \frac{r}{k} + \frac{1}{n-3} + \frac{1}{\Delta} \geq 1 + \frac{r}{\frac{n-r-1}{2}} + \frac{1}{n-3} + \frac{1}{\Delta}$$

$$\geq 1 + \frac{2r}{n-r-1} + \frac{1}{n-3} + \frac{1}{\Delta}$$

$$= \frac{n+r-1}{n-r-1} + \frac{1}{n-3} + \frac{1}{\Delta},$$

a contradiction. This completes the proof of Theorem 4. QED

Proof of Theorem 5. Let G be a graph satisfying the conditions in Theorem 5. Suppose that G is neither pancyclic nor bipartite. Then, from Lemma 5, there exists an integer k such that $1 \leq k < \frac{n}{2}$, $d_k \leq k$, and $d_{n-k} \leq n-k-1$. Therefore, from Lemma 6, we have that

$$\frac{n^2}{2e} + \left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\left(n - 1 - \frac{2e}{n}\right) \geq \sum_{v \in V} \frac{1}{d(v)}$$

$$= \frac{1}{d_1} + \cdots + \frac{1}{d_k} + \frac{1}{d_{k+1}} + \cdots + \frac{1}{d_{n-k}} + \frac{1}{d_{n-k+1}} + \cdots + \frac{1}{d_n}$$

$$\geq \frac{k}{d_k} + \frac{n-2k}{d_{n-k}} + \frac{k}{d_n} \geq \frac{k}{k} + \frac{n-2k}{n-k-1} + \frac{k}{\Delta}$$

$$\geq 1 + \frac{1}{n-2} + \frac{1}{\Delta},$$

a contradiction. This completes the proof of Theorem 5. QED

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