Decompositions of λK_n into LW and OW Graphs

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ABSTRACT. In this paper, we identify LW and OW graphs, find the minimum λ for decomposition of λK_n into these graphs, and show that for all viable values of λ, the necessary conditions are sufficient for LW- and OW-decompositions using cyclic decompositions from base graphs.

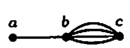
1. Introduction

Decompositions of graphs into subgraphs is a well-known classical problem; for an excellent survey on graph decompositions, see [**1**]. Recently, several people including Chan [**4**], El-Zanati, Lapchinda, Tangsupphathawat and Wannasit [**5**], Hein [**6, 7**], Hurd [**11**], Sarvate [**8, 9, 10**], Winter [**13, 14**] and Zhang [**15**] have worked on decomposing λK_n into multigraphs. In fact, similar decompositions have been attempted earlier in various papers; see [**12**]. Ternary designs also provide such decompositions; see [**2, 3**].

2. Preliminaries

For simplicity of notation, we use the "alphabetic labeling" used in [**6, 7, 8, 9, 10, 13, 14, 15**]:

DEFINITION 1. *An* LW graph *(denoted* $\lceil a, b, c \rceil$*) on* $V = \{a, b, c\}$ *is a graph with 5 edges where the frequencies of edges* $\{a, b\}$ *and* $\{b, c\}$ *are 1 and 4 (respectively).*



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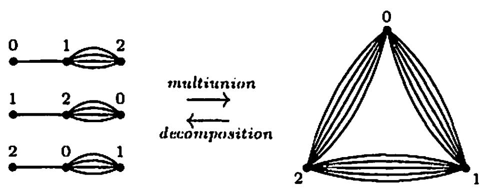
DEFINITION 2. *An* OW graph *(denoted* $\|a, b, c\|$*) on* $V = \{a, b, c\}$ *is a graph with 6 edges where the frequencies of edges* $\{a, b\}$ *and* $\{b, c\}$ *are 2 and 4 (respectively).*

DEFINITION 3. *For positive integers* $n \geq 3$ *and* $\lambda \geq 4$*, an* LW-decomposition *of* λK_n *(denoted* $LW(n, \lambda)$*) is a collection of LW graphs such that the multiunion of their edge sets contains* λ *copies of all edges in a* K_n*.*

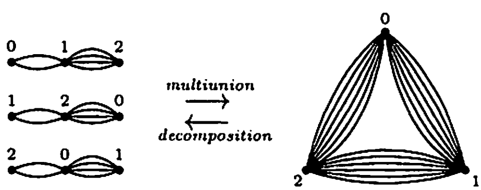
DEFINITION 4. *For positive integers* $n \geq 3$ *and* $\lambda \geq 4$*, an* OW-decomposition *of* λK_n *(denoted* $OW(n, \lambda)$*) is a collection of OW graphs such that the multiunion of their edge sets contains* λ *copies of all edges in a* K_n*.*

One of the powerful techniques to construct combinatorial designs is based on *difference sets* and *difference families*; see [**16**] for details. This technique is modified to achieve our decompositions of λK_n — in general, we exhibit the *base graphs*, which can be developed to obtain the decomposition.

EXAMPLE 1. *Considering the set of points to be* $V = \mathbb{Z}_3$*, the LW base graph* $\lceil 0, 1, 2 \rceil$ *(when developed modulo 3) constitutes an* $LW(3, 5)$*.*

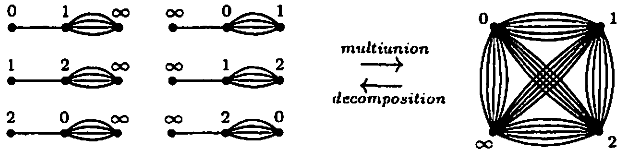


EXAMPLE 2. *Considering the set of points to be* $V = \mathbb{Z}_3$*, the OW base graph* $\|0, 1, 2\|$ *(when developed modulo 3) constitutes an* $OW(3, 6)$*.*

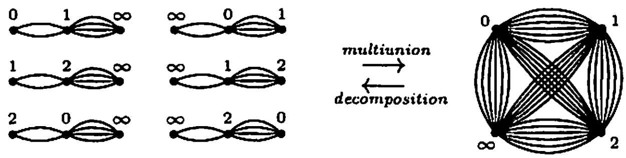


We note that special attention is needed with the base graphs containing the "dummy element" ∞; the non-∞ elements are developed, while ∞ is simply rewritten each time.

EXAMPLE 3. *Considering the set of points to be* $V = \mathbb{Z}_3 \cup \{\infty\}$, *the LW base graphs* $\lceil 0, 1, \infty \rceil$ *and* $\lceil \infty, 0, 1 \rceil$ *(when developed modulo 3) constitute an* $LW(4, 5)$.



EXAMPLE 4. *Considering the set of points to be* $V = \mathbb{Z}_3 \cup \{\infty\}$, *the OW base graphs* $\|0, 1, \infty\|$ *and* $\|\infty, 0, 1\|$ *(when developed modulo 3) constitute an* $OW(4, 6)$.



3. LW–Decompositions

We first address the minimum values of λ in an LW(n, λ). Recall that $\lambda \geq 4$.

THEOREM 3.1. *Let* $n \geq 3$. *The minimum values of* λ *for which an* $LW(n, \lambda)$ *exists are* $\lambda = 4$ *when* $n \equiv 0, 1 \pmod 5$ *and* $\lambda = 5$ *when* $n \not\equiv 0, 1 \pmod 5$.

PROOF. Since there are $\frac{\lambda n(n-1)}{2}$ edges in a λK_n, and 5 edges in an LW graph, we must have that $\lambda n(n-1) \equiv 0 \pmod{10}$ (where $n \geq 3$ and $\lambda \geq 4$) for LW-decompositions. The result follows from cases on $n \pmod{10}$. ■

We are now in a position to prove the main results of the paper. We first remark that an LW graph has 3 vertices; that is, we consider $n \geq 3$. Also, necessarily $\lambda \geq 4$. We note that we use difference sets to achieve our decompositions of λK_n. In general, we exhibit the base graphs, which can be developed (modulo either n or $n-1$) to obtain the decomposition. We also note that the frequency of the edges is fixed by position, as per the LW graph.

THEOREM 3.2. *The minimum number copies of* K_n *(as given in Theorem 3.1) can be decomposed into LW graphs.*

PROOF. Let $n \geq 3$. We proceed by cases on $n \pmod{10}$.

If $n = 10t$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{10t-1} \cup \{\infty\}$. The number of graphs required for LW$(10t, 4)$ is $\frac{4(10t)(10t-1)}{10} = 4t(10t-1)$. Thus, we need $4t$ base graphs (modulo $10t-1$). Then, the differences we must achieve (modulo $10t-1$) are $1, 2, \ldots, 5t-1$. For the first four base graphs, use $\lceil 1, 0, \infty \rceil$, $\lceil 0, 1, 5t \rceil$, $\lceil 0, 1, 5t-1 \rceil$ and $\lceil 0, 1, 5t-2 \rceil$. We also use the $4t-4$ base graphs $\lceil 0, 2, 5t-2 \rceil$, $\lceil 0, 2, 5t-3 \rceil$, $\lceil 0, 2, 5t-4 \rceil$, $\lceil 0, 2, 5t-5 \rceil, \ldots, \lceil 0, t, 2t+4 \rceil$, $\lceil 0, t, 2t+3 \rceil$, $\lceil 0, t, 2t+2 \rceil$ and $\lceil 0, t, 2t+1 \rceil$ if necessary. Hence, in this case, LW$(10t, 4)$ exists.

If $n = 10t + 1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{10t+1}$. The number of graphs required for LW$(10t + 1, 4)$ is $\frac{4(10t+1)(10t)}{10} = 4t(10t + 1)$. Thus, we need $4t$ base graphs (modulo $10t + 1$). Then, the differences we must achieve (modulo $10t+1$) are $1, 2, \ldots, 5t$. We use the base graphs $\lceil 0, 1, 5t+1 \rceil$, $\lceil 0, 1, 5t \rceil$, $\lceil 0, 1, 5t-1 \rceil$, $\lceil 0, 1, 5t-2 \rceil$, $\lceil 0, 2, 5t-2 \rceil$, $\lceil 0, 2, 5t-3 \rceil$, $\lceil 0, 2, 5t-4 \rceil$, $\lceil 0, 2, 5t-5 \rceil, \ldots, \lceil 0, t, 2t+4 \rceil$, $\lceil 0, t, 2t + 3 \rceil$, $\lceil 0, t, 2t + 2 \rceil$ and $\lceil 0, t, 2t + 1 \rceil$. Hence, in this case, LW$(10t + 1, 4)$ exists.

If $n = 10t + 2$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{10t+1} \cup \{\infty\}$. The number of graphs required for LW$(10t + 2, 5)$ is $\frac{5(10t+2)(10t+1)}{10} = (5t + 1)(10t + 1)$. Thus, we need $5t + 1$ base graphs (modulo $10t + 1$). Then, the differences we must achieve (modulo $10t + 1$) are $1, 2, \ldots, 5t$. For the first six base graphs, use $\lceil 1, 0, \infty \rceil$, $\lceil \infty, 0, 1 \rceil$, $\lceil 0, 2, 4 \rceil$, $\lceil 0, 3, 6 \rceil$, $\lceil 0, 4, 8 \rceil$ and $\lceil 0, 5, 10 \rceil$. We also use the $5t - 5$ base graphs $\lceil 0, 6, 12 \rceil$, $\lceil 0, 7, 14 \rceil, \ldots, \lceil 0, 5t, 10t \rceil$ if necessary. Hence, in this case, LW$(10t + 2, 5)$ exists.

If $n = 10t + 3$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{10t+3}$. The number of graphs required for LW$(10t+3, 5)$ is $\frac{5(10t+3)(10t+2)}{10} = (5t+1)(10t+3)$. Thus, we need $5t+1$ base graphs (modulo $10t+3$). Then, the differences we must achieve (modulo $10t + 3$) are $1, 2, \ldots, 5t + 1$. We use the base graphs $\lceil 0, 1, 5t+2 \rceil$, $\lceil 0, 2, 5t+2 \rceil, \ldots, \lceil 0, 5t+1, 5t+2 \rceil$. Hence, in this case, LW$(10t + 3, 5)$ exists.

If $n = 10t + 4$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{10t+3} \cup \{\infty\}$. The number of graphs required for LW$(10t + 4, 5)$ is $\frac{5(10t+4)(10t+3)}{10} = (5t + 2)(10t + 3)$. Thus, we need $5t + 2$ base graphs (modulo $10t + 3$). Then, the differences we must achieve (modulo $10t + 3$) are $1, 2, \ldots, 5t + 1$. For the first two base graphs, we use $\lceil 1, 0, \infty \rceil$ and $\lceil \infty, 0, 1 \rceil$. We also use the $5t$ base graphs $\lceil 0, 2, 4 \rceil$, $\lceil 0, 3, 6 \rceil, \ldots, \lceil 0, 5t+1, 10t + 2 \rceil$ if necessary. Hence, in this case, LW$(10t + 4, 5)$ exists.

If $n = 10t+5$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{10t+4} \cup \{\infty\}$. The number of graphs required for $\mathrm{LW}(10t+5,4)$ is $\frac{4(10t+5)(10t+4)}{10} = (4t+2)(10t+4)$. Thus, we need $4t+2$ base graphs (modulo $10t+4$). Then, the differences we must achieve (modulo $10t+4$) are $1, 2, \ldots, 5t+2$. For the first two base graphs, we use $\lceil 5t+2, 0, \infty \rceil$ and $\lceil 0, 5t+2, 10t+3 \rceil$. We also use the $4t$ base graphs $\lceil 0,1,5t+1 \rceil$, $\lceil 0,1,5t \rceil$, $\lceil 0,1,5t-1 \rceil$, $\lceil 0,1,5t-2 \rceil$, $\lceil 0,2,5t-2 \rceil$, $\lceil 0,2,5t-3 \rceil$, $\lceil 0,2,5t-4 \rceil$, $\lceil 0,2,5t-5 \rceil, \ldots, \lceil 0,t,2t+4 \rceil$, $\lceil 0,t,2t+3 \rceil$, $\lceil 0,t,2t+2 \rceil$ and $\lceil 0,t,2t+1 \rceil$ if necessary. Hence, in this case, $\mathrm{LW}(10t+5,4)$ exists.

If $n = 10t+6$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{10t+6}$. The number of graphs required for $\mathrm{LW}(10t+6,4)$ is $\frac{4(10t+6)(10t+5)}{10} = (4t+2)(10t+6)$. Thus, we need $4t+2$ base graphs (modulo $10t+6$). Then, the differences we must achieve (modulo $10t+6$) are $1, 2, \ldots, 5t+3$. For the first two base graphs, we use $\lceil 0, 5t+3, 10t+5 \rceil$ and $\lceil 0, 5t+3, 10t+4 \rceil$. We also use the $4t$ base graphs $\lceil 0,1,5t+1 \rceil$, $\lceil 0,1,5t \rceil$, $\lceil 0,1,5t-1 \rceil$, $\lceil 0,1,5t-2 \rceil$, $\lceil 0,2,5t-2 \rceil$, $\lceil 0,2,5t-3 \rceil$, $\lceil 0,2,5t-4 \rceil$, $\lceil 0,2,5t-5 \rceil, \ldots, \lceil 0,t,2t+4 \rceil$, $\lceil 0,t,2t+3 \rceil$, $\lceil 0,t,2t+2 \rceil$ and $\lceil 0,t,2t+1 \rceil$ if necessary. Hence, in this case, $\mathrm{LW}(10t+6,4)$ exists.

If $n = 10t+7$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{10t+7}$. The number of graphs required for $\mathrm{LW}(10t+7,5)$ is $\frac{5(10t+7)(10t+6)}{10} = (5t+3)(10t+7)$. Thus, we need $5t+3$ base graphs (modulo $10t+7$). Then, the differences we must achieve (modulo $10t+7$) are $1, 2, \ldots, 5t+3$. We use the base graphs $\lceil 0,1,5t+4 \rceil$, $\lceil 0,2,5t+4 \rceil, \ldots, \lceil 0,5t+3,5t+4 \rceil$. Hence, in this case, $\mathrm{LW}(10t+7,5)$ exists.

If $n = 10t+8$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{10t+7} \cup \{\infty\}$. The number of graphs required for $\mathrm{LW}(10t+8,5)$ is $\frac{5(10t+8)(10t+7)}{10} = (5t+4)(10t+7)$. Thus, we need $5t+4$ base graphs (modulo $10t+7$). Then, the differences we must achieve (modulo $10t+7$) are $1, 2, \ldots, 5t+3$. For the first four base graphs, we use $\lceil 5t+3, 0, \infty \rceil$ $\lceil \infty, 0, 1 \rceil$ $\lceil 0,1,3 \rceil$ and $\lceil 0,2,5 \rceil$. We also use the $5t$ base graphs $\lceil 0,3,7 \rceil$, $\lceil 0,4,9 \rceil, \ldots, \lceil 0, 5t+2, 10t+5 \rceil$ if necessary. Hence, in this case, $\mathrm{LW}(10t+8,5)$ exists.

If $n = 10t+9$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{10t+9}$. The number of graphs required for $\mathrm{LW}(10t+9,5)$ is $\frac{5(10t+9)(10t+8)}{10} = (5t+4)(10t+9)$. Thus, we need $5t+4$ base graphs (modulo $10t+9$). Then, the differences we must achieve (modulo $10t+9$) are $1, 2, \ldots, 5t+4$. We use the base graphs $\lceil 0,1,5t+5 \rceil$, $\lceil 0,2,5t+5 \rceil, \ldots, \lceil 0,5t+4,5t+5 \rceil$. Hence, in this case, $\mathrm{LW}(10t+9,5)$ exists. ■

We now address the sufficiency of existence of LW(n, λ).

THEOREM 3.3. *Let $n \geq 3$ and $\lambda \geq 4$. For LW(n, λ), the necessary condition for n is that $n \equiv 0, 1, 5, 6 \pmod{10}$ when $\lambda \not\equiv 0, 5 \pmod{10}$. There is no condition for n when $\lambda \equiv 0, 5 \pmod{10}$.*

PROOF. Similar to the proof of Theorem 3.1, but by cases on $\lambda \pmod{10}$. ■

LEMMA 3.1. *There exists an LW$(n, 4)$ for the necessary $n \geq 3$.*

PROOF. From Theorem 3.3, the necessary condition is $n \equiv 0, 1, 5, 6 \pmod{10}$. In these cases, LW$(n, 4)$ exists from Theorem 3.2. ■

LEMMA 3.2. *There exists an LW$(n, 5)$ for any $n \geq 3$.*

PROOF. From Theorem 3.3, there is no condition for n. We consider cases when $n \geq 3$ is odd or even.

If $n = 2t + 1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{2t+1}$. The number of graphs required for LW$(2t+1, 5)$ is $\frac{5(2t+1)(2t)}{10} = t(2t+1)$. Thus, we need t base graphs (modulo $2t + 1$). The differences we must achieve (modulo $2t+1$) are $1, 2, \ldots, t$. We use the base graphs $\lceil 0, 1, t+1 \rceil, \ldots, \lceil 0, t, t+1 \rceil$. Hence, in this case, LW$(2t+1, 5)$ exists.

If $n = 2t$ (for $t \geq 2$), we consider the set V as $\mathbb{Z}_{2t-1} \cup \{\infty\}$. The number of graphs required for LW$(2t, 5)$ is $\frac{5(2t)(2t-1)}{10} = t(2t-1)$. Thus, we need t base graphs (modulo $2t - 1$). The differences we must achieve (modulo $2t - 1$) are $1, 2, \ldots, t-1$. For the first two base graphs, we use $\lceil t-1, 0, \infty \rceil$ and $\lceil \infty, 0, t-1 \rceil$. We also use the $t-2$ base graphs $\lceil 0, 1, t-1 \rceil, \ldots, \lceil 0, t-2, t-1 \rceil$ if necessary. Hence, in this case, LW$(2t, 5)$ exists. ■

LEMMA 3.3. *There does not exist an LW$(n, 6)$.*

PROOF. The only edge frequencies in an LW graph are 1 and 4. The only way to write $\lambda = 6$ as a sum of 1s and 4s (both) is as $6 = 4 + 1 + 1$. In an LW$(n, 6)$, the number of times each edge needs to occur with frequency 4 is always less than the number of times it needs to occur with frequency 1. However, as there are equal numbers of single edges and quadruple edges in an LW graph, such a decomposition is not possible. ■

LEMMA 3.4. *There does not exist an LW$(n, 7)$.*

PROOF. The only edge frequencies in an LW graph are 1 and 4. The only way to write $\lambda = 7$ as a sum of 1s and 4s (both) is as

$7 = 4 + 1 + 1 + 1$. In an LW$(n, 7)$, the number of times each edge needs to occur with frequency 4 is always less than the number of times it needs to occur with frequency 1. However, as there are equal numbers of single edges and quadruple edges in an LW graph, such a decomposition is not possible. ■

LEMMA 3.5. *There does not exist an LW$(n, 11)$.*

PROOF. The only edge frequencies in an LW graph are 1 and 4. The only ways to write $\lambda = 11$ as a sum of 1s and 4s (both) are as $11 = 4+4+1+1+1$ and $11 = 4+1+\ldots+1$. In an LW$(n, 11)$, the number of times each edge needs to occur with frequency 4 is always less than the number of times it needs to occur with frequency 1. However, as there are equal numbers of single edges and quadruple edges in an LW graph, such a decomposition is not possible. ■

THEOREM 3.4. *An LW(n, λ) exists for all $\lambda \geq 4$ except $\lambda = 6$ (according to Lemma 3.3), $\lambda = 7$ (according to Lemma 3.4) and $\lambda = 11$ (according to Lemma 3.5), for corresponding necessary $n \geq 3$.*

PROOF. We proceed by cases on $\lambda \pmod 5$.

For $\lambda \equiv 0 \pmod 5$ (so that $\lambda = 5t$ for $t \geq 1$), by taking t copies of an LW$(n, 5)$ (given in Lemma 3.2), we have an LW$(n, 5t)$.

For $\lambda \equiv 1 \pmod 5$ (so that $\lambda = 5t + 1 = 5(t - 3) + 16$ for $t \geq 3$), we first take 4 copies of an LW$(n, 4)$ (given in Lemma 3.1). (This gives us $\lambda = 16$ thus far.) We then adjoin this to $t - 3$ copies of an LW$(n, 5)$ (given in Lemma 3.2) if necessary. Hence, we have an LW$(n, 5t + 1)$.

For $\lambda \equiv 2 \pmod 5$ (so that $\lambda = 5t + 2 = 5(t - 2) + 12$ for $t \geq 2$), we first take 3 copies of an LW$(n, 4)$ (given in Lemma 3.1). (This gives us $\lambda = 12$ thus far.) We then adjoin this to $t - 2$ copies of an LW$(n, 5)$ (given in Lemma 3.2) if necessary. Hence, we have an LW$(n, 5t + 2)$.

For $\lambda \equiv 3 \pmod 5$ (so that $\lambda = 5t+3 = 5(t-1)+8$ for $t \geq 1$), we first take 2 copies of an LW$(n, 4)$ (given in Lemma 3.1). (This gives us $\lambda = 8$ thus far.) We then adjoin this to $t-1$ copies of an LW$(n, 5)$ (given in Lemma 3.2) if necessary. Hence, we have an LW$(n, 5t + 3)$.

For $\lambda \equiv 4 \pmod 5$ (so that $\lambda = 5t + 4$ for $t \geq 0$), we first take an LW$(n, 4)$ (given in Lemma 3.1). (This gives us $\lambda = 4$ thus far.) We then adjoin this to t copies of an LW$(n, 5)$ (given in Lemma 3.2) if necessary. Hence, we have an LW$(n, 5t + 4)$. ■

4. OW–Decompositions

We first address the minimum values of λ in an OW(n, λ). Recall that $\lambda \geq 4$.

THEOREM 4.1. *Let $n \geq 3$. The minimum values of λ for which an OW(n, λ) exists are $\lambda = 4$ when $n \equiv 0, 1 \pmod 3$ and $\lambda = 6$ when $n \equiv 2 \pmod 3$.*

PROOF. Since there are $\frac{\lambda n(n-1)}{2}$ edges in a λK_n, and 6 edges in an OW graph, we must have that $\lambda n(n-1) \equiv 0 \pmod{12}$ (where $n \geq 3$ and $\lambda \geq 4$) for OW–decompositions. The result follows from cases on $n \pmod{12}$. ■

We are now in a position to prove the main results of the paper. We first remark that an OW graph has 3 vertices; that is, we consider $n \geq 3$. Also, necessarily $\lambda \geq 4$. We note that we use difference sets to achieve our decompositions of λK_n. In general, we exhibit the base graphs, which can be developed (modulo either n or $n-1$) to obtain the decomposition. We also note that the frequency of the edges is fixed by position, as per the OW graph.

THEOREM 4.2. *The minimum number copies of K_n (as given in Theorem 4.1) can be decomposed into OW graphs.*

PROOF. Let $n \geq 3$. We proceed by cases on $n \pmod{12}$.

If $n = 12t$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{12t-1} \cup \{\infty\}$. The number of graphs required for OW$(12t, 4)$ is $\frac{4(12t)(12t-1)}{12} = 4t(12t-1)$. Thus, we need $4t$ base graphs (modulo $12t-1$). Then, the differences we must achieve (modulo $12t-1$) are $1, 2, \ldots, 6t-1$. We use the base graphs $\|1, 0, \infty\|$, $\|0, 1, 6t\|$, $\|0, 2, 6t\|$, $\|0, 2, 6t-1\|$, $\|0, 3, 6t-1\|$, $\|0, 3, 6t-2\|, \ldots, \|0, 2t-1, 4t+3\|$, $\|0, 2t-1, 4t+2\|$, $\|0, 2t, 4t+2\|$ and $\|0, 2t, 4t+1\|$. Hence, in this case, OW$(12t, 4)$ exists.

If $n = 12t+1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{12t+1}$. The number of graphs required for OW$(12t+1, 4)$ is $\frac{4(12t+1)(12t)}{12} = 4t(12t+1)$. Thus, we need $4t$ base graphs (modulo $12t+1$). Then, the differences we must achieve (modulo $12t+1$) are $1, 2, \ldots, 6t$. We use the base graphs $\|0, 1, 6t+1\|$, $\|0, 1, 6t\|$, $\|0, 2, 6t\|$, $\|0, 2, 6t-1\|$, $\|0, 3, 6t-1\|$, $\|0, 3, 6t-2\|, \ldots, \|0, 2t-1, 4t+3\|$, $\|0, 2t-1, 4t+2\|$, $\|0, 2t, 4t+2\|$ and $\|0, 2t, 4t+1\|$. Hence, in this case, OW$(12t+1, 4)$ exists.

If $n = 12t+2$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{12t+1} \cup \{\infty\}$. The number of graphs required for OW$(12t+2, 6)$ is $\frac{6(12t+2)(12t+1)}{12} =$

$(6t+1)(12t+1)$. Thus, we need $6t+1$ base graphs (modulo $12t+1$). Then, the differences we must achieve (modulo $12t+1$) are $1, 2, \ldots, 6t$. For the first seven base graphs, use $\|1, 0, \infty\|$, $\|\infty, 0, 1\|$, $\|0, 2, 6t+2\|$, $\|0, 3, 6t+2\|$, $\|0, 4, 6t+2\|$, $\|0, 5, 6t+2\|$ and $\|0, 6, 6t+2\|$. We also use the $6t-6$ base graphs $\|0, 7, 6t+2\|$, $\|0, 8, 6t+2\|, \ldots, \|0, 6t, 6t+2\|$ if necessary. Hence, in this case, $\mathrm{OW}(12t+2, 6)$ exists.

If $n = 12t+3$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{12t+2} \cup \{\infty\}$. The number of graphs required for $\mathrm{OW}(12t+3, 4)$ is $\frac{4(12t+3)(12t+2)}{12} = (4t+1)(12t+2)$. Thus, we need $4t+1$ base graphs (modulo $12t+2$). Then, the differences we must achieve (modulo $12t+2$) are $1, 2, \ldots, 6t+1$. For the first base graph, use $\|6t+1, 0, \infty\|$. We also use the $4t$ base graphs $\|0, 1, 6t+1\|$, $\|0, 1, 6t\|$, $\|0, 2, 6t\|$, $\|0, 2, 6t-1\|, \ldots, \|0, 2t-1, 4t+3\|$, $\|0, 2t-1, 4t+2\|$, $\|0, 2t, 4t+2\|$, $\|0, 2t, 4t+1\|$ if necessary. Hence, in this case, $\mathrm{OW}(12t+3, 4)$ exists.

If $n = 12t+4$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{12t+4}$. The number of graphs required for $\mathrm{OW}(12t+4, 4)$ is $\frac{4(12t+4)(12t+3)}{12} = (4t+1)(12t+4)$. Thus, we need $4t+1$ base graphs (modulo $12t+4$). Then, the differences we must achieve (modulo $12t+4$) are $1, 2, \ldots, 6t+2$. For the first base graph, use $\|0, 6t+2, 12t+3\|$. We also use the $4t$ base graphs $\|0, 1, 6t+1\|$, $\|0, 1, 6t\|$, $\|0, 2, 6t\|$, $\|0, 2, 6t-1\|, \ldots, \|0, 2t-1, 4t+3\|$, $\|0, 2t-1, 4t+2\|$, $\|0, 2t, 4t+2\|$ and $\|0, 2t, 4t+1\|$ if necessary. Hence, in this case, $\mathrm{OW}(12t+4, 4)$ exists.

If $n = 12t+5$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{12t+5}$. The number of graphs required for $\mathrm{OW}(12t+5, 6)$ is $\frac{6(12t+5)(12t+4)}{12} = (6t+2)(12t+5)$. Thus, we need $6t+2$ base graphs (modulo $12t+5$). Then, the differences we must achieve (modulo $12t+5$) are $1, 2, \ldots, 6t+2$. We use the base graphs $\|0, 1, 6t+3\|$, $\|0, 2, 6t+3\|, \ldots, \|0, 6t+2, 6t+3\|$. Hence, in this case, $\mathrm{OW}(12t+5, 6)$ exists.

If $n = 12t+6$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{12t+5} \cup \{\infty\}$. The number of graphs required for $\mathrm{OW}(12t+6, 4)$ is $\frac{4(12t+6)(12t+5)}{12} = (4t+2)(12t+5)$. Thus, we need $4t+2$ base graphs (modulo $12t+5$). Then, the differences we must achieve (modulo $12t+5$) are $1, 2, \ldots, 6t+2$. For the first two base graphs, we use $\|\infty, 0, 6t+2\|$ and $\|\infty, 0, 6t+1\|$. We also use the $4t$ base graphs $\|0, 1, 6t+1\|$, $\|0, 1, 6t\|$, $\|0, 2, 6t\|$, $\|0, 2, 6t-1\|, \ldots, \|0, 2t-1, 4t+3\|$, $\|0, 2t-1, 4t+2\|$, $\|0, 2t, 4t+2\|$ and $\|0, 2t, 4t+1\|$ if necessary. Hence, in this case, $\mathrm{OW}(12t+6, 4)$ exists.

If $n = 12t + 7$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{12t+7}$. The number of graphs required for $\mathrm{OW}(12t+7, 4)$ is $\frac{4(12t+7)(12t+6)}{12} = (4t+2)(12t+7)$. Thus, we need $4t+2$ base graphs (modulo $12t+7$). Then, the differences we must achieve (modulo $12t+7$) are $1, 2, \ldots, 6t+3$. We use the base graphs $\|0, 1, 6t+4\|$, $\|0, 1, 6t+3\|$, $\|0, 2, 6t+3\|$, $\|0, 2, 6t+2\|, \ldots, \|0, 2t, 4t+5\|$, $\|0, 2t, 4t+4\|$, $\|0, 2t+1, 4t+4\|$, $\|0, 2t+1, 4t+3\|$. Hence, in this case, $\mathrm{OW}(12t+7, 4)$ exists.

If $n = 12t+8$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{12t+7} \cup \{\infty\}$. The number of graphs required for $\mathrm{OW}(12t+8, 6)$ is $\frac{6(12t+8)(12t+7)}{12} = (6t+4)(12t+7)$. Thus, we need $6t+4$ base graphs (modulo $12t+7$). Then, the differences we must achieve (modulo $12t+7$) are $1, 2, \ldots, 6t+3$. For the first four base graphs, use $\|6t+3, 0, \infty\|$, $\|\infty, 0, 6t+3\|$, $\|0, 1, 6t+3\|$ and $\|0, 2, 6t+3\|$. We also use the $6t$ base graphs $\|0, 3, 6t+3\|$, $\|0, 4, 6t+3\|, \ldots, \|0, 6t+2, 6t+3\|$ if necessary. Hence, in this case, $\mathrm{OW}(12t+8, 6)$ exists.

If $n = 12t+9$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{12t+8} \cup \{\infty\}$. The number of graphs required for $\mathrm{OW}(12t+9, 4)$ is $\frac{4(12t+9)(12t+8)}{12} = (4t+3)(12t+8)$. Thus, we need $4t+3$ base graphs (modulo $12t+8$). Then, the differences we must achieve (modulo $12t+8$) are $1, 2, \ldots, 6t+4$. For the first three base graphs, we use $\|0, 6t+4, 12t+7\|$, $\|\infty, 0, 6t+2\|$ and $\|\infty, 0, 6t+1\|$. We also use the $4t$ base graphs $\|0, 1, 6t+1\|$, $\|0, 1, 6t\|$, $\|0, 2, 6t\|$, $\|0, 2, 6t-1\|, \ldots, \|0, 2t-1, 4t+3\|$, $\|0, 2t-1, 4t+2\|$, $\|0, 2t, 4t+2\|$ and $\|0, 2t, 4t+1\|$ if necessary. Hence, in this case, $\mathrm{OW}(12t+9, 4)$ exists.

If $n = 12t+10$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{12t+10}$. The number of graphs required for $\mathrm{OW}(12t+10, 4)$ is $\frac{4(12t+10)(12t+9)}{12} = (4t+3)(12t+10)$. Thus, we need $4t+3$ base graphs (modulo $12t+10$). Then, the differences we must achieve (modulo $12t+10$) are $1, 2, \ldots, 6t+5$. For the first three base graphs, we use $\|0, 6t+5, 12t+9\|$ and $\|0, 1, 6t+4\|$, $\|0, 1, 6t+3\|$. We also use the $4t$ base graphs $\|0, 2, 6t+3\|$, $\|0, 2, 6t+2\|$, $\|0, 3, 6t+2\|$, $\|0, 3, 6t+1\|, \ldots, \|0, 2t, 4t+5\|$, $\|0, 2t, 4t+4\|$, $\|0, 2t+1, 4t+4\|$ and $\|0, 2t+1, 4t+3\|$ if necessary. Hence, in this case, $\mathrm{OW}(12t+10, 4)$ exists.

If $n = 12t+11$ (for $t \geq 0$), we consider the set V as $\mathbb{Z}_{12t+11}$. The number of graphs required for $\mathrm{OW}(12t+11, 6)$ is $\frac{6(12t+11)(12t+10)}{12} = (6t+5)(12t+11)$. Thus, we need $6t+5$ base graphs (modulo $12t+11$). Then, the differences we must achieve (modulo $12t+11$) are $1, 2, \ldots, 6t+5$. We use the base graphs $\|0, 1, 6t+6\|$, $\|0, 2, 6t+6\|$,

$\|0, 3, 6t+6\|, \ldots, \|0, 6t+4, 6t+6\|$ and $\|0, 6t+5, 6t+6\|$. Hence, in this case, OW$(12t+11, 6)$ exists. ■

We now address the sufficiency of existence of OW(n, λ).

THEOREM 4.3. *Let $n \geq 3$ and $\lambda \geq 4$. For OW(n, λ) to exist, λ must be even.*

PROOF. The only edge frequencies in an OW graph are 2 and 4. There is no linear combination of 2 and 4 that will equal λ (when λ is odd). Thus, in any OW(n, λ), we must have that λ is even. ■

THEOREM 4.4. *Let $n \geq 3$ and $\lambda \geq 4$. For OW(n, λ) to exist, the necessary conditions for n are that $n \equiv 0, 1, 3, 4 \pmod 6$ when $\lambda \equiv 2, 10 \pmod{12}$ and $n \equiv 0, 1 \pmod 3$ when $\lambda \equiv 4, 8 \pmod{12}$. There is no condition for n when $\lambda \equiv 0, 6 \pmod{12}$.*

PROOF. Similar to the proof of Theorem 4.1, but by cases on even $\lambda \pmod{12}$ (by Theorem 4.3). ■

LEMMA 4.1. *There exists an OW$(n, 4)$ for the necessary $n \geq 3$.*

PROOF. From Theorem 4.4, the necessary condition is $n \equiv 0, 1, 5, 6 \pmod{10}$. In these cases, OW$(n, 4)$ exists from Theorem 4.2. ■

LEMMA 4.2. *There exists an OW$(n, 6)$ for any $n \geq 3$.*

PROOF. From Theorem 4.4, there is no condition for n. We consider cases when $n \geq 3$ is odd or even.

If $n = 2t+1$ (for $t \geq 1$), we consider the set V as $\mathbb{Z}_{2t+1}$. The number of graphs required for OW$(2t+1, 6)$ is $\frac{6(2t+1)(2t)}{12} = t(2t+1)$. Thus, we need t base graphs (modulo $2t+1$). The differences we must achieve (modulo $2t+1$) are $1, 2, \ldots, t$. We use the base graphs $\|0, 1, t+1\|$, $\|0, 2, t+1\|, \ldots, \|0, t, t+1\|$. Hence, in this case, OW$(2t+1, 6)$ exists.

If $n = 2t$ (for $t \geq 2$), we consider the set V as $\mathbb{Z}_{2t-1} \cup \{\infty\}$. The number of graphs required for OW$(2t, 6)$ is $\frac{6(2t)(2t-1)}{12} = t(2t-1)$. Thus, we need t base graphs (modulo $2t-1$). The differences we must achieve (modulo $2t-1$) are $1, 2, \ldots, t-1$. For the first two base graphs, we use $\|t-1, 0, \infty\|$ and $\|\infty, 0, t-1\|$. We also use the $t-2$ base graphs $\|0, 1, t-1\|, \ldots, \|0, t-2, t-1\|$ if necessary. Hence, in this case, OW$(2t, 6)$ exists. ■

THEOREM 4.5. *An OW(n, λ) exists for all even (according to Theorem 4.3) $\lambda \geq 4$, for corresponding necessary $n \geq 3$.*

PROOF. We proceed by cases on even $\lambda \pmod 6$.

For $\lambda \equiv 0 \pmod 6$ (so that $\lambda = 6t$ for $t \geq 1$), by taking t copies of an $\mathrm{OW}(n, 6)$ (given in Lemma 4.2), we have an $\mathrm{OW}(n, 6t)$.

For $\lambda \equiv 2 \pmod 6$ (so that $\lambda = 6t+2 = 6(t-1)+8$ for $t \geq 1$), we first take 2 copies of an $\mathrm{OW}(n, 4)$ (given in Lemma 4.1). (This gives us $\lambda = 8$ thus far.) We then adjoin this to $t-1$ copies of an $\mathrm{OW}(n, 6)$ (given in Lemma 4.2) if necessary. Hence, we have an $\mathrm{OW}(n, 6t+2)$.

For $\lambda \equiv 4 \pmod 6$ (so that $\lambda = 6t+4$ for $t \geq 0$), we first take an $\mathrm{OW}(n, 4)$ (given in Lemma 4.1). (This gives us $\lambda = 4$ thus far.) We then adjoin this to t copies of an $\mathrm{OW}(n, 6)$ (given in Lemma 4.2) if necessary. Hence, we have an $\mathrm{OW}(n, 6t+4)$. ■

5. Conclusion

We have identified LW and OW graphs, found the minimum λ for decomposition of λK_n into these graphs, and showed that for all viable values of λ, the necessary conditions are sufficient for LW- and OW-decompositions.

We leave it as an open problem to find cyclic decompositions of λK_n into so-called EW graphs 

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