k-Zero Divisor Hypergraphs on Commutative Rings.

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October 24, 2015

Abstract

Let R be a commutative ring with identity. For any integer $k > 1$, an element is a k-zero divisor if there are distinct k elements including the given one, such that the product of all is zero but the product of fewer than all is nonzero. Let $Z(R, k)$ denote the set of the k-zero divisors of R. In this paper we consider rings which are not a k integral domains (i. e. $Z(R, k)$ is nontrivial) with finite $Z(R, k)$. We show that a uniform n exists such that $a^n = 0$ for all elements a of the nil-radical N and deduce that a ring R which is not a k-integral domain with more than k minimal prime ideals and whose nil-radical is finitely generated is finite, if $Z(R, k)$ is finite.

Keywords: Hypergraph, chromatic number, commutative rings, ideals, k-zero divisors.

MSC codes:: 13A05, 13E99, 13F15, 5C25

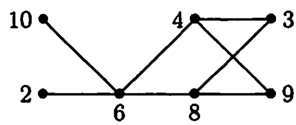
1 Introduction

A simple graph is an ordered pair (V, E), where V is a vertex set and E is an edge set with edges of the form $\{v_1, v_2\}$ where v_1, v_2 are two distinct vertices. Zero divisor graph on a commutative ring R is a simple graph $\Gamma(R)$ whose vertex set is the set of zero divisors $Z(R) = \{a \in R |$ there exists $b \neq 0$ with $ab = 0\}$. Two distinct zero divisors form an edge if $a \cdot b = 0$. The study of a zero divisor graph on a commutative ring was first introduced in 1988 by Beck in [4]. Anderson and Livingston [1] modified this definition by removing zero from the vertex set. We give one example that follows this definition.

Example 1.1. Suppose $R = \mathbb{Z}_{12}$. The vertex set is

$$Z(R)^* = \{2, 3, 4, 6, 8, 9, 10\}$$

and $\Gamma(R)$ is given below, where the edges are the line joining pairs of vertices.



More examples and classifications can be found in [1], [2], [6] and the references cited there. It is clear that the graph is finite if $\Gamma(R)$ has finitely many vertices. Ganesan proved in [8] that for a ring R which is not an integral domain, $\Gamma(R)$ is a finite graph if and only if R is finite.

Edges in a graph contain two vertices. If we drop this restriction and assume that an edge can have any number of vertices, then we get a hyper graph. Thus a hyper graph H is an order pair (V, E), vertex set V and edge set E. Elements of E are called edges and they are subsets of V. A k uniform hyper graph is a hyper graph in which each edge contains precisely k vertices (see [5] for more details). In 2007, Eslahchi and Rahimi (see [7]) used graphs to hyper-graphs relationship to generalize the concept of zero divisors and introduced k-zero divisors. This enables us to associate a k uniform hyper graph $H_k(R)$ to a commutative ring R. For $R = \mathbb{Z}_{12}$, we have shown a hyper-graph in the Example 2.1.

In the same paper, the authors posed a 'finiteness' question similar to one appeared in [8]. In this note, we attempt to answer this question. We follow language and notations from [3] without citation. Section 2 contains main definitions, notations and some basic results. Section 2 ends with a result regarding the nil-radical which is used in section 3 to prove the main results.

2 Notations

All rings considered in this paper are commutative rings with identity. Let R be a commutative ring.

Definition 2.1. A non zero element $a \in R$ is called a *nilpotent* element if $a^r = 0$ for some positive integer r. We define *nilpotent degree* $d_{nil}(a)$ of a nilpotent element a to be the smallest positive integer n for which $a^n = 0$. *Nil radical,* denoted by N is the set of all the nilpotent elements of R. If $N = 0$ then R is called a *reduced* ring.

The nil radical N is an ideal and is equal to intersection of all the prime ideals. If the ring has finitely many minimal prime ideals, then we have a natural imbedding

$$0 \to N = \cap P_i \hookrightarrow R \to \Pi_{i=1}^{n} (R/P_i)$$

where $P_1, \ldots, P_n$ are all the distinct minimal prime ideals. In other words, R/N is isomorphic to a subring of $\Pi_i(R/P_i)$. Further, if R is a reduced ring (*i.e.* N=0), then R is a subring of finite product of integral domains. The last statement is used in the following lemma.

Lemma 2.1. *Suppose R is a reduced ring with finitely many minimal prime ideals and $f(x) \in R[x]$ is a monic polynomial. Then the set of zeros $\{a \in R \mid f(a) = 0\}$ is finite.*

Proof. Suppose $R \hookrightarrow \Pi_{i=1}^{n} R_i$, where R_i are integral domains for $i = 1, \ldots, n$. Consider the projection $f_i(x)$ of $f(x)$ in $R_i[x]$. Since $f(x)$ is monic, each $f_i(x)$ is also a monic polynomial with coefficients in an integral domain. Now if $f(a) = 0$ for some $a \in R$, then $f_i(a_i) = 0$ in R_i for all i, where $a \equiv a_i \mod P_i$. Since each $f_i(x)$ has finitely many zeros, $f(x)$ also has finitely many zeros. □

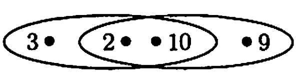
We recall the following definition from [7]

Definition 2.2. Let R be a commutative ring and $k > 1$ be a fixed integer. Element $a_1 \in R$ is called a *k-zero divisor* if there exist $a_2, a_3, \ldots, a_k$ in R such that (1) $\{a_1, a_2, \ldots, a_k\}$ are all distinct elements (2) $\Pi_1^k a_i = 0$ and (3) $\Pi_{i \neq j} a_i \neq 0$ for any $1 \leq j \leq k$. The set of all k-zero divisors is denoted by $Z(R, k)$. A ring with empty $Z(R, k)$ is called *k-integral domain.*

Condition (3) implies that each a_i is nonzero and nonunit. Further, if $\{a'_1, a'_2, \ldots, a'_r\}$ is a proper subset of $\{a_1, a_2, \ldots, a_k\}$, then $\Pi_{i=1}^{r} a'_i \neq 0$. In fact, (3) is equivalent to the statement that the product of fewer than all a_i is nonzero. We will use this fact without justification.

We define a k uniform hypergraph $H_k(R)$ on a commutative ring R as follows. The vertex set is $Z(R, k)$. Elements $a_1, a_2, \ldots, a_k$ which appear in the definition 2.2 form an edge of the hypergraph. Thus, the extension of the concept of zero divisors to that of k-zero divisors in the above definition is purely a graph theoretic. We give one example.

Example 2.1. For $R = \mathbb{Z}_{12}$, the vertex set is $Z(R, 3) = \{2, 3, 9, 10\}$. There are two edges, $\{2, 3, 10\}$ and $\{2, 9, 10\}$ which are enclosed by an ellipse in the following hypergraph:



Note that Z_{12} is a 4-integral domain.

Subrings inherit hypergraph substructure. That is, if R is a subring of S, then $Z(R, k) \subset Z(S, k)$. Further, any edge in $H_k(R)$ is an edge in $H_k(S)$. In this sense, $H_k(R)$ is a subhypergraph of $H_k(S)$. The following proposition will be used in the proof of the main theorem. Since it is an independent result, we have given a status of proposition rather than lemma.

Proposition 2.2. *Let R be a commutative ring with $Z(R, k)$ finite. Then there exists a positive integer n such that $a^n = 0$ for all $a \in N$. Further, if N is finitely generated, then N is nilpotent (i.e. there exists a positive integer n such that $N^n = \{0\}$).*

Proof. Set $n_0 = k(k+1)/2$. If $a^{n_0} = 0$ for all $a \in N$ then we are done. Suppose there exists $a \in N$ with $d_{nil}(a) > n_0$. Now if for any i and j, $1 \le i < j \le d_{nil}(a)$, $a^j = a^i$, then $a^i(a^{j-i} - 1) = 0$. But $a^{j-i} - 1$ is a unit in R, which implies that $a^i = 0$ contradicting the definition of d_{nil}. Let $m = d_{nil}(a) - k(k-1)/2$. Then $m > k-1$. Therefore, a, a^2, $\ldots$, a^{k-1}, a^m are all distinct elements. Further, the product of these elements is $a^{d_{nil}(a)}$ which is zero. Now for any $j \in \{1, 2, \ldots, k-1, m\}$, $\Pi_{i \ne j} a^i = a^r$ with $r < d_{nil}(a)$. Therefore, $\Pi_{i \ne j} a^i \ne 0$. Thus, we see that $a \in Z(R, k)$. Since $Z(R, k)$ is finite, $A = \{x \in N \mid d_{nil}(x) > n_0\}$ is a finite set. Let $m = max_{x \in A}\{d_{nil}(x)\}$, then $b^n = 0$ for all $b \in N$. If N is generated by r elements, then it is easy to show that $N^{rn} = 0$. □

Following the proof we can derive the following corollary.

Corollary. *If there is an $a \in R$ such that $d_{nil}(a) \ge \frac{k(k+1)}{2}$, then R is not a k integral domain.*

Ganesan proved in [8] that a commutative ring with finitely many zero divisors which is not integral domain is finite. Now, all zero divisors are not necessarily 2-zero divisors. For example, 2 in $\mathbb{Z}_4$ is a zero divisor, but not 2-zero divisor. Still we can extend Ganesan's result for 2-zero divisors. The same proof works for a ring which is not a 2-integral domain. We reproduce the proof here.

Proposition 2.3 (Ganesan). *Let R be a commutative ring which is not a 2-integral domain. If $Z(R, 2)$ is finite, then R is finite. Furthermore, if $|Z(R, 2)| = r$ then $|R| \le (r+2)^2$.*

Proof. For any edge $\{x, y\}$ in $H_2(R)$, we have a short exact sequence

$$0 \to ann(x) \to R \to xR \to 0.$$

Now $x \cdot ann(x) = 0$ and $y \cdot (xR) = 0$. Therefore, all the elements of $ann(x)$ other than 0 and x are 2-zero divisors and all the elements of xR other than 0 and y are 2-zero divisors. Hence, $|ann(x)| \le r + 2$ and $|xR| \le r + 2$. This proves that R is finite and has less than $(r+2)^2$ elements. □

For any set S, let $\mathcal{P}(S)$ denote the set of subsets of S and S^m denote the Cartesian product $S \times S \times \ldots \times S$ of m copies of S.

Proposition 2.4. *Suppose R is not a 2-integral domain. Then R^k is not $k+1$ integral domain. Further if $Z(R^k, k+1)$ is finite, then R is finite.*

Proof. Both assertions can be proved together. Define a set theoretic map $\phi : Z(R, 2)^k \to \mathcal{P}(R^k)$ by

$$\phi((x_1, x_2, \dots, x_k)) = \{\bar{x}_1, \bar{x}_2 \dots, \bar{x}_k\},$$

where

$$\bar{x}_1 = (x_1, 1, \dots, 1), \bar{x}_2 = (1, x_2, 1 \dots, 1) \dots,$$
$$\bar{x}_k = (1, 1, \dots, 1, x_k).$$

Clearly, ϕ is injection. Now let $y_i \in ann(x_i)$ be a nonzero element and set $\bar{x}_{k+1} = (y_1, y_2, \dots, y_k)$. Then $k+1$ elements $\bar{x}_i$ forms an edge in $H_{k+1}(R)$. Therefore, $Im(\phi) \subset \mathcal{P}(Z(R^k, k+1))$. This shows that $Z(R^k, k+1)$ is nontrivial, or R^k is not a $k+1$ integral domain. Further, if $Z(R^k, k+1)$ is finite, then one-to-one nature of ϕ implies that $Z(R, 2)$ is finite. Hence by the Proposition 2.3, R is finite. □

3 Main theorem

An r-coloring of a hypergraph $H = (V, E)$ is a map c from V to $\{1, 2, ..., r\}$ such that for every edge e of H, there exist at least two vertices x and y in e with $c(x) \neq c(y)$ (see [7]). The smallest integer r such that H has an r-coloring is called the chromatic number of H and is denoted by $\chi(H)$. When R is a product of n integral domains, [7, Theorem 2.11] gives an estimates for $\chi(H_k(R))$. We state here the precise statement.

Theorem 3.1. *Let $R = R_1 \times R_2 \times \dots \times R_n$, where R_i is an integral domain for each $i = 1, 2, \dots, n$.*

(1) If $n = k$, then $\chi(H_k(R)) = 2$.

(2) If $n = k + t$, then $\chi(H_k(R)) \leq 2 + t$ for all $t \geq 0$.

Now when R is a subring of S, $H_k(R)$ is a sub hypergraph of $H_k(S)$. Therefore, $\chi(H_k(R)) \leq \chi(H_k(S))$ We use this observation in the following proposition to generalize Theorem 3.1 for a reduced rings with finitely many minimal prime ideals. To prove this result (and other results in this section), we will use two well known properties of a prime ideal. Suppose $I_1, I_2, \ldots, I_r$ are ideals in R. If a prime ideal P contains $\cap I_i$, then P must contain one of the I_j. In particular, if P and I_i are all minimal prime ideals, then $P = I_i$ for some i. For the second property which is known as the "prime avoidance theorem", we assume that all I_i are also prime ideals. Then any ideal I contained in $\cup I_i$ must be a subset of one of the I_j (see proposition 1.11 in [3]).

Proposition 3.2. *Suppose R is a reduced ring with n minimal prime ideals. Then*

1. *R is a k-integral domain if and only if $n < k$.*
2. *If $n = k$, then $\chi(H_k(R)) = 2$.*
3. *If $n = k + t$ for some $t \geq 0$, then $\chi(H_k(R)) \leq t + 2$.*

Proof. Suppose $P_1, P_2, \ldots, P_n$ are the minimal prime ideals of R.

To prove (1), assume that $n < k$ and suppose $x_1, x_2, \ldots, x_k$ are distinct elements with $\Pi x_i = 0$. Then $\Pi x_i \in P_j$ for all j. Since P_j are prime ideals, each P_j contains at least one x_i, say x'_j. Let $X = \{x'_j | j = 1, \ldots, n\}$. Then X is a subset of $\{x_1, \ldots, x_k\}$ with at most n elements. Moreover, $\Pi_{x'_j \in X} x'_j \in \cap P_j = \{0\}$. But $n < k$. Therefore $x_1, x_2, \ldots x_k$ is not an edge in $H_k(R)$ or R is a k integral domain.

Now assume that $n \geq k$. Choose $x_1, x_2, \ldots, x_k$ such that $x_i \in P_i - \cup_{j \neq i} P_j$ for $i = 1, 2, \ldots, k-1$ and $x_k \in \cap_k^n P_j - \cup_1^{k-1} P_j$. Then $x_1, x_2, \ldots, x_k$ forms an edge in $H_k(R)$ showing that R is not a k integral domain.

For (2) and (3), assume that $n = k + t$ for some $t \geq 0$. Note that R is a subring of $\Pi_1^n (R/P_i)$. Therefore,

$$\chi(H_k(R)) \leq \chi(H_k(\Pi_1^{n+t}(R/P_i)) \leq t + 2$$

For $n = k$, the above inequality implies that $\chi(H_k(R)) \leq 2$. But by part (1) of the proposition, R contains at least one k-zero divisor. Therefore, $\chi(H_k(R)) = 2$. □

Lemma 3.3. *Let R be a reduced ring with n minimal prime ideals. If $n \geq k$, then R is not a k-integral domain. Further, if $Z(R, k)$ is finite then, R is finite.*

Proof. The first assertion is included in the Proposition 3.2

Now suppose that $Z(R, k)$ is finite. Consider the imbedding $R \hookrightarrow \Pi_i(R/P_i)$. Suppose R is infinite, then R/P_i must be infinite for at least one value of i. Without loss of generality, we can assume that R/P_1 is infinite. Now pick a set of coset representatives of P_1 in R, say T. Then T is infinite.

Let $\{a_1, a_2, \ldots, a_k\}$ be an edge in $H_k(R)$. Since $\Pi_1^k a_i = 0$, $\Pi_1^k a_i \in P_i$ for all i. In particular, there exist a_i which belongs to P_1. By renaming if necessary, we can assume that $a_1 \in P_1$. We will construct infinitely many edges to get a contradiction.

First we choose a nonzero element c in $\cap_2^k P_i$. Since $\cap_1^k P_i = \{0\}$, c is not in P_1. Now for any $a \in R$ and $r_1, r_2 \in T$, if both, $a + cr_1$ and $a + cr_2$ are in the same coset of P_1, then $c(r_1 - r_2) \in P_1$. As $c \notin P_1$, $r_1 - r_2 \in P_1$ or $r_1 + P_1 = r_2 + P_1$. Therefore, $r_1 = r_2$. Thus $a + cr_i$ belong to different cosets of P_1 for different values of i.

We now start our construction. Set $a_1' = a_1$ and choose r_2 in T such that $a_2' = a_2 + cr_2 \notin P_1$. Note that there are infinitely many choices for r_2. Next we choose $r_3 \in T$ such that $a_3' = a_3 + cr_3$ is not in P_1 and is different from a_2'. Similarly we choose $r_4, \ldots, r_k$ such that $\{a_i' = a_i + cr_i\}$ is not in P_1 for $4 \leq i \leq k$ and $a_2', a_3', \ldots, a_k'$ are all distinct. Since a_1 is in P_1, we constructed a set of k distinct elements $a_1', a_2', \ldots, a_k'$. We will show that they form an edge. Since $a_1 c = 0$, $\Pi_1^k a_i' = a_1 \Pi_2^k (a_i + cr_i) = \Pi_1^k a_i = 0$. To show that $\Pi_{i \neq j} a_i' \neq 0$, first observe that if $j \neq 1$, then $\Pi_{i \neq j} a_i' = \Pi_{i \neq j} a_i \neq 0$.

For $j = 1$, $\Pi_{i \neq 1} a_i' = 0$ implies that $\Pi_{i \neq 1} a_i' \in P_1$. Since none of the a_i' belong to P_1, this can not be true. Hence, $\Pi_{i \neq j} a_i' \neq 0$.

Thus there are infinitely many k-zero divisors, which is a contradiction. Therefore our assumption that R is infinite is incorrect. □

Theorem 3.4. *Suppose R is not a k-integral domain such that (1) the nilradical N is finitely generated and (2) R has finitely many and more than k minimal prime ideals. If R has finitely many k-zero divisors, then R is finite.*

Proof. Suppose $P_1, P_2, \ldots, P_n$ are minimal prime ideals of R, then R/N is a reduced ring with $n \geq k$ minimal prime ideals. Therefore, by Lemma 3.3, R/N is not a k-integral domain. We will show now that, $Z(R/N, k)$ is finite. For any $x \in R$, we will use the notation $\bar{x}$ for the image of x in R/N.

Suppose $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$ is an edge in $H_k(R/N)$. Then $\Pi\bar{x}_i = 0$ and for any j, $\Pi_{i \neq j}\bar{x}_i \neq 0$ in R/N. Lifting this to R, we get that $\Pi x_i \in N$ and $\Pi_{i \neq j} x_i \notin N$. By assumption, N is finitely generated and therefore, nilpotent by the Proposition 2.2. Hence there exists a uniform l such that $(\Pi x_i)^l = 0$. Additionally, since $\Pi_{i \neq j} x_i \notin N$, we have that $(\Pi_{i \neq j} x_i)^m \neq 0$ for any m. We claim that, $x_1^l, x_2^l, \ldots, x_k^l$ forms an edge in $H_k(R)$. For that we only need to show that all x_i^l are distinct. To the contrary, suppose $x_r^l = x_s^l$ for some $r < s$. Then,

$$(\Pi_{i \neq s} x_i)^{2l} = (\Pi_1^k x_i)^l \cdot (\Pi_{i \neq r,s} x_i)^l = 0$$

which shows that $\Pi_{i \neq s} x_i \in N$, which is a contradiction. Hence, $x_1^l, x_2^l, \ldots, x_k^l$ is an edge in $H_k(R)$. Thus any k zero divisor in R/N satisfies a monic polynomial of the form $T^l - \bar{a}$ with $a \in Z(R, k)$. Now since $Z(R, k)$ is finite, there are only finitely many such polynomials and each such polynomial has only finitely many zeros by the Lemma 2.1. Therefore, $Z(R/N, k)$ is finite and by the Lemma 3.3, R/N is finite.

Now N/N^2 is a finitely generated R/N module. Since R/N is finite, N/N^2 is also finite. Consider the following exact sequence,

$$0 \longrightarrow \frac{N}{N^2} \xrightarrow{i} \frac{R}{N^2} \xrightarrow{f} \frac{R}{N} \longrightarrow 0$$

where i is inclusion and f is the canonical map induced from the relation $N^2 \subset N$. As both ends are finite, we conclude that R/N^2 is finite. Inductively we can show that R/N^n is finite for all $n > 0$. But N is nilpotent. This completed the proof. □

Corollary. *Suppose R is a noetherian ring, which is not a k-integral domain with at least k minimal prime ideals. If $Z(R, k)$ is finite, then R is finite.*

4 Acknowledgments

This work was done while I was on a sabbatical leave. I sincerely thank the leadership of the Lincoln University, PA and the PTS committee for granting me the opportunity. I thank Dr. Boris Datskovsky (Temple University) for giving me his valuable time to discuss this paper. I also thank the referee of JCMCC for going over this paper and making numerous suggestions to make it more readable. Finally, this work was presented in the MCCCC 2015 conference. I thank the organizers for providing this opportunity.

5 References

[1] D.F. Anderson and P.S. Livingston, "The zero-divisor graph of a commutative ring", *Journal of Algebra* **217** (1999), 434-447.

[2] D.D. Anderson and M. Naseer, Beck's coloring of a commutative ring, *Journal of Algebra* **159** (1993), 500-514.

[3] M.F. Atiyah and I.G. MacDonald, Introduction to Commutative Algebra. Addison-Wesley, Reading, (1969)

[4] I. Beck, Coloring of commutative rings, *Journal of Algebra* **116** (1988), 208-226.

[5] Claude Berge, "Hypergraphs: Combinatorics of finite sets". North-Holland, Reading, 1989.

[6] Jim Coykendall, Sean Sather-Wagstaff, Laura Sheppardson, and Sandra Spiroff, On Zero Divisor Graphs, *Progress in Commutative Algebra* **2** (2012), 241-299.

[7] C. Eslahchi, and A. M. Rahimi, k-Zero-Divisor Hypergraph of A Commutative Ring, *International Journal of Mathematics and Mathematical Sciences* (2007), 329-343.

[8] N. Ganesan, Properties of rings with a finite number of zero divisors-II, *Mathematische Annalen* **161** (1965) no. 4, 241-246.