

Vertex Colorings without Rainbow or Monochromatic Subgraphs

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ABSTRACT. This paper investigates vertex colorings of graphs such that some rainbow subgraph R and some monochromatic subgraph M are forbidden. Previous work focussed on the case that $R = M$. Here we consider the more general case, especially the case that $M = K_2$.

1 Introduction

Let F be a graph. Consider a coloring of the vertices of G . We say that a copy of F (as a subgraph) is *rainbow* (or *heterochromatic*) if all its vertices receive different colors. We say that the copy of F is *monochromatic* if all its vertices receive the same color.

The question of avoiding monochromatic copies of a graph is well studied (see for example the survey [12]). Less studied, but still common, is the question of avoiding rainbow copies (especially for edge-colorings); see for example [1, 2, 3]. In [9, 8] we defined WORM colorings: these forbid both a rainbow and a monochromatic copy of a specific subgraph. But it is more flexible to allow different restrictions. For graphs M and R , we define an (M, R) -WORM coloring of G to be a coloring of the vertices of G with neither a monochromatic subgraph isomorphic to M nor a rainbow subgraph isomorphic to R . Note that such a coloring is not guaranteed to exist. For example, any G with at least one edge does not have a (K_2, K_2) -WORM coloring.

This coloring is a special case of the “mixed hypergraphs” introduced by Voloshin (see for example [15]); see [14] for an overview. A related question studied in the edge case is the *rainbow Ramsey number* (or *constrained Ramsey number*); this is defined as the minimum N such that any coloring of the edges of K_N produces either a monochromatic M or a rainbow R . See [7].

One special case of WORM colorings has a distinguished history. Erdős et al. [6] defined the *local chromatic number* of a graph as the maximum order of a rainbow star that must appear in all proper colorings. In our

notation, this is the minimum r such that the graph has a $(K_2, K_{1,r+1})$ -WORM coloring. For a survey on this parameter, see [11].

One case is trivial: if we forbid a rainbow K_2 , then every component of the graph must be monochromatic. Similarly, if we forbid a rainbow kK_1 , then this is equivalent to using less than k colors. So we will assume that the subgraph R has at least three vertices and at least one edge. On the other hand, taking $M = K_2$ is equivalent to insisting that the coloring is proper. Also, taking $M = kK_1$ is equivalent to using each color less than k times.

Having two competing restrictions leads naturally to considering both the minimum and maximum number of colors in such a coloring. So we define the upper chromatic number $W^+(G; M, R)$ as the maximum number of colors, and the lower chromatic number $W^-(G; M, R)$ as the minimum number of colors, in an (M, R) -WORM coloring of G (if the graph has such a coloring). For bounds, it will be useful to also let $m^-(G; M)$ be the minimum number of colors without a monochromatic M , and $r^+(G; R)$ be the maximum number of colors without a rainbow R . Note that

$$m^-(G; M) \leq W^-(G; M, R) \leq W^+(G; M, R) \leq r^+(G; R),$$

provided G has an (M, R) -WORM coloring.

We proceed as follows. Section 2 contains some general observations. In Section 3 we provide one general upper bound when R is a path. In Section 4 we consider proper colorings without rainbow P_3 , P_4 , or C_4 . Finally, in Section 5 we provide a few results for other cases.

2 Preliminaries

We start with some simple observations. If G is bipartite then the bipartition is immediately an (M, R) -WORM coloring. Indeed, if G is k -colorable with $k < |R|$, then a proper k -coloring of G is an (M, R) -WORM coloring. Also:

Observation 1 *Fix graphs M and R and let G be a graph.*

(a) *If G has an (M, R) -WORM coloring, then so does $G - e$ where e is any edge and $G - v$ where v is any vertex. Further, $W^+(G - e; M, R) \geq W^+(G; M, R)$ and $W^+(G - v; M, R) \geq W^+(G; M, R) - 1$, with similar results for the lower chromatic number.*

(b) *If M and R are connected but G is disconnected, then $W^+(G; M, R)$ is the sum of the parameter for the components, and $W^-(G; M, R)$ is the maximum of the parameter for the components.*

(c) It holds that $W^+(G; M, R) = |V(G)|$ if and only if G is R -free.

(d) It holds that $W^+(G; M, R) \geq |R| - 1$ if G is $|R| - 1$ colorable (and has at least that many vertices).

We will also need the following idea from [10]. We say that a set S *bi-covers* a subgraph H if at least two vertices of H are in S . For positive integer s , define $b_F(s)$ to be the maximum number of copies of F that can be bi-covered by using a set of size s . (Note that by definition $b_F(1) = 0$.)

Lemma 2 [10] *Suppose that graph G of order n contains f copies of R and that $b_R(s) \leq a(s - 1)$ for all s . Then $r^+(G; R) \leq n - f/a$.*

2.1 General M

It should be noted that maximizing the number of colors while avoiding a rainbow subgraph can produce a large monochromatic subgraph. For example:

Observation 3 *For all connected graphs M , there exists a graph G such that $W^+(G; M, P_3) < r^+(G; P_3)$.*

PROOF. In [10] we considered the *corona* $cor(G)$ of a graph G ; this is the graph obtained from G by adding, for each vertex v in G , a new vertex v' and the edge vv' . It was shown that $r^+(G; P_3) = |G| + 1$. In fact, we note here that if G is connected, then one can readily show by induction that the optimal coloring is unique and gives every vertex of G the same color. In particular, it follows that the no-rainbow- P_3 coloring of $cor(M)$ with the maximum number of colors contains a monochromatic copy of M . \diamond

3 A Result on Rainbow Paths

We showed [9] that a nontrivial graph G has a (P_3, P_3) -WORM coloring if and only if it has a (P_3, P_3) -WORM coloring using only two colors. We prove an analogue for general paths. This result is a slight generalization of Theorem 10 in [13].

Theorem 4 *Fix some graph M ; if graph G has an (M, P_r) -WORM coloring, then G has one using at most $r - 1$ colors.*

PROOF. Consider an (M, P_r) -WORM coloring f of G . Let G_M be the spanning subgraph of G whose edges are monochromatic and G_R the spanning subgraph whose edges are rainbow. It follows that G_M does not contain M , and that G_R does not contain P_r . It is well known that a graph without P_r has chromatic number at most $r - 1$.

Now, let g be a proper coloring of G_R using at most $r - 1$ colors and consider g as a coloring of G . Note that the monochromatic edges under g are a subset of those under f . Therefore, g is a (M, P_r) -WORM coloring of G using at most $r - 1$ colors. \diamond

It follows that:

Corollary 5 *For any graph M and $r > 0$, graph G has an (M, P_r) -WORM coloring if and only if $m^-(G, M) \leq r - 1$. If so, $W^-(G; M, P_r) = m^-(G, M)$.*

On the other hand, Theorem 4 does not extend to stars. For example, Erdős et al. [6] constructed a shift graph that has arbitrarily large chromatic number but can be properly colored without a rainbow $K_{1,3}$. That is:

Theorem 6 *For $r \geq 3$ and $k \geq 1$, there exists a graph G with $W^-(G; K_2, K_{1,r}) \geq k$.*

Nor does Theorem 4 generalize to K_3 ; see [16] and [4].

4 Proper Colorings

Recall that $W^+(G; K_2, R)$ is the maximum number and $W^-(G; K_2, R)$ the minimum number of colors in a proper coloring without a rainbow R .

4.1 Two simple cases

Two cases for $M = K_2$ are immediate:

Observation 7 *A graph G has a (K_2, P_3) -WORM coloring if and only if it is bipartite. If so, $W^+(G; K_2, P_3) = W^-(G; K_2, P_3) = 2$, provided G is connected and nonempty.*

PROOF. If we have a (K_2, P_3) -WORM coloring, then for each vertex v all its neighbors must have the same color, which is different to v 's color. It follows that every path must alternate colors. \diamond

In a proper coloring of a graph, all cliques are rainbow. Thus it follows:

Observation 8 *A graph G has a (K_2, K_m) -WORM coloring if and only if it is K_m -free. If so, $W^+(K_2, K_m) = |G|$ while $W^-(K_2, K_m)$ is the chromatic number of G .*

4.2 No rainbow $K_{1,3}$

Consider first that G is bipartite. Then in maximizing the colors, it is easy to see that one may assume the colors in the partite sets are disjoint. (If red is used in both partite sets, then change it to pink in one of the sets.) In particular, unless G is a star, one can use at least two colors in each partite set. (This result generalizes to R any star.) For example, it follows that $W^+(K_{m,m}; K_2, K_{1,3}) = 4$ for $m \geq 2$.

Indeed, it is natural to consider the *open neighborhood hypergraph* $ON(G)$ of the graph G . This is the hypergraph with vertex set $V(G)$ and a hyperedge for every open neighborhood in G . In general, since we have a proper coloring, the requirement of no rainbow $K_{1,r}$ is equivalent to every hyperedge in $ON(G)$ receiving at most $r - 1$ colors. In the case that G is bipartite, the two problems are equivalent:

Observation 9 *For any graph G , the parameter $W^+(G; K_2, K_{1,r})$ is at most the maximum number of colors in a coloring of $ON(G)$ with every hyperedge receiving at most $r - 1$ colors. Furthermore, there is equality if G is bipartite.*

PROOF. When G is bipartite, the $ON(G)$ can be partitioned into two disjoint hypergraphs and so will have disjoint colors in the hypergraphs. It follows that that coloring back in G will be proper. \diamond

Recall that a 2-tree is defined by starting with K_2 and repeatedly adding a vertex that has two adjacent neighbors. For example, this includes maximal outerplanar graphs.

Observation 10 *If G is a 2-tree of order at least 3, then $W^+(G; K_2, K_{1,3}) = 3$.*

PROOF. Any 2-tree is 3-colorable. Furthermore, it follows readily by induction that a $(K_2, K_{1,3})$ -WORM coloring can use only three colors: when we add a vertex v and join it to adjacent vertices x and y , they already have a common neighbor z , and so v must get the same color as z . \diamond

Osang showed that determining whether a graph has a $(K_2, K_{1,3})$ -WORM coloring is hard:

Theorem 11 [11] *Determining whether a graph has a $(K_2, K_{1,3})$ -WORM coloring is NP-complete.*

4.2.1 Cubic graphs

We consider next 3-regular graphs. Since cubic graphs (other than K_4) are 3-colorable, they have a $(K_2, K_{1,3})$ -WORM coloring. And that coloring uses at most three colors. Further, they have a coloring using two colors if and only if they are bipartite. So the only interesting question is the behavior of the upper chromatic number.

Observation 12 *If G is cubic of order n , then $W^+(G; K_2, K_{1,3}) \leq 2n/3$.*

PROOF. Since G is cubic, the hypergraph $ON(G)$ is 3-regular and 3-uniform. Further we need a coloring of $ON(G)$ where every hyperedge has at least one pair of vertices the same color. Consider some color used more than once, say red. If there are r red vertices, then at most $3r/2$ hyperedges can have two red vertices. (Each red can be used at most thrice.)

It follows that if the i^{th} non-unique color is used r_i times, then we need $\sum_i r_i \geq 2n/3$. Let B be the number of vertices that can be discarded and still have one vertex of each color. Then $B = \sum_i (r_i - 1)$ and by above $B \geq n/3$. It follows that the total number of colors is at most $2n/3$. \diamond

Equality in Observation 12 is obtained by taking disjoint copies of $K_{3,3} - e$ and adding edges to make the graph connected. See Figure 1.

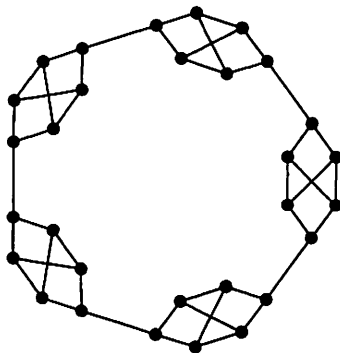


Figure 1: A cubic graph G with $W^+(G; K_2, K_{1,3})$ two-thirds its order

Consider next the minimum value of $W^+(G; K_2, K_{1,3})$ for cubic graphs of order n . We noted above that bipartite graphs in general have a value

of at least 4. Computer search shows that this parameter is at least 3 for $n \leq 18$. Indeed, it finds only three graphs where the parameter is 3: one of order 6 (the prism), one of order 10, and one of order 14, the generalized Petersen graph. These three graphs are shown in Figure 2.

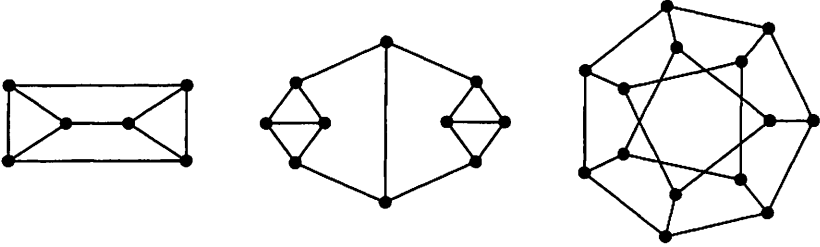


Figure 2: The known cubic graphs with $W^+(G; K_2, K_{1,3}) = 3$

It is unclear what happens in general.

4.3 Forbidding rainbow P_4

We consider proper colorings without rainbow P_4 's. Theorem 4 applies. That is, a graph G has a (K_2, P_4) -WORM coloring if and only if G has chromatic number at most 3. In particular, this means that it is NP-complete to determine if a graph has a (K_2, P_4) -WORM coloring. Further, if such a coloring exists, then $W^-(G; K_2, P_4)$ is the (ordinary) chromatic number of G . So we consider only the upper chromatic number here.

Observation 13 *If graph G is bipartite of order n , then $W^+(G; K_2, P_4) \geq n/2 + 1$.*

PROOF. In the smaller partite set, give all vertices the same color, and in the other partite set, give all vertices unique colors. Note that every copy of P_4 contains two vertices from both partite sets. \diamond

Observation 14 *If connected graph G of order n has a perfect matching, then it holds that $W^+(G; K_2, P_4) \leq n/2 + 1$.*

PROOF. Number the edges of the perfect matching $e_1, \dots, e_{n/2}$ such that for all $i > 1$, at least one of the ends of e_i is connected to some e_j for $j < i$. Then e_i, e_j , and the connecting edge form a P_4 . It follows that e_j and e_i share a color. Thus the total number of colors used is at most $2 + (n/2 - 1) = n/2 + 1$. \diamond

For example, equality is obtained in both observations for any connected bipartite graph with a perfect matching, such as the balanced complete bipartite graph or the path/cycle of even order. Equality is also obtained in Observation 13 for the tree of diameter three where the two central vertices have the same degree. Also, there are nonbipartite graphs that achieve equality in Observation 14; for example, the graph shown in Figure 3.

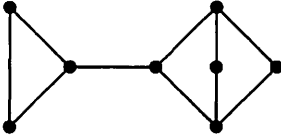


Figure 3: A nonbipartite graph G with a perfect matching and maximum $W^+(G; K_2, P_4)$

We determine next the parameter for the odd cycle:

Observation 15 *If n is odd, then $W^+(C_n; K_2, P_4)$ is 3 for $n \leq 5$, and $(n - 1)/2$ for $n \geq 7$.*

PROOF. The result for $n = 3$ is trivial and for $n = 5$ is easily checked. So assume $n \geq 7$. For the lower bound, color red a maximum independent set, give a new color to every vertex with two red neighbors, and color each vertex with one red neighbor the same color as on the other side of its red neighbor. For example, the coloring for C_{13} is shown in Figure 4 (where the red vertices are shaded).

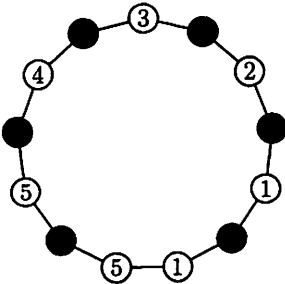


Figure 4: Coloring showing $W^+(C_{13}; K_2, P_4)$

We now prove the upper bound. Two same-colored vertices distance 2 apart bi-cover two copies of P_4 , while two same-colored vertices distance 3 apart bi-cover one copy. It follows that if a color is used k times, it can bi-cover at most $2(k - 1)$ copies of P_4 , except if the vertices of that color form

a maximum independent set, when it bi-covers $2k - 1$ copies. Since there are n copies of P_4 in total, by Lemma 2 it follows that the total number of colors is at most $n/2$, unless some color is a maximum independent set. So say red is a maximum independent set. Let b and e be the two red vertices at distance 3; say the portion of the cycle containing them is $abcdef$. By considering the $a-d$ copy of P_4 , it follows that a must have the same color as c or d . Similarly, f must have the same color as c or d . Thus the total number of colors is at most $1 + (n - (n - 1)/2) - 2 = (n - 1)/2$. \diamond

In contrast to Observation 13, we get the following:

Theorem 16 *If connected graph G has every vertex in a triangle, then $W^+(G; K_2, P_4) = 3$ if such a coloring exists.*

PROOF. Note that every triangle is properly colored. We show that every triangle receives the same three colors. Consider two triangles T_1 and T_2 . If T_1 and T_2 share two vertices, then the third vertex in each share a color. Consider the case that T_1 and T_2 share one vertex. Then by considering the four P_4 's using all vertices but one, it readily follows that the triangles must have the same colors.

Now, assume that T_1 and T_2 are disjoint but joined by an edge e . Suppose they do not have the same three colors. Then there is vertex u_1 in T_1 and u_2 in T_2 that do not share a color with the other triangle. If u_1 and u_2 are the ends of e , then any P_4 starting with e is rainbow. If u_1 and u_2 are not the ends of e , then there is a P_4 whose ends are u_1 and u_2 and that P_4 must be rainbow. Either way, we obtain a contradiction.

Since the graph is connected, it follows that every triangle is colored with the same three colors. Since this includes all the vertices, the result follows. \diamond

For example, it follows that if G is a maximal outerplanar graph, then it follows that $W^+(G; K_2, P_4) = 3$.

4.3.1 Cubic Graphs

There are many cubic graphs with $W^+(G; K_2, P_4) = 3$. These include, for example, the claw-free cubic graphs (equivalently the ones where every vertex is in a triangle). See Theorem 16.

For the largest value of the parameter, computer evidence suggests:

Conjecture 1 *If G is a connected cubic graph of order n , then it holds that $W^+(G; K_2, P_4) \leq n/2 + 1$, with equality exactly when G is bipartite.*

Certainly, by Observations 13 and 14 (and the fact that regular bipartite graphs have perfect matchings), that value is obtained for all bipartite graphs.

4.4 Forbidding rainbow C_4

We conclude this section by considering proper colorings without rainbow 4-cycles.

Observation 17 *If G is a maximal outerplanar graph, then it holds that $W^+(G; K_2, C_4) = 3$.*

PROOF. Consider two triangles sharing an edge. Then to avoid a rainbow C_4 , the two vertices not on the edge must have the same color. It follows that all triangles have the same three colors. \diamond

In particular, we revisit cubic graphs. The parameter $W^-(G; K_2, C_4)$ for a cubic graphs G is uninteresting: the 3-coloring provides such a WORM coloring, and so the parameter is determined by whether G is bipartite or not. Further, the upper bound for $W^+(G; K_2, C_4)$ is trivial: one can have a cubic graph without a 4-cycle.

Computer evidence suggests that:

Conjecture 2 *If G is a connected cubic graph of order n , then it holds that $W^+(G; K_2, C_4) \geq n/2$.*

This lower bound is achievable. Define a *prism* as the cartesian product of a cycle with K_2 . For n even, a *Mobius ladder* is defined by taking the cycle on n vertices and joining every pair of opposite vertices. Note that a prism is bipartite when n is a multiple of 4, and a Mobius ladder is bipartite when n is not a multiple of 4.

Observation 18 *If G is a nonbipartite Mobius ladder or prism of order n , then it holds that $W^+(G; K_2, C_4) = n/2$.*

PROOF. We first exhibit the coloring. Let $m = n/2$. Say the vertices of the prism are u_1, \dots, u_m and v_1, \dots, v_m , where u_i has neighbors u_{i-1}, u_{i+1} , and v_i (arithmetic modulo m) and similarly for v_i . Then for $1 \leq i \leq m$, give vertices u_i and v_{i+1} color i .

Say the vertices of the Mobius ladder are w_1, \dots, w_n where w_i has neighbors w_{i-1}, w_{i+1} , and w_{i+m} (arithmetic modulo n). Then for $2 \leq i \leq m$,

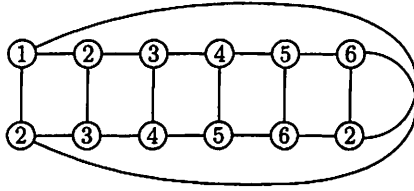


Figure 5: Coloring of Mobius ladder

give vertices w_i and w_{i+m-1} color i , give vertex w_1 color 1 and give vertex w_{n-1} color 2. For example, the coloring for the case $n = 12$ is shown in Figure 5.

Now, for the upper bound, consider a color that is used r times. A color bi-covers a copy of C_4 if it contains vertices from consecutive rungs (where a rung is an edge in two C_4 's). Since the graph is not bipartite, the color cannot be present in every rung. It follows that it can bi-cover at most $r - 1$ copies of C_4 . Now, there are m copies of C_4 (note that the prism of C_4 is bipartite so excluded). It follows from Lemma 2 that the number of colors is at most $n - n/2 = n/2$. \diamond

It appears that this extremal graph is unique for all orders.

5 Other Results

5.1 Paths and paths

The natural strategy to color a long path without a rainbow P_r yields the following result. It can also be obtained as a special case of the result on interval mixed hypergraphs given in [5].

Observation 19 *For any $m \geq 3$, it holds that $W^+(P_n; P_m, P_r) = r^+(P_n; P_r) = \lfloor (r - 2)n / (r - 1) \rfloor + 1$.*

PROOF. Give the first $r - 1$ vertices different colors, then the next vertex the same color as the previous vertex, then the next $r - 2$ vertices different colors, and so on. This coloring has a monochromatic P_2 but not a monochromatic P_3 , and is easily seen to be best possible (as every copy of P_r must contain two vertices of the same color). \diamond

5.2 Biclques and bicliques

Next we revisit the case that G , M , and R are bicliques. For $n \geq b$ it was proved that $W^+(K_{n,n}; K_{1,b}, K_{1,b}) = 2b - 2$ in [9] and it was proved that $W^+(K_{n,n}; K_{b,b}, K_{b,b}) = n + b - 1$ in [8]. The case for stars is special, but it is straight-forward to generalize the latter:

Theorem 20 *Let $m \leq n$ and $2 \leq a \leq b$ with $m \geq a$ and $n \geq b$. Then*

$$r^+(K_{m,n}; K_{a,b}) = \max(a + n - 1, b - 1 + \min(m, b - 1)).$$

PROOF. Consider a coloring $K_{m,n}$ without a rainbow $K_{a,b}$ and assume there are at least $a + b$ colors. If one partite set has at least a colors and the other partite set has at least b colors, then one can choose a colors from the one and b from the other that are disjoint and thus obtain a rainbow $K_{a,b}$. So: either (1) there is a partite set that has at most $a - 1$ colors, or (2) both partite sets have at most $b - 1$ colors. In the first case, the maximum number of colors possible is $a + n - 1$. In the second case, the maximum number of colors possible is $b - 1 + \min(m, b - 1)$. The theorem follows. \diamond

Note that in the above proof, the optimal number of colors can be achieved by making the sets of colors in the two partite sets disjoint. Thus, one obtains a similar value for $W^+(K_{m,n}; M, K_{a,b})$ where M is any non-trivial biclique.

5.3 Grids without rainbow 4-cycles

We conclude this section with a result about forbidden 4-cycles. This result establishes a conjecture proposed in [8]. Let $G_{m,n}$ denote the grid formed by the cartesian product of P_m and P_n .

Observation 21 *For any grid and $s > 0$, $bc_4(s) \leq 2(s - 1)$.*

PROOF. We prove this bound by induction. Let S be a set of s vertices. The bound is immediate when S is contained in only one row. Now suppose S intersects at least two rows. Let S_1 be a maximal set of consecutive vertices of S in the topmost row of S . By the induction hypothesis, the number of C_4 's that contain at least two vertices in $S \setminus S_1$ is at most $2(|S| - |S_1|) - 2$. Further, the number of C_4 's that contain at least one vertex in S_1 and least two vertices in S is at most $2|S_1|$: there are $|S_1| - 1$ possible copies above S_1 and at most $|S_1| + 1$ copies below. Hence, the

number of C_4 's that S bi-covers is at most $2(|S| - |S_1|) - 2 + 2|S_1| = 2|S| - 2$.
◇

In [8] a (C_4, C_4) -WORM coloring is given and it is conjectured that this is best possible. This we now show:

Theorem 22 *For $G_{m,n}$ the $m \times n$ grid, it holds that $W^+(G_{m,n}; C_4, C_4) = \lfloor (m+1)(n+1)/2 \rfloor - 1$.*

PROOF. The lower bound was proved in [8]. The upper bound follows from Lemma 2 and Observation 21: There are $(m-1)(n-1)$ copies of C_4 , and so $r^+(G_{m,n}; C_4) \leq mn - (m-1)(n-1)/2 = (m+1)(n+1)/2 - 1$.
◇

6 Other Directions

We conclude with some thoughts on future directions. Apart from the specific open problems raised here, a direction that looks interesting is the case where M and R are both stars. Also of interest is where the host graph is a product graph.

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