Majestic Colorings of Graphs

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Abstract

For a connected graph G of order at least 3, let $c : E(G) \to \{1, 2, \ldots, k\}$ be an edge coloring of G where adjacent edges may be colored the same. Then c induces a vertex coloring c' of G by assigning to each vertex v of G the set of colors of the edges incident with v. The edge coloring c is called a majestic k-edge coloring of G if the induced vertex coloring c' is a proper vertex coloring of G. The minimum positive integer k for which a graph G has a majestic k-edge coloring is the majestic chromatic index of G and denoted by $\chi'_m(G)$. For a graph G with $\chi'_m(G) = k$, the minimum number of distinct vertex colors induced by a majestic k-edge coloring is called the majestic chromatic number of G and denoted by $\psi(G)$. Thus, $\psi(G)$ is at least as large as the chromatic number $\chi(G)$ of a graph G. Majestic chromatic indexes and numbers are determined for several well-known classes of graphs. Furthermore, relationships among the three chromatic parameters $\chi_m(G)$, $\psi(G)$ and $\chi(G)$ of a graph G are investigated.

Key Words: vertex and edge coloring, majestic edge coloring, majestic chromatic index, majestic number.

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1 Introduction

During the past several decades, there have been a number of research projects dealing with edge colorings of a graph that give rise to vertex colorings of the graph (see [2, 4, 5], for example). Typically, an edge coloring c of a graph G is a function $c : E(G) \to [k] = \{1, 2, \ldots, k\}$ for some positive integer k. Thus, such a coloring c is a k-edge coloring. Among the vertex colorings c' of G obtained from c, the most studied are those for which the color $c'(v)$ of a vertex v of G is either (1) the set of colors of those edges incident with v, (2) the multiset of colors of the edges incident with v or (3) the sum of the colors of the edges incident with v. Here, we consider edge colorings where the vertex colorings are those in (1). In this case then, for a nontrivial connected graph G on which has been defined an edge coloring $c : E(G) \to [k]$, the associated vertex coloring $c' : V(G) \to \mathcal{P}([k]) - \{\emptyset\}$ is defined by $c'(v) = \{c(e) : e \in E_v\}$, where E_v is the set of edges incident with v and $\mathcal{P}([k])$ is the power set of the set $[k]$.

An edge (vertex) coloring is called *unrestricted* if no condition is placed on how edges (vertices) may be colored. In particular, in an unrestricted edge coloring, adjacent edges may be colored the same. In a *proper* vertex coloring, adjacent vertices must be colored differently. A vertex coloring is *vertex-distinguishing* if distinct vertices are assigned distinct colors.

An early example of such an edge coloring was introduced by Harary and Plantholt [3] in 1985. If c is an unrestricted edge coloring of a nontrivial connected graph G and c' is vertex-distinguishing, then c is called a *set irregular edge coloring* of G. The minimum positive integer k for which a graph G has a set irregular edge coloring is the *set irregular chromatic index* of G and is denoted by $\mathrm{si}(G)$. (This parameter was referred to as the *point-distinguishing chromatic index* by Harary and Plantholt.) The set irregular chromatic index does not exist for K_2. Since every two vertices in a connected graph G of order $n \geq 3$ and size $m \geq 2$ are incident with different sets of edges, any edge coloring that assigns distinct colors of $[m]$ to the edges of G is a set irregular edge coloring. Hence, $\mathrm{si}(G)$ exists and $\mathrm{si}(G) \leq m$.

In 2002, Zhang, Liu and Wang [6] introduced an edge coloring $c : E(G) \to [k]$ of a graph G for which both c and the induced set vertex coloring c' are proper. They referred to such a coloring c as an *adjacent strong edge coloring*. The minimum positive integer k for which G has an adjacent strong k-edge coloring is called the *adjacent strong chromatic index* of G.

Inspired by set irregular edge colorings and adjacent strong edge colorings described above, we study in this paper unrestricted edge colorings that induce proper vertex colorings. We refer to the book [2] for graph theory notation and terminology not described in this paper.

2 The Majestic Index of a Graph

Here, we consider unrestricted edge colorings $c : E(G) \to [k]$ of a graph G for which the induced vertex coloring c' (where the color of a vertex v is the set of colors of the edges incident with v) is proper. Such a coloring c is called a *majestic k-edge coloring* (or simply a *majestic edge coloring*). The minimum positive integer k for which a graph G has a majestic k-edge coloring is called the *majestic chromatic index* of G or, more simply, the *majestic index* of G and is denoted by $\chi'_m(G)$. Since there is no majestic edge coloring of K_2, the majestic chromatic index does not exist for K_2. Thus, we consider only connected graphs of order at least 3. Because $\mathrm{si}(G)$ exists for each such graph G, so too does $\chi'_m(G)$. If every edge of a connected graph G is assigned the same color, say 1, then $c'(v) = \{1\}$ for every vertex v of G. Since $c'(x) = c'(y)$ for adjacent vertices x and y of G, there is no majestic 1-edge coloring of G. Thus, we have the following observation.

Observation 2.1 *If G is a connected graph of size $m \geq 2$, then $\chi'_m(G)$ exists and $2 \leq \chi'_m(G) \leq \mathrm{si}(G) \leq m$.*

For complete graphs, an unrestricted edge coloring is majestic if and only if it is vertex-distinguishing. Therefore, every majestic edge coloring of a complete graph is a set irregular coloring. From this observation, the result below follows from a theorem of Harary and Plantholt.

Theorem 2.2 [3] *For every integer $n \geq 3$,*

$$\chi'_m(K_n) = \mathrm{si}(K_n) = \lceil \log_2 n \rceil + 1.$$

This leads to a lower bound for the majestic index of a graph G in terms of its clique number $\omega(G)$ (the largest order of a complete subgraph in G).

Proposition 2.3 *If G is a nontrivial connected graph, then*

$$\chi'_m(G) \geq \lceil \log_2 \omega(G) \rceil + 1.$$

Proof. Let $\omega(G) = p$ and let K_p be a clique of order p in G with $V(K_p) = \{u_1, u_2, \ldots, u_p\}$. Suppose that $\chi'_m(G) = k$ and let $c : E(G) \to [k]$ be a majestic k-edge coloring of G. Since u_i is adjacent to u_j for each integer $j \neq i$ where $1 \leq j \leq p$, it follows that $c(u_iu_j) \in c'(u_i) \cap c'(u_j)$ and so $c'(u_i) \cap c'(u_j) \neq \emptyset$. Consequently, the complement $\overline{c'(u_i)}$ of $c'(u_i)$ $(1 \leq i \leq n)$ is not a color for any vertex of K_p and so there are at most $\frac{1}{2}(2^k) = 2^{k-1}$ choices for the colors of the vertices of K_p. Hence, $p \leq 2^{k-1}$ and so $\log_2 p \leq k - 1$. Thus, $\chi'_m(G) \geq \lceil \log_2 p \rceil + 1$. ■

We now consider the majestic indexes of some graphs belonging to certain well-known classes of graphs, beginning with cycles.

Proposition 2.4 *For each integer $n \geq 3$,*

$$\chi'_m(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 4 \\ 3 & \text{if } n \not\equiv 0 \pmod 4. \end{cases}$$

Proof. First, it is immediate that $\chi'_m(C_3) = \chi'_m(C_5) = 3$ and $\chi'_m(C_4) = 2$. So, we may assume that $n \geq 6$. Suppose that there exists a majestic 2-edge coloring c of $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$. Necessarily, there are two adjacent edges of C_n that are colored differently, say $c(v_n v_1) = 1$ and $c(v_1 v_2) = 2$. Thus, $c'(v_1) = \{1, 2\}$. Therefore, we must have $c(v_2 v_3) = 2$, $c(v_3 v_4) = 1$ and $c(v_4 v_5) = 1$. More generally, $c(v_i v_{i+1}) = 2$ when $i \equiv 1, 2 \pmod 4$ and $c(v_i v_{i+1}) = 1$ when $i \equiv 3, 0 \pmod 4$. If $n \equiv 0 \pmod 4$, then $c'(v_i) = \{1, 2\}$ if i is odd, $c'(v_i) = \{1\}$ if $i \equiv 0 \pmod 4$ and $c'(v_i) = \{2\}$ if $i \equiv 2 \pmod 4$. Hence, c is a majestic 2-edge coloring of C_n and so $\chi'_m(C_n) = 2$ if $n \equiv 0 \pmod 4$.

Next, suppose that n is odd. Thus, $c'(v_n) = c'(v_{n-1}) = \{1\}$ if $n \equiv 1 \pmod 4$ and $c'(v_n) = c'(v_1) = \{1, 2\}$ if $n \equiv 3 \pmod 4$. Hence, c is not a majestic 2-edge coloring of C_n and so $\chi'_m(C_n) \geq 3$. If $n \equiv 1 \pmod 4$, by changing the color of $v_n v_1$ from 1 to 3, we have $c'(v_1) = \{2, 3\}$, $c'(v_n) = \{1, 3\}$ and $c'(v_{n-1}) = \{1\}$. If $n \equiv 3 \pmod 4$, by changing the color of $v_1 v_2$ from 2 to 3, we have $c'(v_1) = \{1, 3\}$, $c'(v_2) = \{2, 3\}$ and $c'(v_n) = \{1, 2\}$. This is a majestic 3-edge coloring and so $\chi'_m(C_n) = 3$ if n is odd.

Finally, suppose that $n \equiv 2 \pmod 4$. Hence, $c'(v_{n-1}) = c'(v_n) = c'(v_1) = \{1, 2\}$ and so c is not a majestic 2-edge coloring of C_n. Therefore, $\chi'_m(C_n) \geq 3$. In this case, changing the colors of both $v_{n-1}v_n$ and $v_n v_1$ to 3 results in $c'(v_{n-1}) = \{1, 3\}$, $c'(v_n) = \{3\}$ and $c'(v_2) = \{2, 3\}$. Since this is a majestic 3-edge coloring, it follows that $\chi'_m(C_n) = 3$ if $n \equiv 2 \pmod 4$. ■

We now turn our attention to bipartite graphs. First, we determine the majestic index of complete bipartite graphs.

Proposition 2.5 *For positive integers r and s where* $\max\{r, s\} = s \geq 2$,

$$\chi'_m(K_{r,s}) = 2.$$

Proof. By Observation 2.1, it suffices to show that $K_{r,s}$ has a majestic 2-edge coloring. Let U and W be the partite sets of $K_{r,s}$, where $|U| = r$ and $W = \{w_1, w_2, \ldots, w_s\}$. Assign the color 1 to each edge incident with w_i for $1 \leq i \leq s-1$ and the color 2 to each edge incident with w_s. Then $c'(w_i) = \{1\}$ for $1 \leq i \leq s-1$, $c'(w_s) = \{2\}$ and $c'(u) = \{1, 2\}$ for each $u \in U$. Thus, c is a majestic 2-edge coloring of $K_{r,s}$ and so $\chi'_m(K_{r,s}) = 2$. ■

It is well known that if $H \subseteq G$, then $\chi(H) \leq \chi(G)$. This, however, is not the case for the majestic index. For example, $C_6 \subseteq K_{3,3}$; nevertheless, $\chi'_m(C_6) = 3$ and $\chi'_m(K_{3,3}) = 2$ by Propositions 2.4 and 2.5. Moreover, for a connected graph G of order at least 3, it is possible that $\chi(G) < \chi'_m(G)$, $\chi(G) = \chi'_m(G)$ and $\chi(G) > \chi'_m(G)$. For example, if $n \equiv 2 \pmod 4$ and $n \geq 6$, then $\chi(C_n) = 2$ and $\chi'_m(C_n) = 3$ by Proposition 2.4; while if $n \equiv 0 \pmod 4$ and $n \geq 4$, then $\chi(C_n) = \chi'_m(C_n) = 2$. Furthermore, if $k \geq 4$, then $\chi(K_k) = k$ and $\chi'_m(K_k) = \lceil \log_2 k \rceil + 1$ by Theorem 2.2.

We saw in Proposition 2.5 that the majestic index of every complete bipartite graph of order at least 3 is 2 and in Proposition 2.4 that the majestic chromatic index of every even cycle is 2 or 3. We now show that the majestic chromatic index of every connected bipartite graph of order 3 or more is either 2 or 3. To verify this, it is convenient to introduce some additional terminology that was introduced and studied in [1].

Let u be a vertex in a nontrivial connected graph G. A vertex v distinct from u is called a *boundary vertex* of u if $d(u,v) = k$ for some positive integer k and no $u - w$ geodesic of length greater than k contains v. In particular, every end-vertex of G different from u is a boundary vertex of u.

Theorem 2.6 *If G is a connected bipartite graph of order 3 or more, then*

$$\chi'_m(G) \leq 3.$$

Proof. Let U and W be the partite sets of G, where U contains at least two vertices. For a vertex u of U, let

$$\begin{aligned} U_1 &= \{v \in V(G) : \ d(u,v) \equiv 0 \pmod 4\} \text{ and} \\ U_2 &= \{v \in V(G) : \ d(u,v) \equiv 2 \pmod 4\}. \end{aligned}$$

Thus, $U = U_1 \cup U_2$ and $W = \{v \in V(G) : \ d(u,v) \text{ is odd}\}$. Assign the color 1 to each edge of G incident with a vertex of U_1 and the color 2 to each edge of G incident with a vertex of U_2. Denote this edge coloring by c and the induced vertex coloring by c'. If no vertex of W is a boundary vertex of u, then every vertex of W has the color $\{1,2\}$. Since each vertex of U has the color $\{1\}$ or $\{2\}$, the coloring c is a majestic 2-edge coloring of G and so $\chi'_m(G) = 2$.

On the other hand, suppose that one or more vertices of W are boundary vertices of u. Let $w \in W$ be a boundary vertex of u. Then $c'(w) = \{1\}$ or $c'(w) = \{2\}$, say the former. For each neighbor x of w on a $u - w$ geodesic, change the color of xw from 1 to 3. Then $c'(w) = \{3\}$ and $c'(x) = \{1,3\}$. This new edge coloring is a majestic 3-edge coloring of G and so $\chi'_m(G) \leq 3$. ∎

The following result describes those bipartite graphs having majestic chromatic index 2.

Theorem 2.7 *Let G be a connected bipartite graph of order 3 or more. Then $\chi'_m(G) = 2$ if and only if there exists a partition $\{U_1, U_2, W\}$ of $V(G)$ such that $U = U_1 \cup U_2$ and W are the partite sets of G and each vertex $w \in W$ has a neighbor in both U_1 and U_2.*

Proof. First, suppose that U and W are the partite sets of G such that U can be partitioned into two sets U_1 and U_2 for which every vertex in W has a neighbor in each of U_1 and U_2. Define the edge coloring $c : E(G) \to \{1, 2\}$ by $c(e) = i$ if e is incident with a vertex in U_i for $i = 1, 2$. Then $c'(u) = \{i\}$ if $u \in U_i$ for $i = 1, 2$ and $c'(w) = \{1, 2\}$ for each $w \in W$. Hence, c is a majestic 2-edge coloring of G and so $\chi'_m(G) = 2$.

For the converse, suppose that G is a connected bipartite graph of order 3 or more such that $\chi'_m(G) = 2$. Let U and W be partite sets of G and let $c : E(G) \to \{1, 2\}$ be a majestic 2-coloring of G. Then U is divided into three sets U_1, U_2, and $U_{1,2}$, where U_i is the set of vertices u with $c'(u) = \{i\}$ for $i = 1, 2$ and $U_{1,2}$ is the set of vertices u with $c'(u) = \{1, 2\}$. Similarly, W is divided into three sets W_1, W_2 and $W_{1,2}$. Observe that the vertices in $U_1 \cup U_2$ can only be adjacent to vertices in $W_{1,2}$ and the vertices in $W_1 \cup W_2$ can only be adjacent to vertices in $U_{1,2}$. Since G is connected and no vertex in $U_{1,2}$ can be adjacent to any vertex in $W_{1,2}$, it follows that either $U_1 \cup U_2 \cup W_{1,2} = \emptyset$ or $W_1 \cup W_2 \cup U_{1,2} = \emptyset$; for otherwise, G would not be connected. We may assume that $W_1 \cup W_2 \cup U_{1,2} = \emptyset$. Then $\{U_1, U_2, W_{1,2}\}$ is the desired partition of the vertex set of G since each vertex $w \in W_{1,2} = W$ must be adjacent to some $u_1 \in U_1$ and some $u_2 \in U_2$ to guarantee that $c'(w) = \{1, 2\}$. ∎

The following is a consequence of the proof of Theorem 2.6.

Corollary 2.8 *Let G be a connected bipartite graph that is not a tree. If G contains a vertex u such that all boundary vertices of u belong to the same partite set of u, then $\chi'_m(G) = 2$.*

The converse of Corollary 2.8 is not true, however. For example, Figure 1 shows a majestic 2-edge coloring of the 3-cube Q_3 (where $c'(v) = \{a\}$ is denoted by a and $c'(v) = \{a, b\}$ is denoted by ab); so $\chi'_m(Q_3) = 2$. (We'll soon say more about the majestic index of the k-cube Q_k in general.) For each vertex u of Q_3, there is a unique boundary vertex v of u such that $d(u, v) = 3$. Thus, u and v do not belong to the same partite set. Therefore, there is no vertex u in Q_3 all of whose boundary vertices belong to the same partite set as u. In fact, if u and v are boundary vertices of each other, then u and v belong to different partite sets of Q_3. Next, we consider the bipartite graph G of Figure 1 that is is not regular. Since G has a majestic 2-edge coloring shown in Figure 1 (where a solid edge is colored 1 and a dashed edge is colored 2), it follows that $\chi'_m(G) = 2$. On the other hand, for $i = 1, 2$, the vertex v_i is a boundary vertex of u_i; while $d(u_i, v_i) = 3$ and so v_i and u_i do not belong to the same partite set. Hence, by symmetry, G has no vertex u all of whose boundary vertices belong to the same partite set as u. For trees, the converse of Corollary 2.8 is true, however, as we show next.

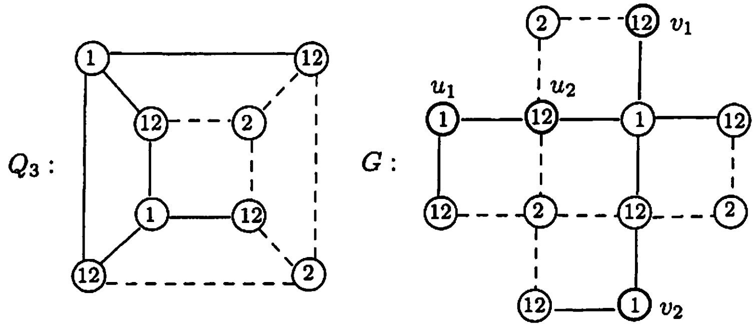

Figure 1: A bipartite graph G with $\chi'_m(G) = 2$

Theorem 2.9 *Let T be a tree of order 3 or more. Then $\chi'_m(T) = 2$ if and only if the distance between every two end-vertices is even. Equivalently, $\chi'_m(T) = 2$ if and only if all end-vertices of T belong to the same partite set of T.*

Proof. Suppose that T is a tree of order 3 or more such that the distance between every two end-vertices is even. Let uv be a pendant edge of T, where u is an end-vertex of T. Assign the color 1 to uv. Let w be any vertex of T such that $d(u,w)$ is even. If $d(u,w) \equiv 2 \pmod 4$, then color all edges incident with w the color 2; while if $d(u,w) \equiv 0 \pmod 4$, then color all edges incident with w the color 1. This is a majestic 2-edge coloring and so $\chi'_m(T) = 2$. Note that for any majestic 2-edge coloring of T, the color of every vertex in one partite set of T is $\{1,2\}$, while the color of a vertex in the other partite set is $\{1\}$ or $\{2\}$.

Next, we verify the converse. Assume to the contrary, that there exists a tree T of order 3 or more such that $\chi'_m(T) = 2$ but T contains a pair u, v of end-vertices for which $d(u,v)$ is odd, say $d(u,v) = 2k+1$ for some positive integer k. Let $P = (u = u_1, u_2, \ldots, u_{2k+2} = v)$ be the $u - v$ path in T and let c be a majestic 2-edge coloring of T. Since u and v are end-vertices, the colors of u and v are either $\{1\}$ or $\{2\}$. Because u_2 is adjacent to u_1, the color of u_1 must be a proper subset of the color of u_2. So, the color of u_2 is $\{1,2\}$. Since u_3 is adjacent to u_2, the color of u_3 must be a singleton. Continuing this process, we obtain that $c'(u_{2i}) = \{1,2\}$ and $c'(u_{2i-1})$ is either $\{1\}$ or $\{2\}$. In particular, $c'(u_{2k+2}) = c'(v) = \{1,2\}$, which is a contradiction. ■

By Theorem 2.9, for each integer $n \geq 3$,

$$\chi'_m(P_n) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 3 & \text{if } n \text{ is even.} \end{cases}$$

We saw that the 3-cube Q_3 has majestic index 2. In fact, every k-cube, $k \geq 2$, has majestic index 2. Since $Q_k = Q_{k-1} \square K_2$ (the Cartesian product of Q_{k-1} and K_2) for $k \geq 3$, this fact is a consequence of the following result. For two disjoint subsets X and Y of vertices of a graph G, let $[X, Y]$ denote the set of edges joining a vertex of X and a vertex of Y in G.

Theorem 2.10 *If G is a nontrivial connected bipartite graph, then*

$$\chi'_m(G \square K_2) = 2.$$

Proof. Let G_1 and G_2 be two copies of G where G_1 has partite sets U_1 and W_1 and G_2 has corresponding partite sets U_2 and W_2. Then $G \square K_2$ has partite sets $U_1 \cup W_2$ and $U_2 \cup W_1$. Let $c : E(G) \to \{1, 2\}$ be defined by

$$c(e) = \begin{cases} 1 & \text{if } e \in [U_1, W_1] \cup [U_1, U_2] \\ 2 & \text{if } e \in [U_2, W_2] \cup [W_1, W_2]. \end{cases}$$

Since the induced vertex coloring c' of $G \square K_2$ satisfies that

$$c'(v) = \begin{cases} \{1\} & \text{if } v \in U_1 \\ \{2\} & \text{if } v \in W_2 \\ \{1, 2\} & \text{if } v \in U_2 \cup W_1, \end{cases}$$

it follows that c' is a proper coloring of $G \square K_2$ and so $\chi'_m(G \square K_2) = 2$. ■

Corollary 2.11 *For each integer $k \geq 2$, $\chi'_m(Q_k) = 2$.*

We have seen that there are connected bipartite graphs G having $\delta(G) \in \{1, 2\}$ and $\chi'_m(G) = 3$. This leads to the following problem concerning connected bipartite graphs having minimum degree at least 3.

Problem 2.12 *If G is a connected bipartite graph with $\delta(G) \geq 3$, does it follow that $\chi'_m(G) = 2$?*

According to Theorem 2.6, if G is a connected bipartite graph of order 3 or more, then $\chi'_m(G)$ is either 2 or 3. We next show for graphs G with $\chi(G) \geq 3$, it is impossible that $\chi'_m(G) = 2$.

Theorem 2.13 *If G is a connected graph with $\chi(G) \geq 3$, then*

$$\chi'_m(G) \geq 3.$$

Proof. Assume, to the contrary, that there exists a graph G with $\chi(G) \geq 3$ such that $\chi'_m(G) = 2$. Thus, there exists a majestic 2-edge coloring c of G, where c' is the induced proper vertex coloring of G. Since each edge of G is colored 1 or 2, each vertex of G is colored $\{1\}$, $\{2\}$ or $\{1,2\}$. Since $\chi(G) \geq 3$, it follows that G contains an odd cycle C, say $C = (v_1, v_2, \ldots, v_{2k+1}, v_{2k+2} = v_1)$, where k is a positive integer. First, observe that for each integer i $(1 \leq i \leq 2k+1)$, $c(v_iv_{i+1}) \in c'(v_i)$ and $c(v_iv_{i+1}) \in c'(v_{i+1})$. Since $c'(v_i) \neq c'(v_{i+1})$, it follows that exactly one of $c'(v_i)$ and $c'(v_{i+1})$ is $\{1,2\}$, say $c'(v_1) = \{1,2\}$. Then $c'(v_i) = \{1,2\}$ for every integer $i \in \{1,3,\ldots,2k+1\}$. However then, $c'(v_1) = c'(v_{2k+1}) = \{1,2\}$, which is a contradiction. ■

3 The Majestic Number of a Graph

Typically, the graph coloring problems of greatest interest have been those of determining the minimum positive integer k for which it is possible to assign colors from the set $[k]$ to the vertices of a graph G in such a way that adjacent vertices are colored differently. For majestic edge colorings of a graph G, here too the goal is to determine the minimum positive integer k but, in this case, we are to assign colors from the set $[k]$ to the edges of G so that two adjacent vertices of G receive distinct induced colors. While the vertex colors are selected from the set $\mathcal{P}([k]) - \{\emptyset\}$ of nonempty subsets of $[k]$, it is of interest here as well to determine the minimum number of vertex colors satisfying these conditions. This leads us to our next topic.

Suppose that G is a connected graph with $\chi'_m(G) = k \geq 2$. Then there exists a majestic k-edge coloring of G where the vertices of G are then colored with the nonempty subsets of $[k]$. Among all majestic k-edge colorings of G, the minimum number of nonempty subsets of $[k]$ needed to color the vertices of G so that two adjacent vertices of G are colored differently is called the *majestic chromatic number* of G or, more simply, the *majestic number* of G and is denoted by $\psi(G)$. First, we present a lower bound for the majestic number of G.

Proposition 3.1 *If G is a connected graph of order at least* 3, *then*

$$\psi(G) \geq \max\{3,\ \chi(G)\}. \tag{1}$$

Proof. Since the induced vertex coloring of a majestic edge coloring of G is a proper vertex coloring, it follows that $\psi(G) \geq \chi(G)$. It remains to show that $\psi(G) \geq 3$. This is certainly the case if $\chi(G) \geq 3$. Hence, we may assume that G is a bipartite graph with $\chi'_m(G) = k \geq 2$. Let c be a majestic k-edge coloring of G for which $\psi(G)$ is minimum. Then the edges of G are colored with at least two colors and there exist two adjacent edges uv and vw that are assigned distinct colors, say 1 and 2, respectively. Thus, $\{1,2\} \subseteq c'(v)$. Now $c'(u) \neq c'(v) \neq c'(w)$. If $c'(u) \neq c'(w)$, then $\psi(G) \geq 3$. If $c'(u) = c'(w)$, then $\{1,2\} \subseteq c'(u) \cap c'(w)$. Therefore, there is an edge incident with one of u and w that is assigned a color different from 1 or 2; say ux is colored 3 but no edge incident with v is colored 3. Therefore, $\{1,2,3\} \subseteq c'(u)$, $\{1,2,3\} \not\subseteq c'(v)$ and $\{3\} \subseteq c'(x)$ but $c'(x) \neq c'(u)$. Hence, $c'(x)$, $c'(u)$ and $c'(v)$ are three distinct colors and $\psi(G) \geq 3$. ■

Since every majestic 2-edge coloring of a graph gives rise to only three distinct vertex colors, we have the following:

Observation 3.2 *If G is a connected graph with $\chi'_m(G) = 2$, then*

$$\psi(G) = 3.$$

Next, we determine the majestic index and majestic number of the Petersen graph. The following lemma will be useful for this purpose.

Lemma 3.3 *Let G be a nonbipartite connected graph such that $\chi'_m(G) = \psi(G) = 3$. If c is a majestic 3-edge coloring whose induced vertex coloring uses exactly three vertex colors, then there is no vertex v of G for which $c'(v)$ is a singleton set.*

Proof. Assume, to the contrary, that there is a majestic 3-edge coloring $c : E(G) \to [3]$ of G such that the induced vertex coloring c' of G uses exactly three colors where $c'(u)$ is a singleton for some vertex u of G. We may assume that $c'(u) = \{1\}$.

Observe that if $e = xy$ is an edge of G, then $c(e) \in c'(x) \cap c'(y)$. Since $\chi'_m(G) = 3$, each of the colors 1, 2, 3 is used to color the edges of G and so each element in $\{1,2,3\}$ belongs to at least two vertex colors. Thus, there is only one possibility for these three vertex colors of c', namely $\{1\}, \{2,3\}$ and $\{1,2,3\}$. For each i with $0 \leq i \leq e(u)$, let

$$V_i = \{x \in V(G) : d(u,x) = i\}.$$

Observe that

$$c'(v) = \begin{cases} \{1,2,3\} & \text{if } v \in V_i \text{ for odd integers } i \text{ with } 1 \leq i \leq e(u) \\ \{1\} \text{ or } \{2,3\} & \text{if } v \in V_i \text{ for even integers } i \text{ with } 1 \leq i \leq e(u). \end{cases}$$

Since c' is a proper vertex coloring and $\{1\} \cap \{2,3\} = \emptyset$, it follows that each set V_i is independent. Let U be the union of those sets V_i where $0 \leq i \leq e(u)$ and i is even and let W be the union of those sets V_i where $1 \leq i \leq e(u)$ and i is odd. Then G is a bipartite graph with partite sets U and W, which is impossible. ■

Proposition 3.4 *For the Petersen graph P, $\chi'_m(P) = 3$ and $\psi(P) = 4$.*

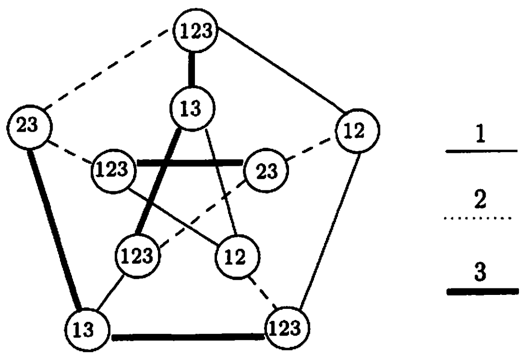

Figure 2: A majestic 3-edge colorings of the Petersen graph with an induced proper 4-vertex coloring

Proof. Since $\chi(P) = 3$, it follows by Theorem 2.13 that $\chi'_m(P) \geq 3$. The majestic 3-edge coloring of P in Figure 2 shows that $\chi'_m(P) = 3$ and $\psi(P) \leq 4$.

In order to verify that $\psi(P) = 4$, we need to show that it is impossible that $\psi(P) = 3$. Assume, to the contrary, that there is a majestic 3-edge coloring c of P with an induced proper 3-vertex coloring c'. By Lemma 3.3, no vertex color is a singleton. Thus, the vertex colors are three of $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ and $\{1,2,3\}$. There is essentially only one proper 3-vertex coloring of the Petersen graph, namely that is shown in Figure 3(a), where the three vertex colors are denoted by p, q, r. Since the vertex v_5 is adjacent to three vertices that are assigned the same color r, the color r cannot be a 2-element set. Consequently, we may assume that $p = \{1,2\}$, $q = \{1,3\}$ and $r = \{1,2,3\}$. Thus, the color of each edge of P joining vertices colored p and q is 1 (see Figure 3(b)). Since $c(u_3u_4) = c(u_4v_4) = 1$, it follows that $c(u_4u_5) = 3$. Since $c(u_1u_2) = c(u_1v_1) = 1$, it follows that $c(u_1u_5) = 2$. Hence, $c(u_5v_5) = 1$. Similarly, $c(v_2v_5) = c(v_3v_5) = 1$, contradicting the assumption that $c'(v_5) = p = \{1,2\}$. Therefore, $\psi(P) \neq 3$ and so $\psi(P) = 4$. ■

The majestic numbers of paths and cycles have been determined. Since the proofs of these two formulas are relatively lengthy, we state these formulas without proofs.

Theorem 3.5 *For each integer $n \geq 3$,*

$$\psi(P_n) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 4 & \text{if } n \text{ is even and } n \neq 6 \\ 5 & \text{if } n = 6. \end{cases}$$

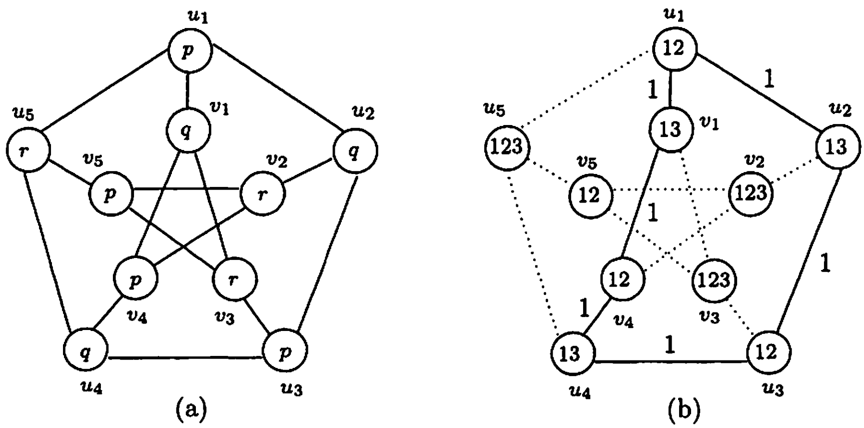

Figure 3: A step in the proof of Proposition 3.4

Theorem 3.6 *For each integer $n \geq 4$,*

$$\psi(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 4 \text{ or } n \equiv 0 \pmod 3 \\ 4 & \text{otherwise.} \end{cases}$$

We saw in Proposition 2.5 that if G is a complete bipartite graph of order at least 3, then $\chi(G) = \chi'_m(G) = 2$. Thus, $\psi(G) = 3$ by Observation 3.2 and so $\psi(G) > \chi(G)$. This is not the case for complete multipartite graphs that are not bipartite, however.

Proposition 3.7 *If G is a complete ℓ-partite graph with $\ell \geq 3$, then*

$$\psi(G) = \chi(G).$$

Proof. Suppose that G is a complete ℓ-partite graph, where $\ell \geq 3$ and $V_1, V_2, \ldots, V_\ell$ are the partite sets of G. By (1), it follows that $\psi(G) \geq \chi(G)$. Thus, it remains to show that $\psi(G) \leq \chi(G)$. To show this, it suffices to show that there is a majestic $\chi(G)$-edge coloring of G whose induced vertex coloring uses $\chi(G)$ colors.

For $\ell = 3$, assign the color 1 to each edge of $[V_1, V_2]$, the color 2 to each edge of $[V_1, V_3]$ and the color 3 to each edge of $[V_2, V_3]$. Thus, $c'(v) = \{1, 2\}$ if $v \in V_1$, $c'(v) = \{1, 3\}$ if $v \in V_2$ and $c'(v) = \{2, 3\}$ if $v \in V_3$. Hence, $\psi(G) = 3$. Suppose next that $\ell \geq 4$. Color the edges in $G[V_1 \cup V_2 \cup V_3]$ as above. For $4 \leq j \leq \ell$, assign the color j to each edge in $[V_j, \cup_{i=1}^{j-1} V_i]$. If $v \in V_1$, then $c'(v) = [\ell] - \{3\}$; if $v \in V_2$, then $c'(v) = [\ell] - \{2\}$; if $v \in V_3$, then $c'(v) = [\ell] - \{1\}$; and if $v \in V_j$ for $4 \leq j \leq \ell$, then $c'(v) = \{j, j+1, \ldots, \ell\}$. In particular, $c'(v) = \{\ell\}$ for each $v \in V_\ell$. Since c' is a proper vertex coloring using ℓ colors, it follows that $\chi(G) = \psi(G) = \ell$. ■

By Proposition 3.7, if G is a complete graph of order $n \geq 3$, then $\psi(G) = \chi(G) = n$.

4 Comparing the Majestic and Chromatic Numbers

We have seen for every connected graph G of order $n \geq 3$ that $\psi(G) \geq \chi(G)$ and have looked at a number of results involving these two parameters when G has small chromatic number. In the case where G is a complete ℓ-partite graph where $\ell \geq 3$, and so $\chi(G) = \ell$, we saw in Proposition 3.7 that $\psi(G) = \chi(G)$. We now consider additional results dealing with majestic and chromatic numbers of graphs where $\chi(G)$ is large. For two graphs G and H, the *composition* $G[H]$ is obtained by replacing each vertex x of G by a copy H_x of H such that every vertex of H_u is adjacent to every vertex of H_v in $G[H]$ if $uv \in E(G)$; that is, $E(G[H]) = \{xy : x \in V(H_x), y \in V(H_y) \text{ and } xy \in E(G)\}$. The graph $C_5[K_2]$ is shown in Figure 4.

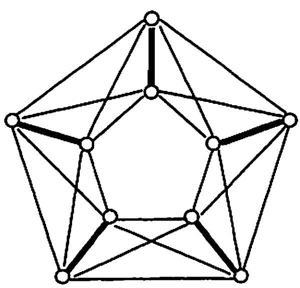

Figure 4: The graph $C_5[K_2]$

The majestic number of $C_n[K_2]$ has been determined for each integer $n \geq 4$. Since the proof of this result is relatively lengthy, we state the result without proof.

Theorem 4.1 *For an integer $n \geq 4$, let $G = C_n[K_2]$.*

- ⋆ *If $n \geq 4$ is even, then $\chi'_m(G) = 3$ and $\psi(G) = 4$.*
- ⋆ *If $n \geq 5$ is odd, then $\chi'_m(G) = 4$ and $\psi(G) = 5$.*

It is not difficult to see that there are infinite classes of graphs G with arbitrarily large chromatic number for which $\psi(G) = \chi(G)$ and for which $\psi(G) = \chi(G) + 1$. We next show the existence of such an infinite class of graphs G for which $\psi(G) = \chi(G) + 2$.

For a given graph G, the *corona* $\text{cor}(G)$ of G is obtained from G by adding a pendant edge to each vertex of G. For each integer $n \geq 3$, let G_n be the graph obtained from the corona $\text{cor}(K_n)$ of the complete graph K_n by subdividing each pendant edge exactly once. Thus, G_n has order $3n$ and exactly n vertices of degree i for each $i \in \{1, 2, n\}$. Suppose that the vertices of the subgraph K_n in G_n are $w_1, w_2, \ldots, w_n$, the vertices of degree 2 in G_n are $v_1, v_2, \ldots, v_n$ and the end-vertices of G_n are $u_1, u_2, \ldots, u_n$, where (u_i, v_i, w_i) is a path of order 3 in G_n for $1 \leq i \leq n$. The graph G_4 is shown in Figure 5.

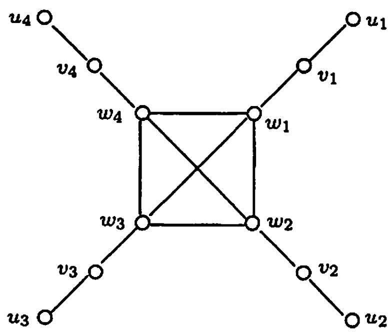

Figure 5: The graph G_4

Theorem 4.2 *For each integer* $n \geq 3$,

$$\chi'_m(G_n) = \lceil \log_2 n \rceil + 1 \text{ and } \psi(G_n) = \chi(G_n) + 2.$$

Proof. For a fixed integer $n \geq 3$, let $k = \lceil \log_2 n \rceil + 1$. By Proposition 2.3, $\chi'_m(G_n) \geq k$. First, we show that $\psi(G_n) \leq \chi(G_n) + 2$. To do this, we show that there is a majestic k-edge coloring $c : E(G_n) \to [k]$ of G_n such that the induced vertex coloring c' uses exactly $\chi(G_n) + 2 = n + 2$ colors in $\mathcal{P}^*([k])$. If $n = 3$ or $n = 4$, then $k = 3$. Majestic 3-edge colorings are shown in Figure 6 for G_3 and G_4. Hence, we may assume that $n \geq 5$.

Thus, $2^{k-2} + 1 \leq n \leq 2^{k-1}$. For $0 \leq i \leq k$, let

$$S_0 = \{k\},\ S_i = \{i, k\} \text{ for } 1 \leq i \leq k-1 \text{ and } S_k = [k].$$

For $k+1 \leq i \leq n-1$, choose the sets $S_i \subseteq [k]$ so that $S_0, S_1, \ldots, S_{n-1}$ are distinct and $k \in S_i$. Now, we assign each vertex w_i $(1 \leq i \leq n)$ the set S_{i-1}. We next define an edge coloring $c_0 : E(K_n) \to [k]$ of K_n as follows: For each integer i with $1 \leq i \leq k-1$, assign the color i to each edge $w_{i+1}w_{t+1}$ if $i \in S_t$ where $k \leq t \leq n-1$ and assign the color k to all other edges of K_n. Figure 7 shows such a 4-edge coloring of K_8, where dashed

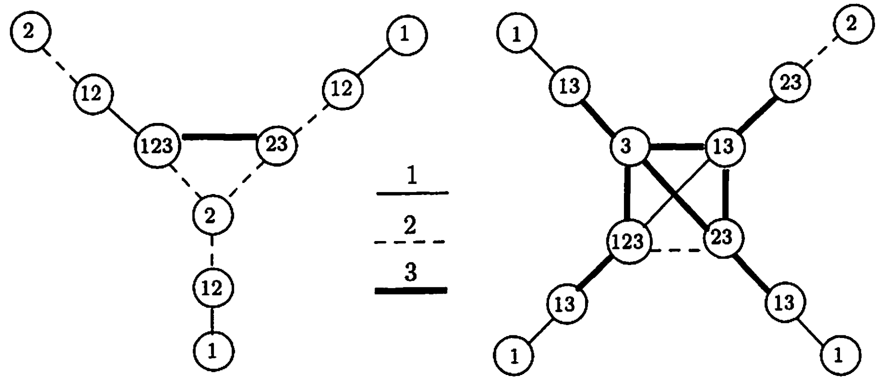

Figure 6: Majestic 3-edge colorings of G_3 and G_4

edges (and edges that are not drawn) are colored 4. Thus $c_0'(w_{j+1}) = S_j$ for all j $(0 \leq j \leq n-1)$ and c_0 is a majestic k-edge coloring of K_n such that $k \in c_0'(v)$ for each vertex v of K_n.

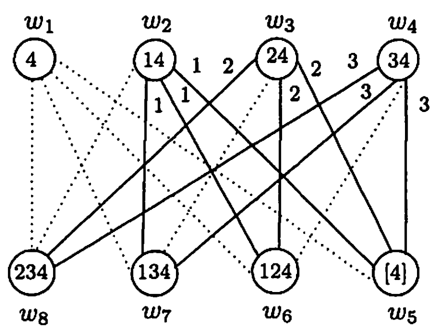

Figure 7: A majestic 4-edge coloring of K_8
where dashed edges and undrawn edges are colored 4

Next, we construct a majestic k-edge coloring $c : E(G_n) \to [k]$ of G_n from the coloring c_0 of K_n as follows:

- ⋆ $c(e) = c_0(e)$ if $e \in E(K_n)$;
- ⋆ $c(w_i v_i) = k$ for $1 \leq i \leq n$;
- ⋆ $c(v_2 u_2) = 2$ and $c(v_i u_i) = 1$ for $1 \leq i \leq n$ and $i \neq 2$.

Figure 8 shows such a 4-edge coloring of G_8, where dashed edges are colored 4. The induced vertex coloring c' then satisfies the following:

- $c'(w) = c_0'(w)$ for each $w \in V(K_n)$,
- $c'(v_2) = \{2,k\} \neq \{1,k\} = c'(w_2)$ and $c'(v_i) = \{1,k\} \neq c'(w_i)$ for $1 \leq i \leq n$ and $i \neq 2$,
- $c'(u_2) = \{1\}$ and $c'(u_i) = \{2\}$ for $1 \leq i \leq n$ and $i \neq 2$.

Thus, c' is proper and so c is a majestic k-edge coloring of G_n. Therefore, $\chi'_m(G_n) = k = \lceil \log_2 n \rceil + 1$. Furthermore, c' uses exactly $n+2$ colors (where $\{1\}$ and $\{2\}$ are the only two new colors added to the vertex coloring c_0' of K_n) and so $\psi(G_n) \leq n+2$.

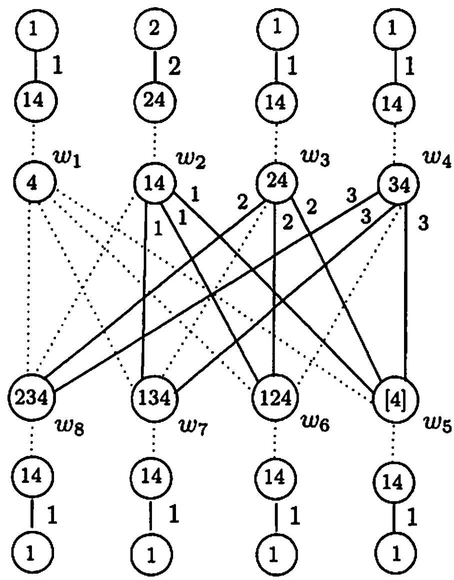

Figure 8: A majestic 4-edge coloring of G_8

It remains to show that $\psi(G_n) \geq \chi(G_n) + 2 = n + 2$. Since $\chi(G_n) = n$, it follows that $\psi(G_n) \geq \chi(G_n) = n$. Suppose that $\chi'_m(G_n) = k$. First, we show that $\psi(G_n) \geq n+1$. Assume, to the contrary, that $\psi(G_n) = n$. Let $c : E(G_n) \to [k]$ be a majestic k-edge coloring of G_n such that the induced vertex coloring c' uses exactly n colors in $\mathcal{P}^*([k])$. We may assume that $c(u_1v_1) = 1$ and $c(v_1w_1) = 2$. Hence, $c'(u_1) = \{1\}$ and $c'(v_1) = \{1,2\}$. Since $\psi(G_n) = n$ and no two vertices of K_n can be colored the same, there are two distinct vertices w_i and w_j, $1 \leq i < j \leq n$, such that $c'(w_i) = \{1\}$ and $c'(w_j) = \{1,2\}$. This implies that no $c'(w_i)$ $(1 \leq i \leq n)$ is a singleton different from $\{1\}$; for otherwise, suppose that $c'(w_t) = \{\ell\}$ for some $t, \ell \in [n]$ and $\ell \neq 1$. Since $c'(w_i) = \{1\}$, there is no appropriate color available for $w_i w_t$, a contradiction. Now either $c(v_j w_j) = 1$ or $c(v_j w_j) = 2$. If $c(v_j w_j) = 1$, then $c(u_j v_j) \notin \{1, 2\}$ and so $c(u_j v_j) = 3$. This implies that $c'(u_j) = \{3\}$. However then, the color of some vertex w_t $(1 \leq t \leq n)$ must be $\{3\}$, which is a contradiction. If $c(v_j w_j) = 2$, then $c(u_j v_j) \notin \{1, 2\}$, which again is a contradiction. Therefore, $\psi(G_n) \geq n + 1$.

Next, we show that $\psi(G_n) \geq n + 2$. Assume, to the contrary, that $\psi(G_n) = n + 1$. Let $c : E(G_n) \to [k]$ be a majestic k-edge coloring of G_n such that the induced vertex coloring c' uses exactly $n+1$ colors in $\mathcal{P}^*([k])$. Again, we may assume that $c(u_1 v_1) = 1$ and $c(v_1 w_1) = 2$. Hence, $c'(u_1) = \{1\}$ and $c'(v_1) = \{1, 2\}$. Since no two vertices of K_n can be colored the same, there is at least one vertex w_i $(1 \leq i \leq n)$ such that $c'(w_i) = \{1\}$ or $c'(w_i) = \{1, 2\}$.

First, suppose that $c'(w_i) = \{1\}$ for some i with $1 \leq i \leq n$. Since $c(v_1 w_1) = 2$, it follows that $i \neq 1$. We may assume that $c'(w_2) = \{1\}$. Thus, $c(v_2 w_2) = 1$ and $c(u_2 v_2) \neq 1$. Assume that $c(u_2 v_2) = a \neq 1$. Then $c'(u_2) = \{a\}$ and $c'(v_2) = \{1, a\}$. Since $c'(w_2) = \{1\}$ and $a \neq 1$, it follows that $c'(w_i) \neq \{a\}$ for all i with $1 \leq i \leq n$ and so $\{a\}$ is the $(n + 1)$th color. Observe that $c'(v_2) = \{1, a\}$ must be used for some w_i where $4 \leq i \leq n$, say $c'(w_4) = \{1, a\}$. Thus, $c(v_4 w_4) \in \{1, a\}$, which implies that $c(v_4 u_4) = \ell \neq a$. Hence, $c'(u_4) = \{\ell\}$ and so there is a vertex w_s such that $c'(w_s) = \{\ell\}$. However then, there is no appropriate color available for $w_2 w_s$, which is a contradiction.

Thus, $c'(w_i) \neq \{1\}$ for all i with $1 \leq i \leq n$ and so $\{1\}$ is the $(n + 1)$th color. Therefore, $c'(w_i) = \{1, 2\}$ for some i with $1 \leq i \leq n$. Since $c(v_1 w_1) = 2$ and $c'(v_1) = \{1, 2\}$, it follows that $i \neq 1$. We may assume that $c'(w_2) = \{1, 2\}$. Thus, $c(w_2 v_2) \in \{1, 2\}$ and so $c(v_2 u_2) = a \notin \{1, 2\}$. Hence, $c'(u_2) = \{a\}$ and $\{a\}$ must be the color of some vertex w_i. Because $c'(w_2) = \{1, 2\}$ and $a \notin \{1, 2\}$, it follows that $c'(w_i) \neq \{a\}$ for all i with $1 \leq i \leq n$, which is impossible. Therefore, $\psi(G_n) \geq n + 2$ and so $\psi(G_n) = n + 2$. ■

Next, we describe an infinite class of graphs G with arbitrarily large chromatic number for which $\psi(G) = \chi(G) + 3$. A *double corona* of G is obtained from G by adding two pendant edges to each vertex of G. Thus, if the order of G is n, then the order of $\text{cor}(G)$ is $2n$ and the order of the double corona of G is $3n$. For each integer $n \geq 3$, let H_n be the graph constructed from the double corona of the complete graph K_n as follows. For each pair of pendant edges at a vertex of K_n, subdivide one pendant edge exactly once and the other pendant edge exactly twice. Thus, H_n has order $6n$, exactly n vertices of degree i for each $i \in \{1, n\}$ and exactly $3n$ vertices of degree 2. The proof of the next theorem is similar to but more complex than the proof of Theorem 4.2, and, thus, we omit it.

Theorem 4.3 *For each integer $n \geq 3$,*

$$\chi'_m(H_n) = \lceil \log_2 n \rceil + 1 \text{ and } \psi(H_n) = \chi(H_n) + 3.$$

We have now seen that there infinitely many connected graphs G satisfying each of the following:

(i) $\psi(G) = \chi(G)$;

(ii) $\psi(G) = \chi(G) + 1$;

(iii) $\psi(G) = \chi(G) + 2$;

(iv) $\psi(G) = \chi(G) + 3$.

This brings up the following two related questions:

Problem 4.4 *For a given positive integer k, does there exist a connected graph F_k such that $\psi(F_k) = \chi(F_k) + k$?*

Problem 4.5 *Does there exist a positive integer K such that $\psi(F) \leq \chi(F) + K$ for every connected graph F?*

We conclude with an additional question.

Problem 4.6 *Does there exist a connected graph G having a majestic k-edge coloring with $k > \chi'_m(G)$ such that the number of vertex colors is p where $p < \psi(G)$?*

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