Changing Views of Ramsey Numbers

– Results and Problems –

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Abstract

In a red-blue coloring of a graph G, every edge of G is colored red or blue. For two graphs F and H, the Ramsey number $R(F, H)$ of F and H is the smallest positive integer n such that every red-blue coloring of the complete graph K_n of order n results in either a subgraph isomorphic to F all of whose edges are colored red or a subgraph isomorphic to H all of whose edges are colored blue. While the study of Ramsey numbers has been a popular area of research in graph theory, over the years a number of variations of Ramsey numbers have been introduced. We look at several of these, with special emphasis on some of those introduced more recently.

Key Words: Ramsey number, arrowing, size Ramsey number, bipartite Ramsey number, monochromatic Ramsey number, balanced complete multipartite graph, k-Ramsey number, rainbow Ramsey number and proper Ramsey number.

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1 Introduction

The famous mathematician Ronald Graham has stated that Ramsey theory is a branch of mathematics dedicated to the proposition that *complete disorder is impossible* (a statement attributed to the mathematician Theodore S. Motzkin) in the sense that within any sufficiently large system, some regularity must occur. Ramsey theory has also been described as the study of unavoidable regularity in large structures, where the primary question is: When is it the case that whenever the elements of some sufficiently large structure are partitioned into a finite number of classes, there is always at least one class within which a prescribed regular structure occurs? When the structures in question are graphs whose edges are colored with a finite number of colors, resulting in a decomposition, and the desired class is a subgraph whose edges are colored the same, then the Ramsey theory being discussed is that in graph theory.

In a *red-blue coloring* of a graph G, every edge of G is colored red or blue. For two graphs F and H, the *Ramsey number* $R(F, H)$ of F and H is the smallest positive integer n such that for every red-blue coloring of the complete graph K_n of order n, there is either a subgraph isomorphic to F all of whose edges are colored red (a *red* F) or a subgraph isomorphic to H all of whose edges are colored blue (a *blue* H). A graph all of whose edges are colored the same is called a *monochromatic graph.* The investigation of Ramsey numbers is one of the best known topics of study within Extremal Graph Theory. A book by Graham, Rothschild and Spencer [31] is devoted to this area of study. In addition, a chapter on Ramsey numbers by Faudree in the *Handbook of Graph Theory* [33, pp. 1002-1025] is devoted, as well, to Ramsey numbers.

Ramsey numbers are named for Frank Ramsey (1903-1930), a British philosopher, economist and mathematician. The theorem for which Ramsey is known was proved only as a minor lemma in a famous paper [43] by Ramsey. This lemma became the basis of the area of graph theory called Ramsey theory.

While the study of Ramsey numbers has been a popular area of research in graph theory, over the years a number of variations of Ramsey numbers have been introduced. We describe several of these here, with special emphasis on some of those introduced more recently. We present several results and open questions in this area of research. While many results obtained on Ramsey numbers and their variations involve bounds, our primary emphasis here is describing some of the exact results obtained. We refer to the book [10] for graph theory notation and terminology not described in this paper.

2 Ramsey Numbers

When F and H are both complete graphs, the Ramsey numbers $R(F, H)$ are often referred to as *classical Ramsey numbers.* For integers $s, t \geq 3$, only a handful of classical Ramsey numbers $R(K_s, K_t)$ are known. The complete list of known classical Ramsey numbers $R(K_s, K_t)$ for $3 \leq s \leq t$ is given below.

$$\begin{array}{lll} R(K_3, K_3) = 6 & R(K_3, K_6) = 18 & R(K_3, K_9) = 36 \\ R(K_3, K_4) = 9 & R(K_3, K_7) = 23 & R(K_4, K_4) = 18 \\ R(K_3, K_5) = 14 & R(K_3, K_8) = 28 & R(K_4, K_5) = 25. \end{array}$$

In particular, the exact value of $R(K_5, K_5)$ is not known. It is only known that $44 \leq R(K_5, K_5) \leq 49$. The best known of the Ramsey numbers listed above is $R(K_3, K_3) = 6$. One interpretation of this number is that in any group of six people every two of which are either acquaintances or strangers, there is always three among them who are mutual acquaintances or mutual strangers. Since the red-blue coloring of K_5 whose red and blue subgraphs are both C_5 does not produce a monochromatic K_3, it follows that $R(K_3, K_3) \geq 6$. To verify that $R(K_3, K_3) \leq 6$, it remains to show that every red-blue coloring of K_6 produces a monochromatic K_3. Let $V(K_6) = \{u, v, w, x, y, z\}$ and let there be given a red-blue coloring of K_6. We may assume that xu, xv, xw are colored the same, say red. If one of the edges uv, vw, uw is red, then there is a red K_3; while if all three edges uv, vw, uw are blue, then there is a blue K_3. Therefore, $R(K_3, K_3) = 6$.

It is a consequence of a theorem of Ramsey [43] that $R(F, H)$ exists for every pair F, H of graphs. Furthermore, it is a result of Erdös and Szekeres [22] that if F is a graph of order s and H is a graph of order t, then

$$R(F, H) \leq R(K_s, K_t) \leq \binom{s+t-2}{s-1}.$$

The exact values of $R(F, H)$ have been determined only for pairs F, H of graphs belonging to relatively few classes. Some of these are listed below (also see [39, 41, 42]).

Theorem 2.1 [13] *Let T be a tree of order $p \geq 2$. For every integer $n \geq 2$,*

$$R(T, K_n) = (p-1)(n-1) + 1.$$

Theorem 2.2 [28] *For integers n and m with $2 \leq m \leq n$,*

$$R(P_n, P_m) = n - 1 + \lfloor m/2 \rfloor.$$

Theorem 2.3 [25] *Let m and n be integers with $3 \leq m \leq n$.*

(1) *If m is odd, where $(m, n) \neq (3, 3)$, then*

$$R(C_m, C_n) = 2n - 1.$$

(2) *If m and n are even, where $(m, n) \neq (4, 4)$, then*

$$R(C_m, C_n) = n + m/2 - 1.$$

(3) *If m is even and n is odd,*

$$R(C_m, C_n) = \max\{n + m/2 - 1,\ 2m - 1\}.$$

(4) $R(C_3, C_3) = R(C_4, C_4) = 6$.

Theorem 2.4 [15, 16] *For integers s and t with $2 \le s \le t$,*

$$R(sK_2, tK_2) = s + 2t - 1.$$

More generally, for every $k \ge 2$ graphs $F_1, F_2, \ldots, F_k$, there exists a least positive integer n such that for every edge coloring of K_n with the colors $1, 2, \ldots, k$, there exists a subgraph of K_n isomorphic to F_i for some i with $1 \le i \le k$ such that every edge of this subgraph is colored i. This integer n is the *Ramsey number* $R(F_1, F_2, \ldots, F_k)$ of $F_1, F_2, \ldots, F_k$, which always exists. The only classical Ramsey numbers whose value is known when $k \ge 3$ and where all complete graphs have order at least 3 is $R(K_3, K_3, K_3) = 17$ (see [32]) and, reportedly, $R(K_3, K_3, K_4) = 30$ (see [17]).

To see that $R(K_3, K_3, K_3) \le 17$, let there be given a red-blue-green coloring of the edges of $G = K_{17}$ and let v be a vertex of G. Therefore, $\deg v = 16$. At least six edges incident with v are colored the same. Hence, we may assume that $vv_1, vv_2, \ldots, vv_6$ are six edges of G, all colored green. If any two vertices of $U = \{v_1, v_2, \ldots, v_6\}$ are joined by a green edge, then G contains a green K_3. Otherwise, every edge of the induced subgraph $H = G[U]$ is colored red or blue. Since $H \cong K_6$ and $R(K_3, K_3) = 6$, it follows that H, and G as well, contains either a red K_3 or a blue K_3. Therefore, $R(K_3, K_3, K_3) \le 17$. Since the complete graph K_{16} has an isomorphic factorization into three factors, each of which is the 5-regular triangle-free graph (called the *Clebsch graph* [14]) shown in Figure 1, it follows that $R(K_3, K_3, K_3) > 16$ and so $R(K_3, K_3, K_3) = 17$.

This more general Ramsey number has also been determined when all graphs F_i are stars.

Theorem 2.5 [7] *Let $s_1, s_2, \ldots, s_k$ be $k \ge 2$ positive integers, t of which are even, and let $s = \sum_{i=1}^{k}(s_i - 1)$. Then*

$$R(K_{1,s_1}, K_{1,s_2}, \ldots, K_{1,s_k}) = \begin{cases} s+1 & \text{if } t \text{ is positive and even} \\ s+2 & \text{otherwise.} \end{cases}$$

If F and H are graphs such that $F \cong H$, then

$$R(F, H) = R(H, F) = R(F, F)$$

is the smallest positive integer n such that if each edge of K_n is colored with one of two colors, then a monochromatic F results. This leads to the following definition. For two graphs F and H, the *monochromatic Ramsey number* $MR(F, H)$ is the smallest positive integer n such that if each edge of K_n is colored with one of two colors, then a monochromatic F or a monochromatic H results. Certainly, $MR(F, H) = MR(H, F)$ for every

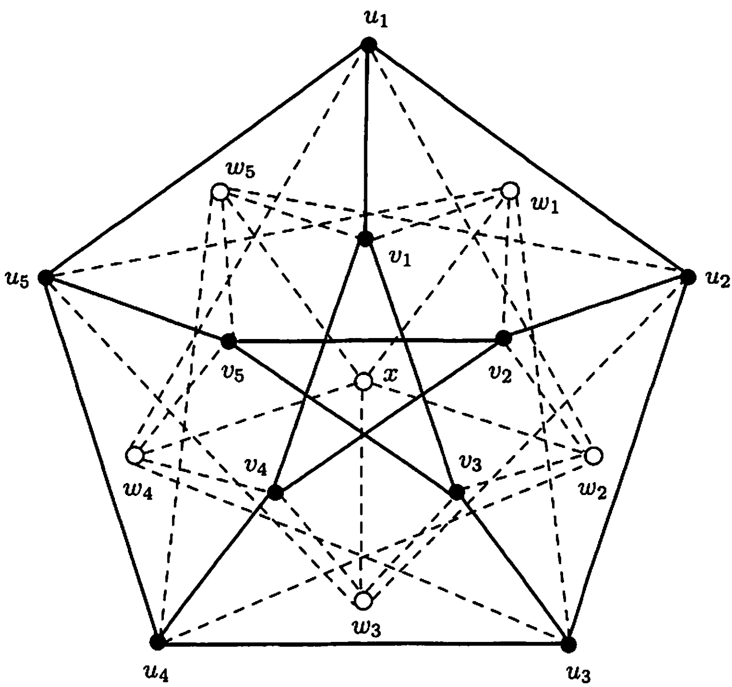


Figure 1: The Clebsch graph

two graphs F and H. Also, $MR(F,H) \leq R(F,H)$. Furthermore, if $F \cong H$, then $MR(F,H) = R(F,H)$ and if $F \subseteq H$, then $MR(F,H) = R(F,F)$ (see [11, pp. 315-320]). By Theorem 2.3, $R(C_3,C_4) = 7$. Next, we show that $MR(C_3,C_4) = 6$. Since the red-blue coloring of K_5 in which both red and blue subgraphs are C_5 avoids both a monochromatic C_3 and a monochromatic C_4, it follows that $MR(C_3,C_4) \geq 6$. Since $R(K_3,K_3) = 6$, it follows that $MR(C_3,C_4) \leq 6$ and so $MR(C_3,C_4) = 6$. Thus, $MR(C_3,C_4) < R(C_3,C_4)$.

3 Arrowing and Size Ramsey Numbers

While the definitions of the Ramsey number $R(F,H)$ of two graphs F and H and that of the more general $R(F_1,F_2,\ldots,F_k)$ of $k \geq 3$ graphs $F_1,F_2,\ldots,F_k$ concern edge colorings of complete graphs, with two colors in the first instance and k colors in the second instance, there has been research dealing with graphs that are not necessarily complete. In this case, different terminology and notation are used.

Let F and H be two graphs. A graph G is said to *arrow* the graphs F and H, written $G \to (F, H)$, if every red-blue coloring of G results in a red F or a blue H. In this case, the primary problem concerns either determining graphs G or properties of graphs G for which $G \to (F, H)$. Obviously, one such graph G with this property is K_r where $r = R(F, H)$. Indeed, any graph G with clique number $\omega(G) \geq r$ has this property. Among the results obtained dealing with this concept are the following (see [9, 26, 38], for example).

Proposition 3.1 *If G is a graph for which $G \to (K_m, K_n)$, where $m, n \geq 2$, then $\omega(G) \geq \max\{m, n\}$.*

Theorem 3.2 *If G is a graph for which $G \to (K_m, K_n)$, where $m, n \geq 2$, then $\chi(G) \geq R(K_m, K_n)$.*

Theorem 3.3 *If G is a connected graph and n is a positive integer, then $G \to (K_{1,n}, K_{1,n})$ if and only if (i) $\Delta(G) \geq 2n - 1$ or (ii) n is even and G is a $(2n-2)$-regular graph of odd order.*

For two graphs F and H, the *size Ramsey number* $\hat{R}(F, H)$ of F and H is the smallest size of a graph G such that $G \to (F, H)$. Bounds on the size Ramsey numbers of paths, cycles or trees have been established in terms of the order and maximum degree of the graphs (see [3, 4, 8, 20], for example).

Proposition 3.4 [20] *For two graphs F and H,*

$$|E(F)| + |E(H)| - 1 \leq \hat{R}(F, H) \leq \binom{R(F,H)}{2}.$$

Theorem 3.5 [20] *For positive integers m, n, s and t,*

(i) $\hat{R}(K_m, K_n) = \binom{R(K_m, K_n)}{2}$

(ii) $\hat{R}(sK_{1,m}, tK_{1,n}) = (m + n - 1)(s + t - 1)$.

4 Bipartite Ramsey Numbers

For two bipartite graphs F and H, the *bipartite Ramsey number* $BR(F, H)$ is defined as the smallest positive integer r such that every red-blue coloring of the r-regular complete bipartite graph $K_{r,r}$ results in either a red F or a blue H. Consequently, if $BR(F, H) = r$ for bipartite graphs F and H, then every red-blue coloring of $K_{r,r}$ results in a red F or a blue H, while there exists a red-blue coloring of $K_{r-1,r-1}$ for which there is neither a red F nor a blue H. To illustrate these concepts, we show that $BR(C_4, C_4) = 5$. Since the red-blue coloring of $K_{4,4}$, both of whose red and blue subgraph

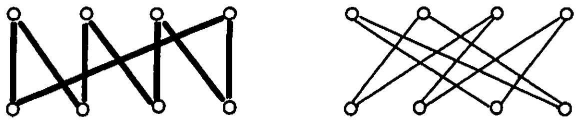

(a) (b)

Figure 2: A red-blue coloring of $K_{4,4}$

are C_8 shown in Figure 2(a) and (b) avoids both a red C_4 and a blue C_4, it follows that $BR(C_4, C_4) \geq 5$.

To verify that $BR(C_4, C_4) \leq 5$, it remains to show that every red-blue coloring of $K_{5,5}$ results in a monochromatic C_4. Let there be given a red-blue coloring of $G = K_{5,5}$ where $U = \{u_1, u_2, \ldots, u_5\}$ and $W = \{v_1, v_2, \ldots, v_5\}$ are the partite sets of $K_{5,5}$. We may assume that the red subgraph G_R of G contains at least 13 edges and so $\Delta(G_R) \geq 3$. If there is a vertex $v \in U$ such that $\deg_{G_R} v = 5$, then $\deg_{G_R} u \leq 1$ for each $u \in U - \{v\}$ and so the size of G_R is at most 9. If there is a vertex $v \in U$ such that $\deg_{G_R} v = 4$, then $\deg_{G_R} u \leq 2$ for each $u \in U - \{v\}$ and so the size of G_R is at most 12. Thus, $\Delta(G_R) = 3$ and at least three vertices in U have degree 3 in G_R, say u_1, u_2, u_3. Furthermore, we may assume that $u_1w_i \in E(G_R)$ for $i = 1, 2, 3$ and $u_2w_i \in E(G_R)$ for $i = 3, 4, 5$. However then, no matter how the red edges incident with u_3 are located in $K_{5,5}$, there is a red C_4. Therefore, $BR(C_4, C_4) = 5$.

It is known that $BR(F, H)$ exists for every two bipartite graphs F and H (see [5]). Indeed, if F is a bipartite graph whose largest partite set contains s vertices and H is a bipartite graph whose largest partite set contains t vertices, then $F \subseteq K_{s,s}$ and $H \subseteq K_{t,t}$, resulting in the following result of Hattingh and Henning.

Theorem 4.1 [34] *If F and H are bipartite graphs such that $F \subseteq K_{s,s}$ and $H \subseteq K_{t,t}$, then*

$$BR(F, H) \leq BR(K_{s,s}, K_{t,t}) \leq \binom{s+t}{s} - 1.$$

The following results and a conjecture were obtained on bipartite Ramsey numbers.

Theorem 4.2 [12] *For integers s and t with $2 \leq s \leq t$,*

$$BR(sK_2, tK_2) = s + t - 1.$$

Theorem 4.3 [5] *For positive integer t,*

$$BR(K_{1,t}, K_{1,t}) = 2t - 1.$$

Conjecture 4.4 [5] *For integers s and t with* $1 \le s \le t$,

$$BR(K_{s,t}, K_{s,t}) = 2^s(t-1) + 1$$

5 k-Ramsey Numbers

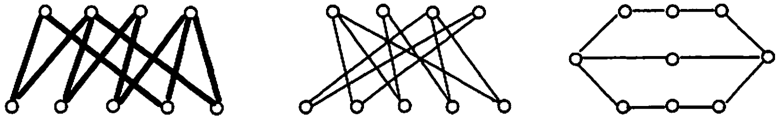
We have seen that if $BR(F, H) = r$ for bipartite graphs F and H, then every red-blue coloring of $K_{r,r}$ results in a red F or a blue H, while there exists a red-blue coloring of $K_{r-1,r-1}$ for which there is neither a red F nor a blue H. This brings up the question of what might occur for red-blue colorings of the intermediate graph $K_{r-1,r}$. This led to a more general concept.

For bipartite graphs F and H, the *2-Ramsey number* $R_2(F, H)$ of F and H is the smallest positive integer n such that every red-blue coloring of the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ of order n results in a red F or a blue H. If the bipartite Ramsey number $BR(F, H)$ of two bipartite graphs F and H is r, then every red-blue coloring of $K_{r,r}$ produces a red F or a blue H, while there exists a red-blue coloring of $K_{r-1,r-1}$ that produces neither. Which of these two situations occurs for the graph $K_{r-1,r}$ depends on the graphs F and H. That is, either

$$R_2(F, H) = 2BR(F, H) \text{ or } R_2(F, H) = 2BR(F, H) - 1. \qquad (1)$$

To illustrate this concept, we show that $R_2(C_4, C_4) = 10$. We saw that $BR(C_4, C_4) = 5$. Hence, $R_2(C_4, C_4) = 10$ or $R_2(C_4, C_4) = 9$. In fact, there is a red-blue coloring of $K_{4,5}$ that results in neither a red C_4 nor a blue C_4. To see this, consider the red-blue coloring of $K_{4,5}$ in which both the red subgraph shown in Figure 3(a) and the blue subgraph shown in Figure 3(b) are isomorphic to the graph in Figure 3(c). Since the graph in Figure 3(c) does not contain C_4 as a subgraph, this red-blue coloring of $K_{4,5}$ avoids both a red C_4 and a blue C_4. Therefore, $R_2(C_4, C_4) \ge 10$ and so $R_2(C_4, C_4) = 10$.

The concept of the 2-Ramsey number of two bipartite graphs is a special case of a more general concept. For an integer $k \ge 2$, a *balanced complete k-partite graph* of order $n \ge k$ is the complete k-partite graph in which every partite set has either $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ vertices. So if $n = kq + r$ where $q \ge 1$ and $0 \le r \le k-1$, then the balanced complete k-partite graph G of order n has r partite sets with $q + 1$ vertices and the remaining $k - r$ partite sets have q vertices. For bipartite graphs F and H and an integer



(a) (b) (c)

Figure 3: A red-blue coloring of $K_{4,5}$

k with $2 \le k \le R(F, H)$, the *k-Ramsey number* $R_k(F, H)$ is defined as the smallest positive integer n such that every red-blue coloring of a balanced complete k-partite graph of order n results in a red F or a blue H.

If F and H are two bipartite graphs for which $R(F, H) = n \ge 3$, then every red-blue coloring of K_n produces either a red F or a blue H. However, such is not the case for the smaller complete graphs K_2, K_3, ..., K_{n-1}. Equivalently, for every red-blue coloring of the complete n-partite graph K_n where each partite set consists of a single vertex, there is either a red F or a blue H. However, for each complete k-partite graph K_k, where $2 \le k \le n-1$ such that every partite set consists of a single vertex, there exists a red-blue coloring that produces neither a red F nor a blue H. On the other hand, for each of the graphs $K_2, K_3, \ldots, K_{n-1}$, we can continue to add vertices to each partite set, resulting in a balanced complete k-partite graph at each step where $2 \le k \le n-1$ until eventually arriving at the balanced complete k-partite graph of smallest order $R_k(F, H)$ having the property that every red-blue coloring of this graph produces a red F or a blue H. Consequently, for every two bipartite graphs F and H and every integer k with $2 \le k \le R(F, H)$, the k-Ramsey number $R_k(F, H)$ exists.

For example, it is known that $R(C_4, C_4) = 6$. Furthermore, we saw that $BR(C_4, C_4) = 5$ and $R_2(C_4, C_4) = 10$. In fact, $R_k(C_4, C_4) = 12 - k$ for $2 \le k \le 6 = R(C_4, C_4)$ (see [1]). As an illustration, we show that $R_3(C_4, C_4) = 9$. Let H be a balanced complete 3-partite graph of order 8. Then $H = K_{2,3,3}$. Figure 4 shows a red-blue coloring of H having neither a red C_4 nor a blue C_4, where the bold edges represent edges colored red. Thus $R_3(C_4, C_4) \ge 9$.

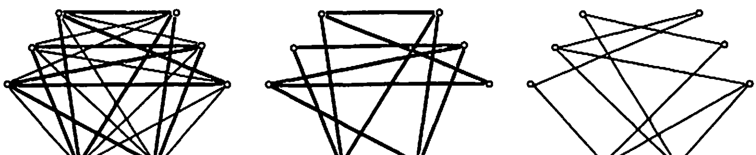


Figure 4: A red-blue coloring of $K_{2,3,3}$

To show that $R_3(C_4, C_4) = 9$, it remains to show that every red-blue coloring of $G = K_{3,3,3}$ results in a monochromatic C_4. Assume, to the contrary, that there is a red-blue coloring of G that produces neither a red C_4 nor a blue C_4. Let G_R and G_B denote the red and blue subgraphs of G, respectively, of sizes m_R and m_B. We may assume that $m_R \geq m_B$. Since $m_R + m_B = 27$, it follows that $m_R \geq 14$. Let V_1, V_2 and V_3 be the three partite sets of G and, for $1 \leq i < j \leq 3$, let $[V_i, V_j]$ denote the nine edges of G joining V_i and V_j. Let G'_R denote the subgraph of size m'_R in G_R with vertex set $V_1 \cup V_2$ such that $E(G'_R) \subseteq [V_1, V_2]$. The subgraphs G''_R and G'''_R with vertex sets $V_2 \cup V_3$ and $V_1 \cup V_3$ and sizes m''_R and m'''_R, respectively, are defined similarly. We may assume that $m'_R \geq m''_R \geq m'''_R$ and so $m'_R + m''_R \geq 10$. Let $V_1 = \{u_1, u_2, u_3\}$, $V_2 = \{v_1, v_2, v_3\}$ and $V_3 = \{w_1, w_2, w_3\}$. Observe that if any of u_1, u_2 and u_3 has degree 3 in G'_R, say u_1, then u_2 and u_3 have degree at most 1 in G'_R and each of w_1, w_2 and w_3 has degree at most 1 in G''_R, for otherwise, a red C_4 is produced. However then, $m'_R + m''_R \leq 8$, a contradiction. Consequently, each of u_1, u_2 and u_3 has degree at most 2 in G'_R. Therefore, $m'_R = 6$ or $m'_R = 5$. In either case, it can be shown that there is a red C_4, producing a contradiction.

The following three results on k-Ramsey numbers were obtained in [2].

Proposition 5.1 *Let F and H be two bipartite graphs. If k is an integer with $2 \leq k \leq R(F, H)$, then $R(F, H) \leq R_k(F, H)$.*

Proposition 5.2 *Let F and H be two bipartite graphs. If k and ℓ are positive integers with $k \geq 2$, then $R_{\ell k}(F, H) \leq R_k(F, H)$.*

Proposition 5.3 *Let F and H be two bipartite graphs. If k is an integer with $k \leq R(F, H)$ for which $R_k(F, H) = R(F, H)$ and $\frac{R_k(F,H)-1}{k} \leq 2$, then*

$$R_\ell(F, H) = R_k(F, H)$$

for each integer ℓ with $k \leq \ell \leq R(F, H)$.

By Theorem 2.5, for two integers $s, t \geq 2$,

$$R(K_{1,s}, K_{1,t}) = \begin{cases} s + t - 1 & \text{if } s \text{ and } t \text{ are both even} \\ s + t & \text{otherwise.} \end{cases} \tag{2}$$

Thus, if $k = R(K_{1,s}, K_{1,t})$, then $R_k(K_{1,s}, K_{1,t})$ is expressed in (2). The k-Ramsey number of stars have been determined for all possible values of k in [1].

Theorem 5.4 *For each integer $t \geq 2$, $R_2(K_{1,2}, K_{1,t}) = 2t + 1$.*

Theorem 5.5 *Let k, s and t be integers with $3 \le k < R(K_{1,s}, K_{1,t})$ and $s + t \ge 5$.*

(a) *If $s + t - 2 = (k-1)q$ for some positive integer q, then*

$$R_k(K_{1,s}, K_{1,t}) = \begin{cases} kq & \text{if } k \text{ and } q \text{ are odd and } s \text{ and } t \text{ are even} \\ kq+1 & \text{otherwise.} \end{cases}$$

(b) *If $s + t - 2 = (k-1)q + r$ for integers q and r where $q \ge 1$ and $1 \le r \le k-2$, then*

$$R_k(K_{1,s}, K_{1,t}) = \begin{cases} kq + r & \text{if } (k-r)q \text{ is odd and } s \text{ and } t \\ & \text{are of opposite parity} \\ kq+r+1 & \text{otherwise.} \end{cases}$$

Consequently, when $3 \le k < R(K_{1,s}, K_{1,t})$ and $s + t \ge 5$, it follows that $R_k(K_{1,s}, K_{1,t})$ is either $s+t-2+\left\lfloor \frac{s+t-2}{k-1} \right\rfloor$ or $s+t-1+\left\lfloor \frac{s+t-2}{k-1} \right\rfloor$, depending on the values of k, s and t in Theorem 5.5.

The bipartite Ramsey number of two stripes was determined in [12].

Theorem 5.6 [12] *For integers s and t with $2 \le s \le t$,*

$$BR(sK_2, tK_2) = s + t - 1.$$

In [2], the k-Ramsey numbers were determined for certain stripes F and H and for certain values of k. By Theorem 2.4 and Proposition 5.1, for integers k, s and t with $2 \le s \le t$ and $2 \le k \le R(sK_2, tK_2)$, it follows that

$$R_k(sK_2, tK_2) \ge s + 2t - 1. \tag{3}$$

By (1), if the bipartite Ramsey number $BR(F, H)$ of two bipartite graphs F and H is r, then $R_2(F, H) = 2r$ or $R_2(F, H) = 2r - 1$. In the case of stripes, $R_2(sK_2, tK_2) = 2BR(sK_2, tK_2)$, which provides the following result [2].

Proposition 5.7 *For integers s and t with $2 \le s \le t$,*

$$R_2(sK_2, tK_2) = 2s + 2t - 2.$$

The k-Ramsey numbers of $R_k(sK_2, tK_2)$ are determined in [2] for (i) all $s = 2, 3$ and $t \ge 2$ and (ii) $k = 3, 4$ and $t \ge s \ge 2$. We state these results next.

Theorem 5.8 *For integers k and t with $2 \le k \le R(2K_2, tK_2)$ and $t \ge 2$,*

$$R_k(2K_2, tK_2) = \begin{cases} 2t+2 & \text{if } k = 2 \\ 2t+1 & \text{otherwise.} \end{cases}$$

Theorem 5.9 *For integers k and t with $2 \le k \le R(3K_2, tK_2)$ and $t \ge 3$,*

$$R_k(3K_2, tK_2) = \begin{cases} 2t+4 & \text{if } k = 2 \\ 2t+2 & \text{otherwise.} \end{cases}$$

Theorem 5.10 *For integers s, t and k with $2 \le s \le t$ and $k \in \{3, 4\}$,*

$$R_k(sK_2, tK_2) = s + 2t - 1.$$

In fact, there is a conjecture on the k-Ramsey number of stripes [2].

Conjecture 5.11 *For integers k, s and t with $2 \le s \le t$, if $5 \le k \le R(sK_2, tK_2)$, then*

$$R_k(sK_2, tK_2) = s + 2t - 1.$$

We have seen in (3) that $R_k(sK_2, tK_2) \ge s + 2t - 1$ for all integers k with $3 \le k \le R(sK_2, tK_2)$. Thus, by Proposition 5.2 and Theorem 5.10, to verify Conjecture 5.11, it suffices to establish the conjecture for primes k with $k \ge 5$.

While the k-Ramsey number $R_k(F, H)$ exists for every two bipartite graphs F and H when $2 \le k \le R(F, H)$, such is not the case when F and H are not bipartite. For graphs F and H that are not bipartite, it was observed in [36] that not only does $R_2(F, H)$ fail to exist but $R_3(F, H)$ and $R_4(F, H)$ also do not exist. To see this, let G be any balanced complete 3-partite graph with partite sets V_1, V_2 and V_3. Assigning the color red to every edge of $[V_1, V_2]$ and blue to all other edges of G results in G_R and G_B both being bipartite. Similarly, if G is a balanced complete 4-partite graph with partite sets V_1, V_2, V_3 and V_4 and the color red is assigned to every edge of $[V_1, V_2] \cup [V_2, V_3] \cup [V_3, V_4]$ and blue to all other edges of G, then G_R and G_B are both bipartite. Indeed, even if $\chi(F) = \chi(H) = 3$, $R_5(F, H)$ need not exist. For example, $R_5(K_3, K_3)$ does not exist. To see this, let G be a balanced complete 5-partite graph with partite sets V_i for $1 \le i \le 5$. If the edges in $[V_1, V_2] \cup [V_2, V_3] \cup [V_3, V_4] \cup [V_4, V_5] \cup [V_5, V_1]$ are colored red and all other edges are colored blue, then G does not contain a monochromatic K_3. Consequently, $R_k(K_3, K_3)$ exists only when $k = R(K_3, K_3) = 6$. On the other hand, $R_5(F, H)$ can exist when $\chi(F) = \chi(H) = 3$ as the following result shows (see [36]).

Theorem 5.12 *If k and ℓ are integers with $k, \ell \geq 2$, then $R_5(C_{2\ell+1}, C_{2k+1})$ exists.*

The k-Ramsey numbers of some well-known class of non-bipartite graphs has been investigated (see [36, 35]).

We have seen that Ramsey numbers are defined for three or more graphs. In particular, for three graphs F_1, F_2 and F_3, the *Ramsey number* $R(F_1, F_2, F_3)$ of F_1, F_2 and F_3 is the smallest positive integer n for which every red-blue-green coloring of the complete graph K_n of order n results in a red F_1, a blue F_2 or a green F_3. This gives rise to the concept of k-Ramsey number of three (or more) graphs. For three graphs F_1, F_2 and F_3 and an integer k with $2 \leq k \leq R(F_1, F_2, F_3)$, the k-Ramsey number $R_k(F_1, F_2, F_3)$ of F_1, F_2 and F_3, if it exists, is the smallest order of a balanced complete k-partite graph G for which every red-blue-green coloring of the edges of G results in a red F_1, a blue F_2 or a green F_3. In particular, if $k = 2$ and $F_i \cong F$ for some graph F where $i = 1, 2, 3$, then the 2-Ramsey number $R_2(F, F, F)$ is the smallest order of a balanced complete bipartite graph G for which every red-blue-green coloring of the edges of G results in a monochromatic F (all of whose edges are colored the same). For example, it was shown in [29] that $BR(C_4, C_4, C_4) = 11$. Furthermore, it was shown in [37] that $R_2(C_4, C_4, C_4) \leq 21$. Therefore, $R_2(C_4, C_4, C_4) = 21$.

6 Rainbow Ramsey Numbers

A subgraph F of an edge-colored graph G is said to be a *rainbow* F if no two edges of F are colored the same. For a graph G, Bialostocki and Voxman [6] defined the *rainbow Ramsey number* $RR(G)$ of G as the smallest positive integer n such that if each edge of the complete graph K_n is colored from any number of colors, then either a monochromatic G or a rainbow G results. The rainbow Ramsey number $RR(G)$ does not exist for all graphs G. While the Ramsey number $R(K_3, K_3) = 6$, the rainbow Ramsey number $RR(K_3)$ does not exist. To see this, let n be an arbitrary positive integer and let $V(K_n) = \{v_0, v_1, \ldots, v_{n-1}\}$. Consider the edge coloring $c : E(K_n) \to [n-1]$ defined by $c(v_iv_j) = j$ if $i < j$. Let T be any triangle of K_n with $V(T) = \{v_i, v_j, v_k\}$ and $i < j < k$. Since $c(v_iv_j) = j$ and $c(v_iv_k) = c(v_jv_k) = k$, the triangle T is neither monochromatic nor rainbow. Consequently, $RR(K_3)$ does not exist.

Bialostocki and Voxman [6] characterized those graphs G for which $RR(G)$ exists.

Theorem 6.1 *The rainbow Ramsey number $RR(G)$ of a graph G is defined if and only if G is acyclic.*

The proof of this result follows from a theorem due to Erdös and Rado. In order to state this theorem, some additional definitions are needed. Let c be an edge coloring of a graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ such that the colors are positive integers. In a *minimum coloring* of G, each edge v_iv_j of G is colored $\min\{i,j\}$; in a *maximum coloring* of G, each edge v_iv_j is colored $\max\{i,j\}$. An edge coloring of G that is either minimum, maximum, monochromatic or rainbow is called a *canonical coloring.* Erdős and Rado [21] proved the following result.

Theorem 6.2 *For every positive integer k, there exists a positive integer n such that every edge coloring of K_n contains a canonically colored complete subgraph of order k.*

Bialostocki and Voxman [6] obtained the following result.

Theorem 6.3 *For every positive integer n,*

$$RR(nK_2) = n(n-1) + 2.$$

Eroh [23, 24] extended the rainbow Ramsey number from one graph to two graphs. For graphs F and H, the *rainbow Ramsey number* $RR(F, H)$ is the smallest positive integer n such that if the edges of K_n are colored with an arbitrary number of colors, either a monochromatic F or a rainbow H results. As expected, $RR(F, H)$ exists only under certain conditions. The following theorem is a consequence of Theorem 6.2.

Theorem 6.4 *The rainbow Ramsey number $RR(F, H)$ of two graphs F and H exists if and only if F is a star or H is a forest.*

Among the exact values of $RR(F, H)$ obtained by Eroh [23, 24] are the following.

Theorem 6.5 *For positive integers s and t,*

$$RR(K_{1,s}, K_{1,t}) = (s-1)(t-1) + 2.$$

Theorem 6.6 *For integers s and t with $2 \le t < s$,*

$$RR(sK_2, tK_2) = t(s-1) + 2.$$

There is another type of rainbow Ramsey number of graphs. Let F and H be two graphs, where H has size m. For a fixed integer $k \geq m$, the *k-rainbow Ramsey number* $RR_k(F, H)$ is the smallest positive integer n such that every k-edge coloring of K_n results in either a monochromatic F or a rainbow H (see [11, pp. 319-320]). Unlike the rainbow Ramsey number $RR(F, H)$, the number $RR_k(F, H)$ always exists. For example, while $RR(K_3, K_3)$ does not exist, $RR_3(K_3, K_3) = 11$. The red-blue-green coloring of K_{10}, where the green subgraph is $K_{5,5}$ and the red and blue subgraphs are two disjoint copies of C_5 produces neither a monochromatic nor a rainbow K_3. Thus, $RR_3(K_3, K_3) \geq 11$. Showing that $RR_3(K_3, K_3) \leq 11$ is more complicated. There is a dynamic survey on this topic by Fujita, Magnant and Ozeki [27].

7 Edge-Chromatic Ramsey Numbers and Proper Ramsey Numbers

While edge colorings of a graph that result in certain monochromatic or rainbow subgraphs have been the subject of much research, the edge colorings receiving the most attention are proper edge colorings, in which every two adjacent edges are assigned different colors. The minimum number of colors required of a proper edge coloring of a graph G is the *chromatic index* of G, denoted by $\chi'(G)$. It is an immediate observation that for every nonempty graph G, the chromatic index of G is at least as large as the maximum degree $\Delta(G)$ of G. The best known and most useful result on edge colorings was obtained by Vizing [44].

Theorem 7.1 (Vizing's Theorem) *For every nonempty graph G,*

$$\chi'(G) \leq \Delta(G) + 1.$$

Thus, by Vizing's theorem, for every nonempty graph G with maximum degree Δ, either $\chi'(G) = \Delta$ or $\chi'(G) = \Delta + 1$. A graph G is said to be of *Class* 1 if $\chi'(G) = \Delta(G)$ and of *Class* 2 if $\chi'(G) = \Delta(G) + 1$. In particular, a regular graph G is of Class 1 if and only if G is 1-factorable. Determining which graphs belong to which class is a major problem of study in this area.

An edge-colored graph G is *properly colored* if every two adjacent edges of G are colored differently. The *edge-chromatic Ramsey number* $CR(F, H)$ of two graphs F and H is the minimum positive integer n such that if the edges of K_n are colored with an arbitrary number of colors, then there is either a monochromatic F or a properly colored H. The edge-chromatic Ramsey number $CR(F, H)$ exists for exactly the same pairs F, H of graphs for which rainbow Ramsey numbers exist (see Theorem 6.4). The following result is due to Eroh [23].

Theorem 7.2 *The edge-chromatic Ramsey number $CR(F, H)$ of two graphs F and H exists if and only if F is a star or H is a forest.*

As is usually the case for results for Ramey numbers and its variations, most results are bounds. Among the exact results obtained on edge-chromatic Ramsey numbers are the following.

Theorem 7.3 [23] *For integers $m \geq 2$ and $n \geq 2$,*

$$CR(C_n, P_3) = n \quad \text{and} \quad CR(C_3, P_m) = m.$$

Theorem 7.4 [23] *For every integer $n \geq 3$,*

$$CR(K_{1,n}, P_4) = n + 1 \quad \text{and} \quad CR(P_n, P_4) = n + 1.$$

We now consider a related Ramsey number where the number of colors assigned to edges is prescribed. Let F and H be two nonempty graphs such that $\chi'(H) = t$. The *proper Ramsey number* $PR(F, H)$ of F and H is the smallest positive integer n such that every t-edge coloring of K_n results in either a monochromatic F or a properly colored H. Since the Ramsey number $R(F_1, F_2, \ldots, F_t)$, where $F_t \cong F$ for all $1 \leq i \leq t$, exists and $PR(F, H) \leq R(F_1, F_2, \ldots, F_t)$, it follows that the proper Ramsey number $PR(F, H)$ exists for every two graphs F and H. Here, we investigate the proper Ramsey number $PR(F, H)$ for several pairs F, H of connected graphs of order at least 3 where $\chi'(H) = 2$. For each such pair then,

$$|V(F)| \leq PR(F, H) \leq R(F, F). \tag{4}$$

To illustrate these concepts, we show that $PR(P_5, P_6) = 6$. First, the red-blue coloring of K_5 in which the red subgraph is $K_{1,4}$ and the blue subgraph is K_4 avoids both a monochromatic P_5 and a properly colored P_6. Hence, $PR(P_5, P_6) \geq 6$. Next, we show that $PR(P_5, P_6) \leq 6$. Assume, to the contrary, that there is a red-blue coloring of $G = K_6$ that avoids both a monochromatic P_5 and a properly colored P_6. Let $V(K_6) = \{v_1, v_2, \ldots, v_6\}$. First, show that G contains a properly colored P_4. It is immediate that there is a properly colored P_3, say (v_1, v_2, v_3) where v_1v_2 red and v_2v_3 blue. If v_1 is joined to a vertex in $\{v_4, v_5, v_6\}$ by a blue edge or v_3 is joined to a vertex in $\{v_4, v_5, v_6\}$ by a red edge, then there is a properly colored P_4. Thus, we may assume that v_1v is red and v_3v is blue for each $v \in \{v_4, v_5, v_6\}$. Since there are at least two edges in $G[\{v_4, v_5, v_6\}]$ of the same color, we may assume that v_4v_5 and v_5v_6 are red. However then, $(v_2, v_1, v_4, v_5, v_6)$ is a red P_5, which contradicts our assumption. Thus, G contains a properly colored P_4, say (v_1, v_2, v_3, v_4) is a properly colored P_4, where v_1v_2 and v_3v_4 red and v_2v_3 blue. Let x and y be the remaining two vertices of G. If xv_1 and xv_4 are both red, then (v_2, v_1, x, v_4, v_3) is a red P_5, which is impossible. Thus, at least one of xv_1 and xv_4 is blue, say xv_1 is blue. Hence, (x, v_1, v_2, v_3, v_4) is a properly colored P_5. We may assume, without loss of generality, that xv_4 is red. If xy is red or v_4y is blue, then G contains a properly colored P_6. Thus, xy is blue and v_4y is red.

- ⋆ If v_1y is red, then (v_2, v_1, y, v_4, v_3) is a red P_5; so v_1y is blue.
- ⋆ If v_2y is blue, then (v_1, x, y, v_2, v_3) is a blue P_5; so v_2y is red.

However then, (v_1, v_2, y, v_4, v_3) is a red P_5, a contradiction. Therefore, $PR(P_5, P_6) \leq 6$ and $PR(P_5, P_6) = 6$.

In general, for integers n and m with $n \geq m \geq 2$, the proper Ramsey number of $PR(P_n, P_m)$ can be determined with the aid of (4) and the Ramsey number $R(P_n, P_m)$ for $2 \leq m \leq n$. In fact, more can be said. By Theorem 2.2, if n and m are integers with $2 \leq m \leq n$, then $R(P_n, P_m) = n - 1 + \lfloor \frac{m}{2} \rfloor$. In particular, if $n = m \geq 2$, then

$$R(P_n, P_n) = n - 1 + \left\lfloor \frac{n}{2} \right\rfloor . \tag{5}$$

The following result [18] is a consequence of (4) and (5).

Theorem 7.5 *If P is a path of order 5 or more and C is an even cycle, then*

$$PR(P_n, P) = PR(P_n, C) = n - 1 + \left\lfloor \frac{n}{2} \right\rfloor .$$

Proof. Let $N = n - 1 + \lfloor \frac{n}{2} \rfloor$. It follows by (4) that $PR(P_n, P) \leq N$ and $PR(P_n, C) \leq N$. On the other hand, consider the red-blue coloring of K_{N-1} that assigns the color red to each edge of a subgraph K_{n-1} and the color blue to the remaining edges of K_{N-1}. Since there is no monochromatic P_n, no properly colored P and no properly colored C, it follows that $PR(P_n, P) \geq N$ and $PR(P_n, C) \geq N$, producing the desired results. ∎

In [19] the proper Ramsey number $PR(F, H)$ was investigated for certain pairs F, H of connected graphs when $t = 2$, namely when F is a complete graph, star or path and when H is a path or even cycle of small order. In particular, $PR(F, H)$ is determined when (1) F is a complete graph and H is a path of order 6 or less, (2) F is a complete graph and H is a 4-cycle, (3) F is a star and H is a 4-cycle or a 6-cycle and (4) F is a star and H is a path of order 8 or less. We state these results as follows (see [19]).

Theorem 7.6 *For each integer $n \geq 3$,*

$$PR(K_n, P_k) = \begin{cases} n & \text{if } k = 3 \\ n+1 & \text{if } k = 4 \\ k & \text{if } n = 3 \text{ and } k \in \{5,6\} \\ 2n-2 & \text{if } n \geq 4 \text{ and } k \in \{5,6\}. \end{cases}$$

Theorem 7.7 *For each integer $n \geq 3$, $PR(K_n, C_4) = 2n - 2$.*

Theorem 7.8 *For every integer $n \geq 3$,*

(1) $PR(K_{1,n}, C_4) = n + 1$,

(2) $PR(K_{1,3}, C_6) = 6$ *and* $PR(K_{1,n}, C_6) = 2n - 1$ *if* $n \geq 4$.

Theorem 7.9 *For each integer $n \geq 3$,*

(1) *if* $k \in \{3, 4\}$, *then* $PR(K_{1,n}, P_k) = n + 1$;

(2) *if* $n \geq 4$, *then* $PR(K_{1,n}, P_5) = n + 1$;

(3) *if* $k \in \{6, 7, 8\}$ *and* $n \geq k - 1$, *then* $PR(K_{1,n}, P_k) = n + k - 5$.

It can be shown for integers m and n with $m \geq 4$ and $n \geq \lceil \frac{m}{2} \rceil + 1$ that

$$PR(K_{1,n}, P_m) \geq n + \left\lfloor \frac{m-3}{4} \right\rfloor + \left\lceil \frac{m-3}{4} \right\rceil.$$

In fact, the results obtained in [19] suggest the following conjecture.

Conjecture 7.10 *For integers m and n with $m \geq 4$ and $n \geq \lceil \frac{m}{2} \rceil + 1$,*

$$PR(K_{1,n}, P_m) = n + \left\lfloor \frac{m-3}{4} \right\rfloor + \left\lceil \frac{m-3}{4} \right\rceil.$$

8 Closing Comments

There is a general setting for Ramsey numbers. Let $S=\{G_1, G_2, G_3, \ldots\}$ be an infinite set of graphs with the property that G_i is a proper induced subgraph of G_{i+1} for $i = 1, 2, 3, \ldots$. Let F and H be two graphs with the property that $F \subseteq G_k$ and $H \subseteq G_k$ for some $k \in \mathbb{N}$. Therefore, $F \subseteq G_n$ and $H \subseteq G_n$ for every $n \geq k$.

- ⋆ If $G_i = K_i$ for each $i \in \mathbb{N}$, then for every two graphs F and H, there exist positive integers n such that for every red-blue coloring of G_n, there is either a red F in G_n or a blue H in G_n. Of course, the smallest such positive integer n with this property is the Ramsey number $R(F, H)$.

- ⋆ If $G_i = K_{i,i}$ for each $i \in \mathbb{N}$, then for every two *bipartite* graphs F and H, there exist positive integers r such that for every red-blue coloring of G_r, there is either a red F in G_r or a blue H in G_r. The smallest such positive integer r with this property is the bipartite Ramsey number $BR(F, H)$.

⋆ If $G_2 = K_{1,1}, G_3 = K_{1,2}, G_4 = K_{2,2}, G_5 = K_{2,3}, G_6 = K_{3,3}, \ldots$, that is, if $G_i = K_{\lfloor \frac{i}{2} \rfloor, \lceil \frac{i}{2} \rceil}$ for each integer $i \geq 2$, then for every two *bipartite* graphs F and H, there exist positive integers n such that for every red-blue coloring of G_n, there is either a red F in G_n or a blue H in G_n. The smallest such positive integer n with this property is the 2-Ramsey number $R_2(F, H)$. In a similar way, the k-Ramsey number $R_k(F, H)$ of two *bipartite* graphs F and H can be defined for every integer $k \geq 2$. For example, if $k = 3$, then let $G_3 = K_{1,1,1}, G_4 = K_{1,1,2}, G_5 = K_{1,2,2}, G_6 = K_{2,2,2}, G_7 = K_{2,2,3}, \ldots$ and so on.

This suggests looking at other collections S of graphs G_i and pairs F, H of graphs that are subgraphs of $G_i \in S$ for some $i \in \mathbb{N}$ and study the *S-Ramsey number $R_S(F, H)$ of F and H* defined as the smallest positive integer n such that for every red-blue coloring of G_n, there is either a red F in G_n or a blue H in G_n. Furthermore, there are also corresponding concepts of monochromatic S-Ramsey number, rainbow S-Ramsey number and proper S-Ramsey number of graphs.

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