Oriented graphs with minimal skew energy *

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Abstract

The concept of the skew energy of a digraph was introduced by Adiga, Balakrishnan and So in 2010. An *oriented graph* G^σ is a simple undirected graph G with an orientation, which assigns to each edge a direction so that G^σ becomes a directed graph. Then G is called the underlying graph of G^σ. Let $S(G^\sigma)$ be the skew-adjacency matrix of G^σ and $\lambda_1, \lambda_2 \cdots \lambda_n$ denote all the eigenvalues of the $S(G^\sigma)$. The skew energy of G^σ is defined as the sum of the absolute values of all eigenvalues of $S(G^\sigma)$. Recently, Gong, Li and Xu determined all oriented graphs with minimal skew energy among all connected oriented graphs on n vertices with m $(n \leq m \leq 2(n-2))$ arcs. In this paper, we determine all oriented graphs with the second and the third minimal skew energy among all connected oriented graphs with n vertices and m $(n \leq m < 2(n-2))$ arcs. In particularly, when the oriented graphs is unicyclic digraphs or bicyclic digraphs, the second and the third minimal skew energy is determined.

Keywords: Oriented graph; skew energy; skew-adjacency matrix.

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1 Introduction

An important quantum-chemical characteristic of a conjugated molecule is its total π-electron energy. The energy of a graph has closed links to chemistry. Since the concept of the energy of simple undirected graphs was introduced by Gutman in [4], there have been lots of research papers on this topic. For the energy of graphs, Li, Shi and Gutman published a paper on this subject; see [10].

Let G^σ be a digraph of order n with vertex set $V(G^\sigma) = \{v_1, v_2, \ldots, v_n\}$, and arc set $\Gamma(G^\sigma) \subset V(G^\sigma) \times V(G^\sigma)$. The *skew-adjacency matrix* of G^σ is the $n \times n$ matrix $S(G^\sigma) = [s_{ij}]$, where the (i, j) entry satisfies:

$$s_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in G^\sigma \\ -1, & \text{if } (v_j, v_i) \in G^\sigma \\ 0, & \text{otherwise} \end{cases}$$

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The skew energy of an oriented graph G^σ, introduced by Adiga, Balakrishnan and So in [1]. If we denote the skew energy of G^σ by $\varepsilon_s(G^\sigma)$, then $\varepsilon_s(G^\sigma) = \sum_{i=1}^{n} |\lambda_i|$. For more detail on the skew energy of oriented graphs, we refer to the survey paper by Li and Lian [11].

There have been lots of research papers on this topic of skew energy. Shen and Hou [13] showed that bicyclic digraphs with extremal skew energy. More results on the energy of the adjacency matrix of a graph, such as skew energy, Laplacian energy, Distance energy see e.g. [2, 4, 8, 12, 14, 15, 16, 17, 18].

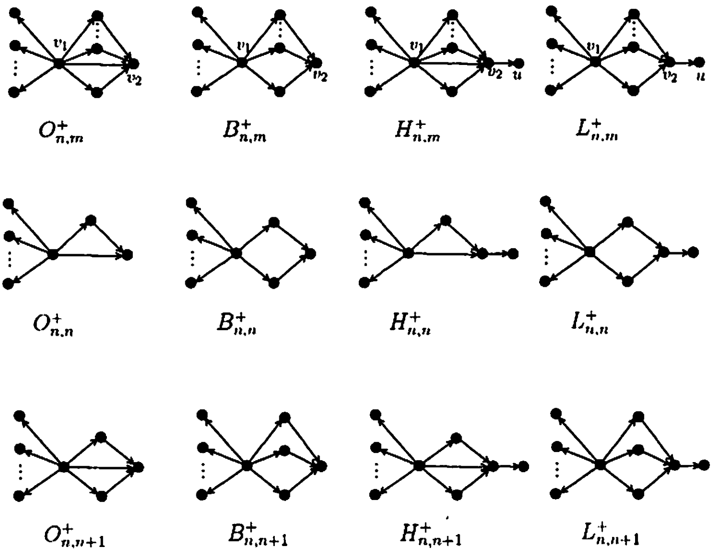

Figure 1: Graphs $O^+_{n,n}, B^+_{n,n}, H^+_{n,n}, L^+_{n,n}$.

Let $O^+_{n,m}$ be the oriented graph on n vertices which is obtained from the oriented star S^σ_n by adding $m - n + 1$ arcs such that all those arcs have a common vertex, where v_1 is the tail of each arc incident to it and v_2 is the head of each arc incident to it, and $B^+_{n,m}$ be the oriented graph obtained from $O^+_{n,m+1}$ by deleting the arc (v_1, v_2). Let $H^+_{n,m}$ be the graph obtained from $O^+_{n-1,m-1}$ by adding a new vertex u and a new arc (v_2, u), and $L^+_{n,m}$ be the graph obtained from $B^+_{n-1,m-1}$ by adding a new vertex u and a new arc (v_2, u) (see Figure 1). A connected graph with n vertices and n edges is called a *unicyclic graph*; a connected graph with n vertices and $n + 1$ edges is called a *bicyclic graph*. Clearly, $O^+_{n,n}, B^+_{n,n}, H^+_{n,n}, L^+_{n,n}$ are all unicyclic graphs, and $O^+_{n,n+1}, B^+_{n,n+1}, H^+_{n,n+1}, L^+_{n,n+1}$ are all bicyclic graphs, see Figure 1.

In [6], Gong, Li and Xu determined all oriented graphs with minimal skew energy among all connected oriented graphs on n vertices with m ($n \leq m \leq 2(n-2)$) arcs.

Theorem 1.1 [6] *Let $n \geq 5$ and $G^\sigma \in G^\sigma(n,m)$ be an oriented graph with maximum degree $n-1$.Suppose that $n \leq m < 2(n-2)$ and $G^\sigma \neq O^+_{n,m}$.Then $G^\sigma \succ O^+_{n,m}$.*

Theorem 1.2 [6] *Let $n \geq 5$ and $G^\sigma \in G^\sigma(n,m)$ be an oriented graph with $\Delta(G^\sigma) \leq (n-2)$.Suppose that $n \leq m < 2(n-2)$ and $G^\sigma \neq B^+_{n,m}$.Then $G^\sigma \succ B^+_{n,m}$.*

Theorem 1.3 [6] *Let G^σ be an oriented graph with minimal skew energy among all oriented graphs with n vertices and $m(n \leq m < 2(n-2))$ arcs. Then, up to isomorphism, G^σ is*

(1) $O^+_{n,m}$ if $m < \frac{3n-5}{2}$

(2) either $B^+_{n,m}$ or $O^+_{n,m}$ if $m = \frac{3n-5}{2}$ and

(3) $B^+_{n,m}$ otherwise.

Energy sequencing problem is important. For example, the authors [3] investigated the unicyclic graphs with maximal energy. Later, Gutman et al. [5] studied the unicyclic graphs with studied the unicyclic graphs with the second-maximal and third-maximal energy.

In this paper, we are interested in studying the orientations of oriented graphs with the second and third minimal skew energy, and obtain the following result.

Theorem 1.4 *Among all oriented graphs with n vertices and m $(n \leq m < 2(n-2))$ arcs, we have the following results.*

- *$O^+_{n,m}$ has the minimal skew energy;*
- *$B^+_{n,m}$ has the second minimal skew energy;*
- *$H^+_{n,m}$ has the third skew energy for $m < \frac{3n-6}{2}$;*
- *$B^+_{n,m}$ has the second minimal skew energy and $H^+_{n,m}$ or $L^+_{n,m}$ has the third skew energy for $m = \frac{3n-6}{2}$;*
- *$O^+_{n,m}$ has the minimal skew energy, $B^+_{n,m}$ has the second minimal skew energy and $L^+_{n,m}$ has the third skew energy for $\frac{3n-6}{2} < m < \frac{3n-5}{2}$;*
- *$O^+_{n,m}$ or $B^+_{n,m}$ has the minimal skew energy, $L^+_{n,m}$ has the second minimal skew energy and $H^+_{n,m}$ has the third skew energy for $m = \frac{3n-5}{2}$;*
- *$B^+_{n,m}$ has the minimal skew energy, $O^+_{n,m}$ has the second minimal skew energy and $L^+_{n,m}$ has the third skew energy for $\frac{3n-5}{2} < m < \frac{5n-10}{3}$;*
- *$B^+_{n,m}$ has the minimal skew energy, $O^+_{n,m}$ or $L^+_{n,m}$ has the second minimal skew energy and $H^+_{n,m}$ has the third skew energy for $m = \frac{5n-10}{3}$;*

- $B_{n,m}^+$ *has the minimal skew energy,* $L_{n,m}^+$ *has the second minimal skew energy and* $O_{n,m}^+$ *has the third skew energy for* $\frac{5n-10}{3} < m < 2(n-2)$.

From the above theorem, we derive the following results.

Corollary 1.5 *Let* G^σ *be unicyclic digraphs. Then* $O_{n,n}^+ \prec B_{n,n}^+ \prec H_{n,n}^+ \prec L_{n,n}^+$ *for* $n \geq 6$, *and* $O_{n,n}^+ = B_{n,n}^+ = L_{n,n}^+ \prec H_{n,n}^+$ *for* $n = 5$. *Namely, among all unicyclic digraphs with* n *vertices and* n *arcs, we have the following results.*

- $O_{n,n}^+$ *has the minimal skew energy;*
- $B_{n,n}^+$ *has the second minimal skew energy;*
- $H_{n,n}^+$ *has the third skew energy for* $n \geq 6$;
- $O_{n,n}^+$, $B_{n,n}^+$ *or* $L_{n,n}^+$ *has the minimal skew energy;*
- $H_{n,n}^+$ *has the second minimal skew energy for* $n = 5$.

Corollary 1.6 *Let* G^σ *be bicyclic digraphs. Then* $O_{n,n+1}^+ \prec B_{n,n+1}^+ \prec H_{n,n+1}^+ \prec L_{n,n+1}^+$ *for* $n \geq 9$; $O_{n,n+1}^+ \prec B_{n,n+1}^+ \prec H_{n,n+1}^+ = L_{n,n+1}^+$ *for* $n = 8$; $O_{n,n+1}^+ = B_{n,n+1}^+ \prec L_{n,n+1}^+ \prec H_{n,n+1}^+$ *for* $n = 7$, *and* $B_{n,n+1}^+ \prec L_{n,n+1}^+ \prec O_{n,n+1}^+ \prec H_{n,n+1}^+$ *for* $n = 6$. *Namely, among all bicyclic digraphs with* n *vertices and* $n+1$ *arcs, we have the following results.*

- $O_{n,n+1}^+$ *has the minimal skew energy;*
- $B_{n,n+1}^+$ *has the second minimal skew energy and* $H_{n,n+1}^+$ *has the third skew energy for* $n > 8$;
- $O_{n,n+1}^+$ *has the minimal skew energy,* $B_{n,n+1}^+$ *has the second minimal skew energy, both* $H_{n,n+1}^+$ *and* $L_{n,n+1}^+$ *have the third skew energy for* $n = 8$;
- *Both* $O_{n,n+1}^+$ *and* $B_{n,n+1}^+$ *has the minimal skew energy, and* $L_{n,n+1}^+$ *has the second minimal skew energy for* $n = 7$;
- $B_{n,n+1}^+$ *has the minimal skew energy,* $L_{n,n+1}^+$ *has the second minimal skew energy, and* $O_{n,n+1}^+$ *has the third skew energy for* $n = 6$.

2 Preliminary

A *basic oriented graph* is an oriented graph whose components are even cycles and/or complete oriented graphs with exactly two vertices. If C be any undirected even cycle of G^σ, we say C is *evenly oriented* relative to G^σ if it has an even number of edges oriented in the direction of the routing. Otherwise C is oddly oriented.

Lemma 2.1 [6] *Let* G^σ *be an oriented graph on* n *vertices, and let the skew characteristic polynomial of* G^σ *be*

$$\begin{aligned} \phi(G^\sigma, \lambda) &= \sum_{i=0}^{n}(-1)^i a_i \lambda^{n-i} \\ &= \lambda^n - a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + (-1)^{n-1}a_{n-1}\lambda + (-1)^n a_n \end{aligned}$$

Then $a_i = 0$ *if* i *is odd; and*

$$a_i = \sum_{\mathscr{H}} (-1)^{c^+} 2^c \text{ if } i \text{ is even,}$$

where the summation is over all basic oriented subgraphs $\mathscr{H}$ *of* G^σ *having* i *vertices and* c^+ *and* c *are respectively the number of evenly oriented even cycles and even cycles contained in* $\mathscr{H}$.

Let $G = (V, E)$ be a graph, directed or not, on n vertices. Then we denote by $\Delta(G)$ be the maximum degree of G and set $\Delta(G) = \Delta(G^\sigma)$. An *$r$-matching* M in G is a subset with r edges such that every vertex of $V(G)$ is incident with exactly one edge in M. Denote by $M(G, r)$ be the number of all r-matchings in G and set $M(G, 0) = 1$.

Lemma 2.2 [6] *Let* G^σ *be an oriented graph containing* n *vertices and* m *arcs. Suppose*

$$\phi(G^\sigma, \lambda) = \sum_{i=0}^{n} (-1)^i a_i(G^\sigma) \lambda^{n-i}$$

Then $a_0(G^\sigma) = 1, a_2(G^\sigma) = m$ *and* $a_4(G^\sigma) \geq M(G^\sigma, 2) - 2q(G^\sigma)$ *with equality if and only if all oriented quadrangles of* G^σ *are evenly oriented.*

Let G^{σ_1} and G^{σ_2} be two oriented graphs of order n. If $a_{2i}(G^{\sigma_1}) \leq a_{2i}(G^{\sigma_2})$ for all i with $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$, then we write that $G^{\sigma_1} \preceq G^{\sigma_2}$. Furthermore, if $G^{\sigma_1} \preceq G^{\sigma_2}$ and there exists at least one index i such that $a_{2i}(G^{\sigma_1}) < a_{2i}(G^{\sigma_2})$, then we write that $G^{\sigma_1} \prec G^{\sigma_2}$. If $a_{2i}(G^{\sigma_1}) = a_{2i}(G^{\sigma_2})$ for all i, we write $G^{\sigma_1} \sim G^{\sigma_2}$. According to the integral formula, we have, for two oriented graphs G^{σ_1} and G^{σ_2} of order n, that

$$G^{\sigma_1} \preceq G^{\sigma_2} \Rightarrow \varepsilon_s(G^{\sigma_1}) \leq \varepsilon_s(G^{\sigma_2}) \text{ and } G^{\sigma_1} \prec G^{\sigma_2} \Rightarrow \varepsilon_s(G^{\sigma_1}) < \varepsilon_s(G^{\sigma_2})$$

By a directly calculation, we have

$$\begin{aligned}
a_4(O^+_{n,m}) &= M(G,2) - 2q(G) = (m-n+1)(n-3) - 2\binom{m-n+1}{2} \\
&= (m-n+1)(2n-m-3); \\
a_4(B^+_{n,m}) &= M(G,2) - 2q(G) = (m-n+2)(n-3) - 2\binom{m-n+2}{2} \\
&= (m-n+2)(2n-m-4); \\
a_4(H^+_{n,m}) &= M(G^\sigma,2) - 2q(G^\sigma) \\
&= (m-n+1)(n-4) + (n-3) - 2\binom{m-n+1}{2} \\
&= (m-n+2)(2n-m-3) - 1; \\
a_4(L^+_{n,m}) &= M(G^\sigma,2) - 2q(G^\sigma) \\
&= (m-n+2)(n-4) + (n-3) - 2\binom{m-n+2}{2} \\
&= (m-n+3)(2n-m-4) - 1.
\end{aligned}$$

Therefore, we have

$$\phi(O_{n,m}^+) = \lambda^n + m\lambda^{n-2} + (m-n+1)(2n-m-3)\lambda^{n-4} \quad (2.1)$$
$$\phi(B_{n,m}^+) = \lambda^n + m\lambda^{n-2} + (m-n+2)(2n-m-4)\lambda^{n-4} \quad (2.2)$$
$$\phi(H_{n,m}^+) = \lambda^n + m\lambda^{n-2} + [(m-n+2)(2n-m-3)-1]\lambda^{n-4} \quad (2.3)$$
$$\phi(L_{n,m}^+) = \lambda^n + m\lambda^{n-2} + [(m-n+3)(2n-m-4)-1]\lambda^{n-4} \quad (2.4)$$

Lemma 2.3 [6] *Let $n \geq 5$ and $G \in G(n,m)$ be an arbitrary connected undirected graph containing n vertices and $m(n \leq m \leq 2(n-2))$ edges. Then $q(G) \leq \binom{m-n+2}{2}$, where $q(G)$ denotes the number of quadrangles contained in G.*

Lemma 2.4 [7] *Let G^σ be an oriented graph with an arc $e = (u, v)$, suppose that e is not contained in any even cycle. Then*

$$\phi(G^\sigma, \lambda) = \phi(G^\sigma \backslash e, \lambda) + s_{uv}^2 \phi(G^\sigma \backslash uv, \lambda). \quad (2.5)$$

By equating the coefficients of polynomials in Eq (2.5), we have

Lemma 2.5 [6] *Let G^σ be an oriented graph on n vertices and $e = (u, v)$ a pendant arc of G^σ with pendant vertex v. Suppose $\phi(G^\sigma, \lambda) = \sum_{i=0}^{n}(-1)^i a_i(G^\sigma)\lambda^{n-i}$. Then*

$$a_i(G^\sigma, \lambda) = a_i(G^\sigma - v, \lambda) + a_{i-2}(G^\sigma - u - v, \lambda).$$

3 Proof of Theorem 1.4

We now in a position to give our main result.

Lemma 3.1 *Let $n \geq 5$ and let $G^\sigma \in G^\sigma(n,m)$ be an oriented graph with maximal degree $n-2$. If $n \leq m < 2(n-2)$ and $G^\sigma \notin \{H_{n,m}^+, B_{n,m}^+\}$, then $G^\sigma \succ H_{n,m}^+$.*

Proof. To prove this theorem, it would be sufficient to prove that $a_i(G^\sigma) \geq a_i(H_{n,m}^+)$ for i $(0 \leq i \leq n)$. From Lemma 2.1, $a_i(G^\sigma) = a_i(H_{n,m}^+) = 0$ for i is odd. Observe that $a_i(G^\sigma) \geq a_i(H_{n,m}^+) = 0$ for $i \geq 6$. By Lemma 2.1, we have $a_0(G^\sigma) = a_0(H_{n,m}^+) = 1$ and $a_2(G^\sigma) = a_2(H_{n,m}^+) = m$. Thus, it suffices to prove that $a_4(G^\sigma) > a_4(H_{n,m}^+)$.

First, we show that $M(G^\sigma, 2) \geq M(H_{n,m}^+, 2)$. Suppose that v_1 is the vertex with degree $n-2$ in $H_{n,m}^+$. For convenience, all arcs incident to v_1 are colored as white, the pendant arc (u, v_2) in $H_{n,m}^+$ with pendant vertex u is colored as red and all other arcs are colored as black. Then there are $n-2$ white arcs and $m-n+1$ black arcs. We estimate the cardinality of 2-matchings in $H_{n,m}^+$ as follows. Noticing that all white arcs are incident to v_1, each pair of white arc can not form a 2-matching. Since $d(v_1) = n - 2$ and each black arc incident to exactly two white arcs, each black arc together with a white arcs except its neighbors forms a 2-matching, the red arc with a white arcs except arc (v_1, v_2) forms a 2-matching, that is, there are $(m-n+1)(n-4)+(n-3)$ 2-matchings in $H^+_{n,m}$.

Note that there exists a vertex, say v_1, such that $d_{G^\sigma}(v) = n - 2$. Then there exists a vertex u such that $(u_1, u) \notin E(G)$. For convenience, all arcs in G^σ incident to v_1 are colored as white, a arc incident to the vertex u in G^σ is colored as red and all other arcs are colored as black. Observe that each pair of white arc can not form a 2-matching. Since $d(v_1) = n - 2$ and each black arc incident to exactly two white arcs, each black arc together with a white arcs except its neighbors forms a 2-matching, the red arc with a white arcs except arc (v_1, v_2) forms a 2-matching, that is, there are $(m-n+1)(n-4)+(n-3)$ 2-matchings in G^σ.

Moreover, noticing that $G^\sigma \neq H^+_{n,m}$, $G^\sigma - v_1$ does not contain the directed star S_{m-n+3} as its subgraph, and thus there is at least one 2-matching formed by a pair of disjoint black arcs and the red arc, or G^σ is an oriented graph of the following graph F.

If it is the first case, then the number of 2-matchings in G^σ satisfies

$$M(G^\sigma, 2) \geq (m-n+1)(n-4)+(n-3).$$

When $G^\sigma \neq B^+_{n,m}$ and $G^\sigma \in G^\sigma(n,m)$ be an oriented graph with maximal degree $n-2$, we have $q(G^\sigma) \leq \binom{m-n+1}{2}$, and then by applying Lemma 2.2 again, we have

$$\begin{aligned} a_4(G^\sigma) &\geq M(G^\sigma, 2) - 2q(G^\sigma) \\ &\geq (m-n+1)(n-4)+(n-3)-(m-n+1)(m-n) \\ &= a_4(H^+_{n,m}). \end{aligned}$$

Therefore, $a_i(G^\sigma) \geq a_i(H^+_{n,m})$ for i $(0 \leq i \leq n)$. The proof is now complete.

Lemma 3.2 *Let n be an integer with $n \geq 5$ and let $G^\sigma \in G^\sigma(n,m)$ be an oriented graph with $\Delta(G^\sigma) \leq n-3$. If $n \leq m < 2(n-2)$ and $G^\sigma \neq L^+_{n,m}$, then $G^\sigma \succ L^+_{n,m}$.*

Proof. To prove this lemma, it would be sufficient to prove that $a_i(G^\sigma) > a_i(L^+_{n,m})$. We apply induction on n to prove it. By Lemma 2.2, we have $a_0(G^\sigma) = a_0(L^+_{n,m}) = 1$ and $a_2(G^\sigma) = a_2(L^+_{n,m}) = m$. It suffices to prove that $a_4(G^\sigma) > a_4(L^+_{n,m})$.

By a direct calculation, the result is true for $n = 5$. Since $n = 5$, it follows that $5 \leq m < 2(5-2) = 6$, and hence there exists exactly three graphs in $G^\sigma(5,5)$, that is, the oriented cycle C_3 together with two pendant arcs attached to two different vertices of the C_3, the oddly oriented cycle C_4 together with a pendant arc, and the oriented cycle C_5. We now assume $n \geq 6$ and suppose the result is true for smaller n.

Case 1. There is a pendant arc (u, v) in G^σ with pendant vertex v.

By Lemma 2.5, we have

$$a_4(G^\sigma,\lambda)=a_4(G^\sigma-v,\lambda)+a_2(G^\sigma-u-v,\lambda)=a_4(G^\sigma-v,\lambda)+e(G^\sigma-u-v,\lambda)$$

Since $\Delta(G^\sigma)\le n-3$, it follows that $e(G^\sigma-u-v,\lambda)\ge m-\Delta(G^\sigma)\ge m-n+3$.

By induction hypothesis, $a_4(G^\sigma-v)>a_4(L^+_{n-1,m-1})$ with equality if and only if $G^\sigma-v=L^+_{n-1,m-1}$. Then

$$a_4(G^\sigma,\lambda)\ge a_4(L^+_{n-1,m-1})+(m-n+3)=a_4(L^+_{n-1,m-1})+e(S_{m-n+4}).$$

Since $a_4(L^+_{n,m})=a_4(L^+_{n-1,m-1})+e(S_{m-n+4})$, it follows that $a_4(G^\sigma)\ge a_4(L^+_{n,m})$ with equality if and only if $G^\sigma\cong L^+_{n,m}$.

Case 2. There is no pendant vertex in G^σ.

We first claim that there exists an oriented graph $L^+_{n,m}$ containing pedant vertices such that

$$\sum_{v\in V(L^+_{n,m})}\binom{d(v)}{2}>\sum_{v\in V(G^\sigma)}\binom{d(v)}{2}.$$

Let $(d)_{G^\sigma}=(d_1,d_2,\cdots,d_i,d_{i+1},\cdots,d_n)$ be the non-increasing degree sequence of G^σ. We label the vertices of G^σ corresponding to the degree sequence $(d)_{G^\sigma}$ as $v_1,v_2,\cdots,v_i$,
$\cdots,v_n$ such that $d_{G^\sigma}(v_i)=d_i$ for each i. Assume $d_1<n-3$. Then there exists a vertex v_k that is not adjacent to v_1, but is adjacent to one neighbor, say v_i of v_1. Thus $(d_1+1,d_2,\cdots,d_i-1,d_{i+1},\cdots,d_n)$ is the degree sequence of the oriented graph G^{σ_1} obtained from G^σ by deleting the arc (v_k,v_1), regardless the orientation of the arc (v_k,v_1). Rewriting the sequence above such that

$$(d)_{G^{\sigma_1}}=(d'_1,d'_2,\cdots,d'_i,d'_{i+1},\cdots,d'_n)$$

is also a non-increasing sequence. Thus we have

$$\sum_{i=1}^{n}\binom{d'(v)}{2}>\sum_{i=1}^{n}\binom{d(v)}{2}$$

since $\sum_{i=1}^{n}\binom{d'(v)}{2}-\sum_{i=1}^{n}\binom{d(v)}{2}=\binom{d_1+1}{2}+\binom{d_i-1}{2}-\binom{d_1}{2}-\binom{d_i}{2}=d_1-d_i+1>0$

Notice that $d_1\ge d_2\ge\cdots\ge d_i\ge\cdots\ge d_n\ge 2$ and $\sum_{i=1}^{n}d_i=2m$.

Repeating this procession, we can obtain the sequence

$$(d)_{G^{\sigma_2}}=(d''_1,d''_2,\cdots,d''_i,d''_{i+1},\cdots,d''_n)$$

such that $\Delta(G^{\sigma_2})=d''_1=n-3$ and

$$\sum_{v\in V(G^{\sigma_2})}\binom{d''(v)}{2}>\sum_{v\in V(G^{\sigma_1})}\binom{d'(v)}{2}>\cdots>\sum_{v\in V(G^\sigma)}\binom{d(v)}{2}.$$

Similarly, we can assume that there exists a vertex v_k that is not adjacent to v_i, but is adjacent to one neighbor, say v_j of v_i. Thus $(d_1, d_2, \cdots, d_i + 1, d_{i+1}, \cdots, d_j - 1, \cdots, d_n)$ is the degree sequence of the oriented graph G^{σ_3} obtained from G^{σ_2} by deleting the arc (v_k, v_j) and adding the arc (v_k, v_i), regardless the orientation of the arc (v_k, v_j). By a similar proof, we can get

$$\sum_{v\in V(G^{\sigma_3})} \binom{d'''(v)}{2} > \sum_{v\in V(G^{\sigma_2})} \binom{d''(v)}{2}$$

Then by applying the above procedure repeatedly, we eventually obtain the degree sequence $(d)_{L^+_{n,m}}$,

$$(d)_{L^+_{n,m}} = (n-3, m-n+3, 2, 2, \cdots, 2, 1, 1, \cdots, 1)$$

where the number of vertices of degree 2 is $m-n-2$, and the number of vertices of degree 1 is $2n - m + 4$. Finally, we get

$$\begin{aligned}\sum_{v\in V(G^{\sigma_3})} \binom{d'''(v)}{2} &> \sum_{v\in V(G^{\sigma_2})} \binom{d''(v)}{2} > \sum_{v\in V(G^{\sigma_1})} \binom{d'(v)}{2} \\ &> \cdots > \sum_{v\in V(G^{\sigma})} \binom{d(v)}{2}\end{aligned}$$

For a simple graph G, we have $M(G,2) = \binom{m}{2} - \sum_{v\in V(G)} \binom{d(v)}{2}$.
By Lemma 2.2 we know that

$$\begin{aligned}a_4(G^\sigma) &\geq M(G^\sigma, 2) - 2q(G^\sigma) = \binom{m}{2} - \sum_{v\in V(G^\sigma)} \binom{d(v)}{2} - 2q(G^\sigma) \\ &> \binom{m}{2} - \sum_{v\in V(G^{\sigma_3})} \binom{d'''(v)}{2} - 2q(G^\sigma) = a_4(L^+_{n,m})\end{aligned}$$

The result thus follows. ∎

Proof of Theorem 1.4: Combining with Theorem 1.3, Lemma 3.1 and Lemma 3.2, the oriented graph with minimal skew energy among all oriented graphs of $G^\sigma \in G^\sigma(n,m)$ is $O^+_{n,m}$ or $B^+_{n,m}$. Furthermore, from Eq (2.2), (2.3), (2.4) and Eq (2.5), we have

$$a_4(O^+_{n,m}) = (m-n+1)(2n-m-3); a_4(B^+_{n,m}) = (m-n+2)(2n-m-4)$$

$$a_4(H^+_{n,m}) = (m-n+2)(2n-m-3)-1; a_4(L^+_{n,m}) = (m-n+3)(2n-m-4)-1$$

Then, by a direct calculation, we have

1. $O^+_{n,m} \prec B^+_{n,m} \prec H^+_{n,m} \prec L^+_{n,m}$ if $m < \frac{3n-6}{2}$;

2. $O^+_{n,m} \prec B^+_{n,m} \prec H^+_{n,m} = L^+_{n,m}$ if $m = \frac{3n-6}{2}$;

3. $O^+_{n,m} \prec B^+_{n,m} \prec L^+_{n,m} \prec H^+_{n,m}$ if $\frac{3n-6}{2} < m < \frac{3n-5}{2}$;

4. $O^+_{n,m} = B^+_{n,m} \prec L^+_{n,m} \prec H^+_{n,m}$ if $m = \frac{3n-5}{2}$;

5. $B^+_{n,m} \prec O^+_{n,m} \prec L^+_{n,m} \prec H^+_{n,m}$ if $\frac{3n-5}{2} < m < \frac{5n-10}{3}$;

6. $B^+_{n,m} \prec O^+_{n,m} = L^+_{n,m} \prec H^+_{n,m}$ if $m = \frac{5n-10}{3}$;

7. $B^+_{n,m} \prec L^+_{n,m} \prec O^+_{n,m} \prec H^+_{n,m}$ if $\frac{5n-10}{3} < m \leq 2(n-2)$.

The result follows. ∎

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