A short note on the number of spanning trees in lexicographic product of graphs

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Abstract

An explicit formula for the number of spanning trees of the lexicographic product $G[H]$ of two arbitrary graphs G and H is deduced in terms of structure parameters of G and H. Some properties on the number of spanning trees of $G[H]$ are revealed. Sharp lower and upper bounds for the number of spanning trees of lexicographic product of graphs are established. In particular, simple formulae for the number of spanning trees of the lexicographic product of some special graphs are derived, which extend some previously known results in the literature.

Key words: spanning tree, Laplacian spectrum, lexicographic product, Matrix-Tree Theorem

1 Introduction

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. A *spanning tree* of G is a subgraph of G which is a tree and which contains all the vertices of G. The number of spanning trees of G, denoted by $t(G)$, is the total number of distinct spanning subgraphs that are trees. The number of spanning trees of graphs has been studied for a long time, dating back to the celebrating work of Kirchhoff [1]. Evaluating the number of spanning trees of graphs (as well as the number of rooted spanning trees of digraphs, see e.g. [2]) is interesting not only because it is an important graph invariant but also because it has been widely used to analyze the reliability of networks, design electrical circuits, and analyze energy of masers, etc.

There are two elegant methods for counting spanning trees of graphs One is the combinatorial method, which states that for any edge $e \in E(G)$,

$$t(G) = t(G-e) + t(G/e),$$

where $G-e$ is the graph obtained by deleting e from G and G/e is the graph obtained from G by contracting e. The other method is known as Kirchhoff's "Matrix-Tree Theorem", which is one of the earliest and most impressive contributions of spectral graph theory. The Matrix-Tree Theorem relies on the eigenvalues of the Laplacian matrix of G. The *Laplacian matrix* of G is the $n \times n$ matrix $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of G and $A(G)$ is the (0, 1) adjacency matrix of G. The spectrum of the Laplacian matrix of G is called the *Laplacian spectrum* of G and denoted by $S(G)$. We write $S(G) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with the understanding that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. It is well known that $\lambda_1 = 0$, and $\lambda_2 > 0$ if and only if G is connected.

Theorem 1.1 (Matrix-Tree Theorem). *[1] Let G be a connected graph on n vertices with $S(G) = \{0 = \lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then*

$$t(G) = \frac{1}{n}\prod_{i=2}^{n} \lambda_i. \tag{1.1}$$

Since graph products form a basis for many network topologies, it is natural and interesting to study the number of spanning trees in various graph products. In [3], Huang and Li formulated the number of spanning trees in join and cartesian product of two arbitrary graphs. In [4], Azarija gave sharp lower and upper bounds for the number of spanning trees in cartesian product of graphs. In [5], Li and Shiu obtained formula for the number of spanning trees of corona of graphs. In this paper, we study the number of spanning trees of lexicographic product of graphs. The *lexicographic product* (also known as *composition*) of two graphs G and H, denoted by $G[H]$, is defined as the graph such that the vertex set of $G[H]$ is the cartesian product $V(G) \times V(H)$, and any two vertices (u, v) and (x, y) are adjacent in $G[H]$ if and only if either u is adjacent with $x \in V(G)$ or $u = x$ and v is adjacent with y in H. By the definition, two simple facts can be seen. First, the lexicographic product is in general noncommutative, that is, $G[H] \neq H[G]$. Second, $G[H]$ is connected if and only if G is connected, regardless of the connectivity of H. Till now, the number of spanning trees of lexicographic product of some special graphs have been obtained. In [3], Huang and Li gave formula for the number of spanning trees of $K_m[G]$, $G[K_m]$, $K_{p,q}[G]$, and $G[K_{p,q}]$, where K_m is the complete graph of order m and $K_{p,q}$ denotes the complete bipartite graph such that two partite sets have p and q vertices, repectively. In [6], Li, Peng and Zhao developed a formula for the number of spanning trees of a class of multi-lexicographic product of graphs. In [7], Li et al. established a closed formulae for the number of spanning trees of $G[K_{l_1,l_2,\cdots,l_s}]$, where $K_{l_1,l_2,\cdots,l_s}$ is complete multipartite graph with s parts and for each i, the i-th part has l_i vertices. In [8], Li formulated the number of spanning trees of $K_{s_1,s_2,s_3}[G]$. Then Liang, Li and Xu [9] extended the result of Li and obtained a closed formula for the number of spanning trees of $K_{l_1,l_2,\cdots,l_s}[G]$. In [10], Li et al. obtained a sharp lower bound for the maximum number of edge-disjoint spanning trees in lexicographic product graphs. However, as proposed in [9], it still remain unsolved to derive a formula for $t(G_1[G_2])$, in which G_1 and G_2 are arbitrary graphs. In this paper, we solve this problem and give closed-form formula for the number of spanning trees of lexicographic product of two arbitrary graphs. Using this formula, some properties on the number of spanning trees of $G[H]$ are revealed. Sharp lower and upper bounds for the number of spanning trees of lexicographic product of graphs are established. In particular, simple formulae for the number of spanning trees of the lexicographic product of some special graphs are derived, which extend some previously known results in the literature.

2 The number of spanning trees in $G[H]$

Our formula for the number of spanning trees of $G[H]$ relies on the Laplacian spectrum of $G[H]$, which has been completely characterized by Barik, Bapat and Pati [11].

Theorem 2.1. *[11] Let G be a connected graph with vertex set $\{v_1, ..., v_n\}$ and H be any graph of order m. Suppose that $S(G) = (\lambda_1, \lambda_2, ..., \lambda_n)$ and $S(H) = (\mu_1, \mu_2, ..., \mu_m)$. Then*

(1) $m\lambda_i \in S(G[H])$ for $i = 1, 2, \cdots, n$,

(2) $md_i + \mu_j \in S(G[H])$ for $i = 1, \ldots, n$ and $j = 2, \ldots, m$, where d_i is the degree of v_i in G.

Then by Theorems 1.1 and 2.1, it is straightforward to obtain the formula for the number of spanning trees of $G[H]$, which is given in terms of structure parameters of G and H.

Theorem 2.2. *Let G be a connected graph and let H be an arbitrary graph. Then the number of spanning trees in $G[H]$ can be computed as*

$$t(G[H]) = m^{n-2}t(G)\prod_{i=1}^{n}\prod_{j=2}^{m}(md_i + u_j). \tag{2.1}$$

According to Eq, (2.1), it is interesting to observe that the ratio of $t(G[H])$ over $t(G)$ could be expressed in an explicit way.

Proposition 2.3. *Let G be a connected graph and let H be an arbitrary graph. Then*

$$\frac{t(G[H])}{t(G)} = m^{n-2} \prod_{i=1}^{n} \prod_{j=2}^{m} (md_i + u_j). \tag{2.2}$$

The *degree sequence* of a graph is the sequence of vertex degrees in non-decreasing order. For a given graph H, from Eq. (2.1), it is interesting to note that $t(G[H])$ depends only on $t(G$ and the degree sequence of G, which leads to the following proposition.

Proposition 2.4. *Let G_1 and G_2 be connected graphs of order n with the same degree sequence. Then for any graph H, we have*

$$\frac{t(G_1[H])}{t(G_2[H])} = \frac{t(G_1)}{t(G_2)}. \tag{2.3}$$

As an immediate consequence of Eq. (2.1), if G is regular, then the expression for the number of spanning trees of $G[H]$ can be further simplified.

Corollary 2.5. *Let G be a connected r-regular graph and let H be an arbitrary graph. Then*

$$t(G[H]) = m^{n-2} t(G) \prod_{j=2}^{m} (mr + u_j)^n. \tag{2.4}$$

We proceed to give two alternative expressions for $t(G[H])$ – one takes the coefficients of the characteristic polynomial of the Laplacian matrix of H into consideration, and the other takes the spanning forests of H into consideration. Suppose that the characteristic polynomial of the Laplacian matrix of H is

$$C(\lambda) = \lambda^m + c_1 \lambda^{m-1} + \cdots + c_{m-1}\lambda.$$

Then by the relation between roots and coefficients, we have

$$c_k = (-1)^k \sum_{2 \le l_1 \le l_2 \le \cdots \le l_k \le m} \mu_{l_1} \mu_{l_2} \cdots \mu_{l_k}. \tag{2.5}$$

In addition, Kel'mans [12, p. 38] has established the connection between the coefficients of the Laplacian characteristic polynomial and the structure of the respective graph,

$$c_k = (-1)^k \sum_{F \in \mathcal{F}_{m-k}} \gamma(F) \tag{2.6}$$

where F is a spanning forest and the summation goes over the set $\mathcal{F}_{m-k}$ of all spanning forests of H with exactly $m-k$ components, and $\gamma(F)$ is the product of the number of vertices of the components of F .

Now we are ready to give two alternative expressions for the number of spanning tree of $G[H]$.

Theorem 2.6. *Let G be a connected graph and let H be an arbitrary graph. Then the number of spanning trees in $G[H]$ can be computed as*

$$\begin{aligned} t(G[H]) &= m^{n-2}t(G)\prod_{i=1}^{n}\left[\sum_{k=1}^{m-1}(-1)^k(md_i)^{m-1-k}c_k\right] &(2.7)\\ &= m^{n-2}t(G)\prod_{i=1}^{n}\left[\sum_{k=1}^{m-1}\left[(md_i)^{m-1-k}\sum_{F\in\mathcal{F}_{m-k}}\gamma(F)\right]\right].&(2.8)\end{aligned}$$

Proof. Eq. (2.7) can be deduced from Eq. (2.1) by noticing that

$$\begin{aligned}\prod_{i=1}^{n}\prod_{j=2}^{m}(md_i+u_j) &= \prod_{i=1}^{n}\left[\sum_{k=1}^{m-1}\left[(md_i)^{m-1-k}\sum_{2\le l_1\le\cdots\le l_k\le m}\mu_{l_1}\cdots\mu_{l_k}\right]\right]\\ &= \prod_{i=1}^{n}\left[\sum_{k=1}^{m-1}(md_i)^{m-1-k}(-1)^k c_k\right].\end{aligned}$$

Then substitute Eq. (2.6) into Eq. (2.7) to get Eq. (2.8). □

3 Sharp lower and upper bounds

In this section, we derive lower and upper bounds for the number of spanning tree in the lexicographic product $G[H]$ of G and H.

Let $\delta(G)$ and $\Delta(G)$ be the minimum degree and maximum degree of G, respectively. Then by Eq. (2.1), it is straightforward to obtain the following bounds.

Theorem 3.1. *Let G be a connected graph and let H be an arbitrary graph. Then*

$$m^{n-2}t(G)\prod_{j=2}^{m}\left[m\delta(G)+u_j\right]^n \le t(G[H]) \le m^{n-2}t(G)\prod_{j=2}^{m}\left[m\Delta(G)+\mu_j\right]^n, \tag{3.1}$$

with both equalities holding if and only if G is regular.

Next, we give a lower and upper bounds for $t(G[H])$ by the inequality of arithmetic and geometric means. We first introduce a result that will be used later.

Lemma 3.2. *[13] Let G be a connected graph with $S(G) = \{0, \lambda_2, \dots, \lambda_n\}$. Then $\lambda_2 = \lambda_3 = \dots = \lambda_n$ if and only if G is the complete graph K_n.*

Theorem 3.3. *Let G be a connected graph and let H be an arbitrary graph. Then*

$$t(G[H]) \geq 2^{(m-1)n} m^{\frac{mn+2n-4}{2}} t(G) \, [t(H)]^{\frac{n}{2}} \prod_{i=1}^{n} d_i^{\frac{m-1}{2}}, \tag{3.2}$$

where the equality holds if and only if $G = K_2$ and $H = K_m$.

Proof. Bearing in mind that

$$t(G[H]) = m^{n-2} t(G) \prod_{i=1}^{n} \prod_{j=2}^{m} (md_i + \mu_j).$$

Then by the inequality of arithmetic and geometric means that $md_i + \mu_j \geq 2\sqrt{md_i\mu_j}$ for every i, j, we have

$$\begin{aligned} t(G[H]) &\geq m^{n-2} t(G) \prod_{i=1}^{n} \prod_{j=2}^{m} (2\sqrt{md_i\mu_j}) \\ &= m^{n-2} t(G) 2^{(m-1)n} m^{\frac{(m-1)n}{2}} \prod_{i=1}^{n} \prod_{j=2}^{m} (\sqrt{d_i\mu_j}) \\ &= 2^{(m-1)n} m^{\frac{mn+n-4}{2}} t(G) (\sqrt{\mu_2\mu_3 \cdots \mu_m})^n (\prod_{i=1}^{n} d_i)^{\frac{m-1}{2}} \\ &= 2^{(m-1)n} m^{\frac{mn+n-4}{2}} t(G) \left[\sqrt{mt(H)}\right]^n \prod_{i=1}^{n} d_i^{\frac{m-1}{2}} \\ &= 2^{(m-1)n} m^{\frac{mn+2n-4}{2}} t(G) \, [t(H)]^{\frac{n}{2}} \prod_{i=1}^{n} d_i^{\frac{m-1}{2}}, \end{aligned}$$

as desired.

Note that equality holds if and only

$$md_1 = md_2 = \cdots = md_n = \mu_2 = \cdots = \mu_m. \tag{3.3}$$

Obviously, it first requires G to be regular. Now suppose that G is r-regular ($r \geq 1$). Then it follows that $S(H)$ have eigenvalues mr of multiplicity $m-1$. Hence by Lemma 3.2, we conclude that either $H = K_m$ or H is disconnected and all the components of H are complete graphs of the same order. But the later case can not happen since otherwise $\mu_2 = 0$, contradicting the fact that $\mu_2 = mr \geq m$. So Eq. (3.3) can hold only when $H = K_m$. In this case, $\mu_2 = \cdots = \mu_m = m$. which further requires that G is 1-regular, that is, $G = K_2$. □

Theorem 3.4. *Let G be a connected graph and let H be an arbitrary graph. Then*

$$t(G[H]) \leq 2^{(m-1)n} m^{n-2} t(G) \left[\frac{m(m-1)|E(G)| + n|E(H)|}{(m-1)n} \right]^{(m-1)n}, \quad (3.4)$$

where the equality holds if and only if G is regular and either $H = K_m$ or H is disconnected with all of its components being complete graphs of the same order.

Proof. By applying applying the inequality of geometric and arithmetic means on the factors of Eq. (2.1), we have

$$\begin{aligned} t(G[H]) &= m^{n-2} t(G) \prod_{i=1}^{n} \prod_{j=2}^{m} (md_i + \mu_j) \\ &\leq m^{n-2} t(G) \left[\frac{\sum_{i=1}^{n} \sum_{j=2}^{m} (md_i + \mu_j)}{(m-1)n} \right]^{(m-1)n} \\ &\leq m^{n-2} t(G) \left[\frac{(m-1)\sum_{i=1}^{n} md_i + n \sum_{j=2}^{m} \mu_j}{(m-1)n} \right]^{(m-1)n}. \end{aligned}$$

Observe that $\sum_{i=1}^{n} d_i = 2|E(G)|$ and $\sum_{j=2}^{m} \mu_j = 2|E(H)|$. It follows that

$$t(G[H]) \leq 2^{(m-1)n} m^{n-2} t(G) \left[\frac{m(m-1)|E(G)| + n|E(H)|}{(m-1)n} \right]^{(m-1)n}.$$

Note that equality holds if and only if

$$md_i + \mu_j = md_k + \mu_l, \quad \text{for } i,k = 1,2,\ldots,n \text{ and } j,l = 2,3,\ldots,m.$$

It means that $d_1 = d_2 = \cdots = d_n$ and $\mu_2 = \mu_3 = \cdots = \mu_m$, which, by Lemma 3.2, means that G is regular and either $H = K_m$ or H is disconnected with all of its components being complete graphs of the same order. □

We end this section by giving another upper bound for $t(G[H])$.

Theorem 3.5. *Let G be a connected graph and let H be an arbitrary graph. Then*

$$t(G[H]) \leq m^{n-2} t(G) \prod_{i=1}^{n} \left[md_i + \frac{2E|H|}{m-1} \right]^{m-1} \quad (3.5)$$

where the equality holds if and only if $H = K_m$ or H is disconnected with all of its components being complete graphs of the same order.

Proof. By Eq. (2.1),

$$t(G[H]) = m^{n-2}t(G)\prod_{i=1}^{n}\prod_{j=2}^{m}(md_i + u_j).$$

Then by the inequality of geometric and arithmetic means, we have

$$\begin{aligned}(md_i+\mu_2)(md_i+\mu_3)\cdots(md_i+\mu_m) &\le \left[\frac{\sum_{j=2}^{m}(md_i+\mu_j)}{m-1}\right]^{m-1}\\ &= \left[md_i + \frac{2|E(H)|}{m-1}\right]^{m-1},\end{aligned}$$

with equality if and only if $\mu_2 = \mu_3 = \cdots = \mu_m$. Hence the desired bound is obtain by substituting this result into Eq. (2.1). As observed before, equality holds if and only if $H = K_m$ or H is disconnected with all of its components being complete graphs of the same order. □

4 Special cases

In this section, we choose G and H be some special graphs, so that some simple formulae are obtained, with some previously known results being reestablished.

4.1 The lexicographic product of a graph with complete multipartite graph

Now we consider the case that G or H is the complete multipartite graph $K_{l_1,l_2,\cdots,l_s}$. Suppose that $l_1 \ge l_2 \ge \cdots \ge l_s$ and $l_1 + l_2 + \cdots + l_s = p$. Then the Laplacian spectrum of $K_{l_1,l_2,\cdots,l_s}$ is [14]:

$$(0, (p-l_1)^{l_1-1}, \cdots, (p-l_s)^{l_s-1}, p^{s-1}),$$

where the power on each eigenvalue denotes the multiplicity of the eigenvalue. Then by Theorem 2.1, we have

Theorem 4.1. *The number of spanning trees in $G[K_{l_1,l_2,\cdots,l_s}]$ and $K_{l_1,l_2,\cdots,l_s}[H]$ can be computed as*

$$t(G[K_{l_1,l_2,\cdots,l_s}]) = p^{sn-2}t(G)\prod_{i=1}^{n}(d_i+1)^{s-1}\prod_{k=1}^{s}\prod_{i=1}^{n}(pd_i+p-l_k)^{l_k-1}. \quad (4.1)$$

$$t(K_{l_1,l_2,\cdots,l_s}[H]) = m^{p-2}p^{s-2}\prod_{i=1}^{s}(p-l_i)^{l_i-1}\prod_{j=2}^{m}\prod_{i=1}^{s}\left[m(p-l_i)+\mu_j\right]^{l_i}. \quad (4.2)$$

Remark 1. Eq. (4.1) coincides with the result in [7]. Eq. (4.2) verifies the result of [9], which generalizes the result in [8].

4.2 The generalized double graphs

The composite graphs $G[\overline{K_m}]$ are also known as the generalized double graphs of G, because these graphs turn out to be a generalization of the *double graph* [15] $D(G)$ obtained by taking two copies of G (including the initial edge set of each) and joining each vertex in one copy with the neighbours of the corresponding vertex in the other copy. Clearly $D(G) = G[\overline{K_2}]$. Noticing that $S(\overline{K_m}) = (0, 0, \cdots, 0)$, by Theorem 2.1, we have

Theorem 4.2. *The number of spanning tress in the generalized double graph can be computed as*

$$t(G[\overline{K_m}]) = m^{mn-2} t(G) \prod_{i=1}^{n} d_i^{m-1}. \tag{4.3}$$

In particular, if $m = 2$ (i.e. $G[\overline{K_m}]$ is the double graph $D(G)$ of G), then

$$t(D(G)) = 4^{n-1} t(G) \prod_{i=1}^{n} d_i. \tag{4.4}$$

Remark 2. From Theorem 4.2, one may find that $t(G[\overline{K_m}])$ is divisible by $t(G)$, which seems to be interesting.

4.3 The graphs $P_n[H]$ and $C_n[H]$

Let P_n and C_n be the path graph and cycle graph of order n, respectively. The graph $P_n[H]$ can be viewed as the graph obtained in the following ways: first take n copies of H, then connect all the vertices in the i-th copy with those in the $(i+1)$-th copy for $1 \le i \le n-1$. Similarly, the graph $C_n[H]$ can be obtained by first taking n copies of H, and then connecting all the vertices in the i-th copy with those in the $(i+1)$-th (mod n) copy for all $1 \le i \le n$. By Theorem 2.1, we readily have

Theorem 4.3. *The number of spanning tress in $P_n[H]$ and $C_n[H]$ can be computed as*

$$t(P_n[H]) = m^{n-2} \prod_{j=2}^{m} \left[(m+\mu_j)^2 (2m+\mu_j)^{n-2}\right]. \tag{4.5}$$

$$t(C_n[H]) = m^{n-2} n \prod_{j=2}^{m} (2m+\mu_j)^n. \tag{4.6}$$

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