On the (super) edge-magic deficiency of chain graphs

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Abstract

A graph G of order $|V(G)|$ and size $|E(G)|$ is called *edge-magic* if there exists a bijection $f : V(G) \cup E(G) \to \{1, 2, 3, \cdots, |V(G)| + |E(G)|\}$ such that $f(x) + f(xy) + f(y)$ is a constant for every edge $xy \in E(G)$. An edge-magic graph G is said to be *super* if $f(V(G)) = \{1, 2, 3, \cdots, |V(G)|\}$. Furthermore, the *edge-magic deficiency* of a graph G, $\mu(G)$, is defined as the minimum nonnegative integer n such that $G \cup nK_1$ is edge-magic. Similarly, the *super edge-magic deficiency* of a graph G, $\mu_s(G)$, is either the minimum nonnegative integer n such that $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer n. In this paper, we investigate the (super) edge-magic deficiency of chain graphs. Based on these, we propose some open problems.

1 Introduction

Let G be a finite and simple graph, where $V(G)$ and $E(G)$ are its vertex set and edge set, respectively. In [9], Kotzig and Rosa introduced the concepts of edge-magic labelings and edge-magic graphs as follows. An *edge-magic labeling* of a graph G is a bijection $f : V(G) \cup E(G) \to \{1, 2, 3, \cdots, |V(G)| + |E(G)|\}$ such that $f(x) + f(xy) + f(y)$ is a constant k, called the *magic constant* of f, for every edge xy of G. A graph that admits such a labeling is called an *edge-magic graph*. Motivated by the concept of edge-magic labelings, Enomoto, Lladó, Nakamigawa, and Ringel [3] introduced the concept of super edge-magic labelings and super edge-magic graphs as follows. A *super edge-magic labeling* of a graph G is an edge-magic labeling f of G with the extra property that $f(V(G)) = \{1, 2, 3, \cdots, |V(G)|\}$. A *super edge-magic graph* is a graph that admits a super edge-magic labeling. The following lemma proved by Figueroa-Centeno *et al.* [4] provides necessary and sufficient conditions for a graph to be super edge-magic.

Lemma 1 *A graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1, 2, \cdots, |V(G)|\}$ such that the set $S = \{f(x) + f(y) : xy \in E(G)\}$ consists of $|E(G)|$ consecutive integers. In this case, f can be extended to a super edge-magic labeling of G with the magic constant $|V(G)| + |E(G)| + min(S)$.*

The next lemma taken from [3] gives sufficient condition for non-existence of super edge-magic labeling of a graph.

Lemma 2 *If G is a super edge-magic graph, then $|E(G)| \leq 2|V(G)| - 3$.*

In [9], also Kotzig and Rosa proved that for every graph G there exists a nonnegative integer n such that $G \cup nK_1$ is edge-magic. This fact motivated them to introduced the concept of edge-magic deficiency of a graph. The *edge-magic deficiency* of a graph G, $\mu(G)$, is defined as the minimum nonnegative integer n such that $G \cup nK_1$ is an edge-magic graph. Motivated by Kotzig and Rosa's concept of edge-magic deficiency, Figueroa-Centeno *et al.* [5] introduced the concept of super edge-magic deficiency of a graph analogously. They defined the *super edge-magic deficiency* of a graph G, $\mu_s(G)$, as the minimum nonnegative integer n such that $G \cup nK_1$ is a super edge-magic graph or $+\infty$ if there exists no such n. Several authors have studied the super edge-magic deficiency of some classes of graphs. Ngurah *et al.* in two separate papers [11, 13] studied the super edge-magic deficiency of some classes of chain graphs, wheels, fans, double fans, and disjoint union of particular type of complete bipartite graphs. Ichishima and Oshima [8] investigated the super edge-magic deficiency of disjoint union of complete bipartite graphs. Recently, Ngurah and Simanjuntak [12] studied the super edge-magic deficiency of join products of a path, a star, and a cycle, respectively, with isolated vertices. Other resuls can be found in [5, 6] and the latest developments in these and other types of graph labelings can be found in the survey paper of graph labelings by Gallian [7] .

In 2002, Barrientos [2] defined a *chain graph* as a graph with blocks $B_1, B_2, \cdots, B_k$ such that for every i, B_i and B_{i+1} have a common vertex in such a way that the block-cut-vertex graph is a path. We will denote the chain graph with k blocks $B_1, B_2, \cdots, B_k$ by $C[B_1, B_2, \cdots, B_k]$. If $B_1 = \cdots = B_t = B$, we will write $C[B_1, B_2, \cdots, B_k]$ as $C[B^{(t)}, B_{t+1}, \cdots, B_k]$. If for every i, $B_i = H$ for a given graph H, then $C[B_1, B_2, \cdots, B_k]$ is denoted by kH-path. Hence, kC_4-path is a chain graph with k blocks where each block is identical and isomorphic to the cycle C_4. Suppose that $c_1, c_2, \ldots, c_{k-1}$ are the consecutive cut vertices of $C[B_1, B_2, \cdots, B_k]$. The *string* of $C[B_1, B_2, \cdots, B_k]$ is $(k-2)$-tuple $(d_1, d_2, \ldots, d_{k-2})$ where d_i is the distance between c_i and c_{i+1}, $1 \leq i \leq k-2$. We will write $(d_1, d_2, \ldots, d_{k-2})$ as $(d^{(t)}, d_{t+1}, \ldots, d_{k-2})$ if $d_1 = \ldots = d_t = d$. Some authors have studied the super edge-magic deficiency of chain graphs. In 2003, Lee and Wang [10] proved that some classes of chain graphs whose blocks are complete graphs are super edge-magic. In other words, they showed that some classes of chain graphs whose blocks are complete graphs have zero super edge-magic deficiency. In [11], Ngurah *et al.* studied the super edge-magic deficiency of kK_3-paths and kK_4-paths.

In this paper, we present some new results on the (super) edge-magic deficiency of chain graphs. Particularly, we investigate the super edge-magic deficiency of kDL_m-paths and $C[K_4^{(k)}, DL_m, K_4^{(n)}]$, where DL_m, $m \geq 2$, is a graph obtained from the ladder $L_m = P_m \times P_2$ by adding two diagonals in each rectangle of L_m. Also, we studied the edge-magic deficiency of kC_4-paths. Additionally, we propose some open problems related to the (super) edge-magic deficiency of these graphs.

2 The Results

First, we consider the super edge-magic deficiency of a graph whose obtained from the ladder $L_m = P_m \times P_2$, $m \geq 2$, by adding two diagonals in each rectangle of L_m. Such a graph is denoted by DL_m. Hence, DL_m can be defined as a graph with the vertex set $V(DL_m) = \{u_j, v_j | 1 \leq j \leq m\}$ and edge set $E(DL_m) = \{u_j v_j | 1 \leq j \leq m\} \cup \{u_j v_{j+1}, v_j u_{j+1}, u_j u_{j+1}, v_j v_{j+1} | 1 \leq j \leq m-1\}$. Notice that Ahmad et al. [1] called this graph diagonal ladder and proved the following theorem.

Theorem 1 *[1] For every integer $m \geq 2$, $\mu_s(DL_m) = \lfloor \frac{m}{2} \rfloor$.*

Let us turn our attention to the graph $DL_m^* = DL_m \cup \lfloor \frac{m}{2} \rfloor K_1$. For odd m, this graph satisfies $2|V(DL_m^*)| - 3 = 5m - 4 = |E(DL_m^*)|$. Consequently, the magic constant of any super edge-magic labelings of DL_m^*, for odd m, are $|V(DL_m^*)| + |E(DL_m^*)| + 3 = 3|V(G_m^*)| = \frac{1}{2}(15m - 3)$. For even m, we have $2|V(DL_m^*)| - 3 = 5m - 3 = |E(DL_m^*)| + 1$, implying that the magic constant of any super edge-magic labelings of DL_m^* are $3|V(DL_m^*)| - 1 = \frac{1}{2}(15m-2)$ or $3|V(DL_m^*)| = \frac{15m}{2}$. As we can see, the labeling $f : V(DL_m^*) \longrightarrow \{1, 2, 3, \ldots, \lfloor \frac{5m}{2} \rfloor\}$, (see Theorem 2, [1]), defined by

$$f(x) = \begin{cases} \lfloor \frac{1}{2}(5i-3) \rfloor, & \text{if } x = u_i,\ 1 \le i \le m, \\ \lfloor \frac{5i}{2} \rfloor, & \text{if } x = v_i,\ 1 \le i \le m. \end{cases}$$

can be extended to a super edge-magic labeling of DL_m^* with the magic constant $\frac{1}{2}(15m-3)$ for odd m and $\frac{1}{2}(15m-2)$ for even m. Also, it is easy to verify that, for even m, the labeling $f'(x) = \frac{1}{2}(5m+2) - f(x)$ for every $x \in V(DL_m^*)$ can be extended to a super edge-magic labeling of DL_m^* with the magic constant $\frac{15m}{2}$. Hence, there exists a super edge-magic labeling of DL_m^* witgh the magic constant k for every $k \in \{\frac{1}{2}(15m-3), \frac{1}{2}(15m-2), \frac{15m}{2}\}$.

Next, we study the super edge-magic deficiency of some classes of chain graphs. First, we study the super edge-magic deficiency of kDL_m-path. Our first result gives the lower bound of the super edge-magic deficiency of any kDL_m-path as we state in the following lemma.

Lemma 3 *Let $k \geq 3$ and $m \geq 2$ be integers. If $H = kDL_m$-path with string $(d_1, d_2, \ldots, d_{k-2})$, where $d_i \in \{1, 2, \ldots, m-1\}$, $1 \le i \le k-2$, then $\mu_s(H) \geq \lfloor \frac{1}{2}(m-2)k+1 \rfloor$.*

Proof DL_m is a graph of order $2m$ and size $5m-4$. So, H is a graph of order $(2m-1)k+1$ and size $(5m-4)k$. By Lemma 2, H is not a super edge-magic graph. Again, by Lemma 2, it is not hard to verify that $\mu_s(H) \geq \lfloor \frac{1}{2}(m-2)k+1 \rfloor$. □

Notice that, the lower bound presented in Lemma 3 is tight. We found that the chain graph kDL_m-path with string $(m-1, m-1, \ldots, m-1)$ has the super edge-magic deficiency $\lfloor \frac{1}{2}(m-2)k+1 \rfloor$ as we state in following theorem.

Theorem 2 *Let $k \geq 3$ and $m \geq 2$ be integers. If $G = kDL_m$-path with string $(m-1, m-1, \ldots, m-1)$, then $\mu_s(G) = \lfloor \frac{1}{2}(m-2)k+1 \rfloor$.*

Proof By Lemma 3, $\mu_s(G) \geq \lfloor \frac{1}{2}(m-2)k+1 \rfloor$. Now, we have to show $\mu_s(G) \leq \lfloor \frac{1}{2}(m-2)k+1 \rfloor$. First, we define G as a graph having

$$V(G) = \{c_i | 1 \le i \le (m-1)k+1\} \cup \{u_j^t | 1 \le j \le m, 1 \le t \le k\}$$

and
$E(G) = \{u_j^t u_{j+1}^t, u_j^t c_{(t-1)m+j+2-t}, c_{(t-1)m+j+1-t} u_{j+1}^t | 1 \le j \le m-1, 1 \le$

$t \le k\} \cup \{u_j^t c_{(t-1)m+j+1-t} | 1 \le j \le m, 1 \le t \le k\} \cup \{c_i c_{i+1} | 1 \le i \le (m-1)k\}$. Notice that under this definition, $c_m, c_{2(m-1)+1}, \ldots, c_{(k-1)(m-1)+1}$ are the cut vertices of G.

Next, define a bijection $g : V(G) \cup (\lfloor \frac{1}{2}(m-2)k+1 \rfloor)K_1 \longrightarrow \{1, 2, 3, \ldots, \lfloor \frac{1}{2}((5m-4)k+4) \rfloor\}$ as follows.
Case: m is even

$$g(x) = \begin{cases} \lfloor \frac{1}{2}(5j-3) \rfloor, & \text{if } x = u_j^1,\ 1 \le j \le m, \\ \lfloor \frac{1}{2}(5m+5j-7) \rfloor, & \text{if } x = u_j^2,\ 1 \le j \le m, \\ \lfloor \frac{5i}{2} \rfloor, & \text{if } x = c_i,\ 1 \le i \le m, \\ \lfloor \frac{1}{2}(5i+1) \rfloor, & \text{if } x = c_i,\ m+1 \le i \le 2m-1. \end{cases}$$

Case: m is odd

$$g(x) = \begin{cases} \lfloor \frac{1}{2}(5j-3) \rfloor, & \text{if } x = u_j^1,\ 1 \le j \le m-1, \\ \frac{1}{2}(5m-1), & \text{if } x = u_m^1, \\ \lfloor \frac{1}{2}(5m+5j-4) \rfloor, & \text{if } x = u_j^2,\ 1 \le j \le m-1, \\ \frac{1}{2}(10m-8), & \text{if } x = u_m^2, \\ \lfloor \frac{5i}{2} \rfloor, & \text{if } x = c_i,\ 1 \le i \le m-1, \\ \frac{1}{2}(5m-3), & \text{if } x = c_m, \\ \lfloor \frac{1}{2}(5i-2) \rfloor, & \text{if } x = c_i,\ m+1 \le i \le 2m-2, \\ \frac{1}{2}(10m-4), & \text{if } x = c_{2m-1}. \end{cases}$$

For $1 \le j \le m$ and $3 \le t \le k$,

$$g(u_j^t) = \begin{cases} g(u_j^1) + \frac{1}{2}(t-1)(5m-4), & \text{if } t \text{ is odd}, \\ g(u_j^2) + \frac{1}{2}(t-2)(5m-4), & \text{if } t \text{ is even}. \end{cases}$$

For $1 \le j \le m-1$ and $3 \le t \le k$, label the remaining vertices as follows.

$$g(c_{(t-1)m+j+2-t}) = \begin{cases} g(c_{j+1}) + \frac{1}{2}(t-1)(5m-4), & \text{if } t \text{ is odd}, \\ g(c_{m+j}) + \frac{1}{2}(t-2)(5m-4), & \text{if } t \text{ is even}. \end{cases}$$

We can verify that no labels are repeated. The largest vertex labels used are $\frac{1}{2}((5m-4)k+3)$, if m and k are odd, and $\frac{1}{2}((5m-4)k+4)$, if both m and k are not odd. Hence, if m and k are odd, there exist $\frac{1}{2}((5m-4)k+3)-((2m-1)k+1) = \frac{1}{2}(m-2)k+\frac{1}{2}$ labels that are not utilized. Otherwise, there exist $\frac{1}{2}((5m-4)k+4)-((2m-1)k+1) = \frac{1}{2}(m-2)k+1$ labels that are not utilized.

Next, label isolated vertices in the following way.

Case m is odd and k is even.

If $m = 3$, we denote the isolated vertices with $\{z_i | 1 \le i \le \frac{1}{2}k+1\}$ and set $g(z_i) = 11i - 7$ for $1 \le i \le \frac{1}{2}k$, and $g(z_{\frac{1}{2}k+1}) = \frac{11}{2}k + 1$.

If $m \geq 5$, we denote the isolated vertices with $\{z_0\} \cup \{ z_i^t | 1 \leq i \leq \frac{1}{2}(m-1), t \text{ is odd}\} \cup \{z_j^t | 1 \leq j \leq \frac{1}{2}(m-3), t \text{ is even}\}$ and set

$$g(x) = \begin{cases} \frac{1}{2}(5m-4)(t-1)+5i-1, & \text{if } x = z_i^t,\ 1 \leq i \leq \frac{1}{2}(m-1), \\ & t \text{ is odd}, \\ \frac{1}{2}(5m-1)+5j, & \text{if } x = z_j^2, 1 \leq j \leq \frac{1}{2}(m-3), \\ \frac{1}{2}((5m-4)(t-1)+3)+5j, & \text{if } x = z_j^t,\ 1 \leq j \leq \frac{1}{2}(m-3), \\ & t \geq 4 \text{ is even}, \\ \frac{1}{2}(5m-4)k+1, & \text{if } x = z_0. \end{cases}$$

Case m is odd and k is odd.

If $m = 3$, we denote the isolated vertices with $\{z_i | 1 \leq i \leq \lfloor \frac{1}{2}k+1 \rfloor\}$ and set $g(z_i) = 11i - 7$ for $1 \leq i \leq \lfloor \frac{1}{2}k+1 \rfloor$.

If $m \geq 5$, we denote the isolated vertices with $\{z_i^t | 1 \leq i \leq \frac{1}{2}(m-1), t \text{ is odd}\} \cup \{z_j^t | 1 \leq j \leq \frac{1}{2}(m-3), t \text{ is even}\}$ and set

$$g(x) = \begin{cases} \frac{1}{2}(5m-4)(t-1)+5i-1, & \text{if } x = z_i^t,\ 1 \leq i \leq \frac{1}{2}(m-1), \\ & t \text{ is odd}, \\ \frac{1}{2}(5m-1)+5j, & \text{if } x = z_j^2, 1 \leq j \leq \frac{1}{2}(m-3), \\ \frac{1}{2}((5m-4)(t-1)+3)+5j, & \text{if } x = z_j^t,\ 1 \leq j \leq \frac{1}{2}(m-3), \\ & t \geq 4 \text{ is even}, \end{cases}$$

Case m is even and for any k.

In this case, we denote the isolated vertices with $\{z_0\} \cup \{z_i^t | 1 \leq i \leq \frac{1}{2}(m-2),\ 1 \leq t \leq k\}$ and set $g(z_o) = \frac{1}{2}(5m-4)k+1$ and $g(z_i^t) = \frac{1}{2}(5m-4)(t-1)+5j-1$ for $1 \leq i \leq \frac{1}{2}(m-2)$ and $1 \leq t \leq k$.

We can also verify that, under the vertex labeling g, the set $\{g(x)+g(y) : xy \in E(G)\} = \{3, 4, 5, \ldots, (5m-4)k+2\}$. By Lemma 1, g can be extended to a super edge-magic labeling of $G \cup \lfloor \frac{1}{2}(m-2)k+1 \rfloor K_1$ with the magic constant $\lfloor \frac{1}{2}((15m-12)k+10) \rfloor$. Hence, $\mu_s(G) \leq \lfloor \frac{1}{2}(m-2)k+1 \rfloor$. This completes the proof. □

An example of the labeling defined in the proof of Theorem 2 is shown in Figure 1.

Let $G^* = G \cup \lfloor \frac{1}{2}(m-2)k+1 \rfloor K_1$. It can be checked that G^* satisfies $2|V(G^*)|-3 = |E(G^*)|$ for m and k are odd, and $2|V(G^*)|-3 = |E(G^*)|+1$ for otherwise. Concequently, the magic constant of any super edge-magic labelings of G^* are $\frac{1}{2}((15m-12)k+9)$, $\frac{1}{2}((15m-12)k+10)$, or $\frac{1}{2}((15m-12)k+12)$. It is not hard to check that, for m and k are not both odd, the

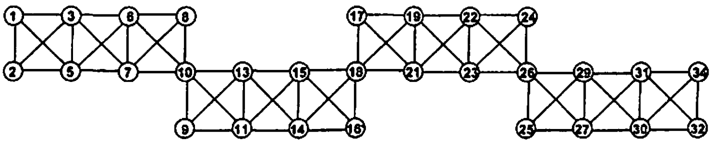

Figure 1: The vertex labelings of a $4DL_4$-path with string (3, 3).

labeling $g'(x) = \frac{1}{2}((5m-4)k+6) - g(x)$ for every $x \in V(G^*)$, where g is the labeling in the proof of Theorem 2, can be extended to a super edge-magic labeling of G^* with the magic constant $\frac{1}{2}((15m-12)k+12)$. Hence, for every $c \in \{\frac{1}{2}((15m-12)k+9), \frac{1}{2}((15m-12)k+10), \frac{1}{2}((15m-12)k+12)\}$, there exists a super edge-magic labeling of G^* with the magic constant c.

Referring to the afore-mentioned results, we propose the following problems.

Open problem 1 *For any integers $k \geq 3$ and $m \geq 2$, find the upper bound of the super edge-magic deficiency of every $G = kDL_m$-path. Further, for any integers $k \geq 3$ and $m \geq 2$, find the exact value of the super edge-magic deficiency of $G = kDL_m$-path with particular string $(d_1, d_2, \ldots, d_{k-2}) \neq (m-1, m-1, \ldots, m-1)$.*

Notice that $DL_2 = K_4$ and from Theorem 2 we knew that $\mu_s(kK_4\text{-path}) = 1$ (see also [11]). Next, we study the super edge-magic deficiency of chain graphs whose blocks are combination of K_4 and DL_m. Particularly, we study the super edge-magic deficiency of $F = C[K_4^{(k)}, DL_m, K_4^{(n)}]$. First, we give the lower bound of the super edge-magic deficiency of F. This lower bound is a consequence of Lemma 2. So, we state the result without proof.

Lemma 4 *Let $k, n \geq 1$ and $m \geq 2$ be integers. If $F = C[K_4^{(k)}, DL_m, K_4^{(n)}]$ with string $(1^{(k-1)}, d, 1^{(n-1)})$, where $d \in \{1, 2, 3, \ldots, m-1\}$, then $\mu_s(F) \geq \lfloor \frac{m}{2} \rfloor$.*

Now, we consider the super edge-magic deficiency of F with string $(1^{(k-1)}, m-1, 1^{(n-1)})$. As we can see, the super edge-magic deficiency of F with string $(1^{(k-1)}, m-1, 1^{(n-1)})$ does not depend on k and n. We define the vertex and edge sets of F as follows. $V(F) = \{a_i, b_i | 1 \leq i \leq k\} \cup \{c_i | 1 \leq i \leq k+1\} \cup \{u_j, v_j | 1 \leq j \leq m\} \cup \{x_t, y_t | 1 \leq t \leq n\} \cup \{z_t | 1 \leq t \leq n+1\}$ and $E(F) = \{a_i b_i, a_i c_i, a_i c_{i+1}, b_i c_i, b_i c_{i+1}, c_i c_{i+1} | 1 \leq i \leq k\} \cup \{u_j v_j | 1 \leq j \leq m\} \cup \{u_j u_{j+1}, v_j v_{j+1}, u_j v_{j+1}, v_j u_{j+1} | 1 \leq j \leq$

$m-1\} \cup \{x_ty_t, x_tz_t, x_tz_{t+1}, y_tz_t, y_tz_{t+1}, z_tz_{t+1} | 1 \le t \le n\}$, where $c_{k+1} = u_1$ and $v_m = z_1$. Hence, F is a graph of order $3(k+n)+2m$ and size $6(k+n)+5m-4$. The cut vertices of F are $c_2, c_3, \ldots, c_{k+1}, z_1, z_2, \ldots, z_n$.

Theorem 3 *Let $k, n \ge 1$ and $m \ge 2$ be integers. If $F = C[K_4^{(k)}, DL_m, K_4^{(n)}]$ with string $(1^{(k-1)}, m-1, 1^{(n-1)})$, then $\mu_s(F) = \lfloor \frac{m}{2} \rfloor$.*

Proof By Lemma 4, $\mu_s(F) \ge \lfloor \frac{m}{2} \rfloor$. Let $F^* = F \cup \lfloor \frac{m}{2} \rfloor K_1$ and define F^* as graph with $V(F^*) = V(F) \cup \{w_s | 1 \le s \le \lfloor \frac{m}{2} \rfloor\}$ and $E(F^*) = E(F)$. Next, define a bijective function $h : V(F^*) \longrightarrow \{1, 2, 3, \ldots, \lfloor 3(k+n) + \frac{1}{2}(5m) \rfloor\}$ as follows.

$$h(x) = \begin{cases} 3i-2, & \text{if } x = a_i,\ 1 \le i \le k, \\ 3i, & \text{if } x = b_i,\ 1 \le i \le k, \\ 3i-1, & \text{if } x = c_i,\ 1 \le i \le k+1, \\ \frac{1}{2}(6k+5j-1), & \text{if } x = u_j,\ j \text{ is odd}, \\ \frac{1}{2}(6k+5j-4), & \text{if } x = u_j,\ j \text{ is even}, \\ \frac{1}{2}(6k+5j-3), & \text{if } x = v_j,\ j \text{ is odd}, \\ \frac{1}{2}(6k+5j), & \text{if } x = v_j,\ j \text{ is even}, \\ 3k+5s-1, & \text{if } x = w_s,\ 1 \le s \le \lfloor \frac{m}{2} \rfloor. \end{cases}$$

To label the remaining vertices, we consider two cases depending on the value of m.
Case m is even

$$h(x) = \begin{cases} 3(k+t) + \frac{1}{2}(5m-4), & \text{if } x = x_t,\ 1 \le t \le n, \\ 3(k+t) + \frac{1}{2}(5m-8), & \text{if } x = y_t,\ 1 \le t \le n, \\ 3(k+t) + \frac{1}{2}(5m-6), & \text{if } x = z_t,\ 1 \le t \le n+1. \end{cases}$$

Case m is odd

$$h(x) = \begin{cases} 3(k+t) + \frac{1}{2}(5m-1), & \text{if } x = x_t,\ 1 \le t \le n, \\ 3(k+t) + \frac{1}{2}(5m-5), & \text{if } x = y_t,\ 1 \le t \le n, \\ 3(k+t) + \frac{1}{2}(5m-9), & \text{if } x = z_t,\ 1 \le t \le n+1. \end{cases}$$

Under the labeling h, it can be checked that $h(c_{k+1}) = h(u_1)$ and $h(v_m) = h(z_1)$. Also, it can be checked that $\{h(x) + h(y) | xy \in E(F^*)\}$ is a set of $6(k+n)+5m-4$ consecutive integers. By Lemma 1, h can be extended to a super magic edge labeling of F^* with the magic constant $9(k+n)+7m+\lfloor \frac{m}{2} \rfloor - 1$. Hence, $\mu_s(F) \le \lfloor \frac{m}{2} \rfloor$. □

The example of the vertex labeling defined in the proof of Theorem 3 can be seen in Figure 2.

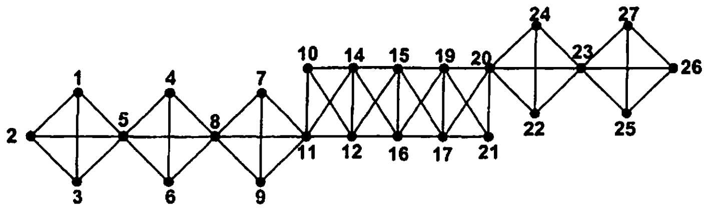

Figure 2: The vertex labelings of $C[K_4^{(3)}, DL_5, K_4^{(2)}]$ with string (1, 1, 4, 1).

It can be checked that the magic constant of any super edge-magic labelings of $F^* = F \cup \lfloor \frac{m}{2} \rfloor K_1$ are $9(k+n)+7m+\frac{1}{2}(m-1)-1$, $9(k+n)+7m+\frac{m}{2}-1$, or $9(k+n)+7m+\frac{m}{2}$. Also, it is easy to verify that there exists a super edge-magic labeling of F^* with the magic constant c for every $c \in \{9(k+n)+7m+\frac{1}{2}(m-1)-1, 9(k+n)+7m+\frac{m}{2}-1, 9(k+n)+7m+\frac{m}{2}\}$.

The open problems related the super edge-magic deficiency of $C[K_4^{(k)}, DL_m, K_4^{(n)}]$ are presented bellow.

Open problem 2 *For any integers $k, n \geq 1$ and $m \geq 2$, find the upper bound of the super edge-magic deficiency of $C[K_4^{(k)}, DL_m, K_4^{(n)}]$. Further, for any integers $k, n \geq 1$ and $m \geq 2$, find the exact value of the super edge-magic deficiency of $C[K_4^{(k)}, DL_m, K_4^{(n)}]$ with particular string $(1^{(k-1)}, d, 1^{(n-1)})$ for $d \in \{1, 2, 3, \ldots, m-2\}$.*

In the next two theorems, we study the edge-magic deficiency of chain graphs kC_4-path for some positive integer k.

Theorem 4 *Let $p, q \geq 2$ be integers. If $H = (p+q)C_4$-path with string $(2^{(p-2)}, 1, 1, 2^{(q-2)})$, then $\mu(H) = 0$.*

Proof First, define the vertex and edge sets of H as follows. $V(H) = \{a_i, b_i | 1 \leq i \leq p\} \cup \{c_i | 1 \leq i \leq p+1\} \cup \{x_j, y_j | 1 \leq j \leq q\} \cup \{z_j | 1 \leq j \leq q+1\}$, where $b_p = x_1$. $E(H) = \{c_i a_i, c_i b_i, a_i c_{i+1}, b_i c_{i+1} | 1 \leq i \leq p\} \cup \{z_j x_j, z_j y_j, x_j z_{j+1}, y_j z_{j+1} | 1 \leq j \leq q\}$.

Next, define a bijection $h : V(H) \cup E(H) \rightarrow \{1, 2, 3, \ldots, 7(p+q)+1\}$

as follows.

$$g(u) = \begin{cases} i, & \text{if } u = c_i,\ 1 \le i \le p+1, \\ 6p+7q+1+i, & \text{if } u = a_i,\ 1 \le i \le p, \\ 3p+q+2+i, & \text{if } u = b_i,\ 1 \le i \le p, \\ 3p+q+4-2i, & \text{if } u = a_ic_i,\ 1 \le i \le p, \\ 3p+q+3-2i, & \text{if } u = a_ic_{i+1},\ 1 \le i \le p, \\ 6p+7q+3-2i, & \text{if } u = b_ic_i,\ 1 \le i \le p, \\ 6p+7q+2-2i, & \text{if } u = b_ic_{i+1},\ 1 \le i \le p, \\ p+1+j, & \text{if } u = z_j,\ 1 \le j \le q+1, \\ 4p+q+1+j, & \text{if } u = x_j,\ 1 \le j \le q, \\ 4p+4q+1+j, & \text{if } u = y_j,\ 1 \le j \le q, \\ 4p+7q+3-2j, & \text{if } u = x_jz_j,\ 1 \le j \le q, \\ 4p+7q+2-2j, & \text{if } u = x_jz_{j+1},\ 1 \le j \le q, \\ 4p+4q+3-2j, & \text{if } u = y_jz_j,\ 1 \le j \le q, \\ 4p+4q+2-2j, & \text{if } u = y_jz_{j+1},\ 1 \le j \le q. \end{cases}$$

Is is a routine procedure to verify that h is an edge-magic labeling of H with the magic constant $9p+8q+5$. This complete the proof. □

Theorem 5 *Let $p \ge 2$ and $q \ge 1$ be integers. If $G = (p+q)C_4$-path with string $(2^{(p-2)}, 1, 2^{(q-1)})$, then $\mu(G) = 0$.*

Proof Let G be a graph with $V(G) = \{a_i, b_i | 1 \le i \le p\} \cup \{c_i | 1 \le i \le p+1\} \cup \{x_j, y_j | 1 \le j \le q\} \cup \{z_j | 1 \le j \le q+1\}$, where $b_p = z_1$. $E(G) = \{c_ia_i, c_ib_i, a_ic_{i+1}, b_ic_{i+1} | 1 \le i \le p\} \cup \{z_jx_j, z_jy_j, x_jz_{j+1}, y_jz_{j+1} | 1 \le j \le q\}$.

Next, define a labeling $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, 7(p+q)+1\}$ as follows.

$$f(u) = \begin{cases} i, & \text{if } u = c_i,\ 1 \le i \le p+1, \\ 6p+7q+1+i, & \text{if } u = a_i,\ 1 \le i \le p, \\ 3p+3q+1+i, & \text{if } u = b_i,\ 1 \le i \le p, \\ 3p+3q+3-2i, & \text{if } u = a_ic_i,\ 1 \le i \le p, \\ 3p+3q+2-2i, & \text{if } u = a_ic_{i+1},\ 1 \le i \le p, \\ 6p+7q+3-2i, & \text{if } u = b_ic_i,\ 1 \le i \le p, \\ 6p+7q+2-2i, & \text{if } u = b_ic_{i+1},\ 1 \le i \le p, \\ 4p+3q+j, & \text{if } u = z_j,\ 1 \le j \le q+1, \\ p+1+j, & \text{if } u = x_j,\ 1 \le j \le q, \\ 4p+4q+1+j, & \text{if } u = y_j,\ 1 \le j \le q, \\ 4p+7q+3-2j, & \text{if } u = x_jz_j,\ 1 \le j \le q, \\ 4p+7q+2-2j, & \text{if } u = x_jz_{j+1},\ 1 \le j \le q, \\ p+3q+3-2j, & \text{if } u = y_jz_j,\ 1 \le j \le q, \\ p+3q+2-2j, & \text{if } u = y_jz_{j+1},\ 1 \le j \le q. \end{cases}$$

It is easy to check that f is an edge-magic labeling of G with the magic constant $9p + 10q + 4$. Hence, $\mu(G) = 0$. □

The examples of the labelings defined in Theorem 4 and Theorem 5 are showed in Figure 3(a) and Figure 3(b), respectively.

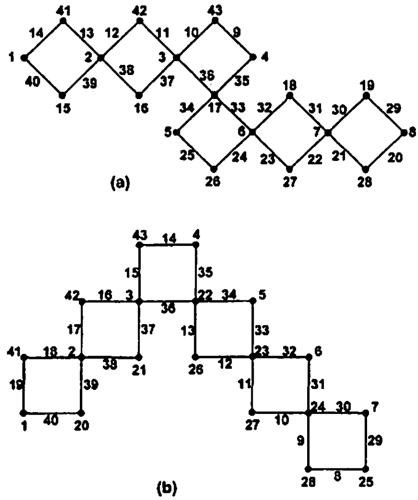

Figure 3: (a) The vertex and edge labelings of a $6C_4$-path with string (2, 1, 1, 2) with the magic constant 56. (b) The vertex and edge labelings of a $6C_4$-path with string (2, 1, 2, 2) with the magic constant 61.

Open problem 3 *For any integers $k \geq 3$, find the edge-magic deficiency of any kC_4-paths.*

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