The log-behavior of some sequences related to the Catalan-Larcombe-French sequence and the Fennessey-Larcombe-French sequence[1]

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Abstract

In this paper, for the Catalan-Larcombe-French sequence $\{P_n\}_{n\geq 0}$ and the Fennessey-Larcombe-French sequence $\{V_n\}_{n\geq 0}$, we mainly discuss the log-behavior of some sequences related to $\{P_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$. For example, we study the log-behavior of some sequences such as $\{P_n^2\}_{n\geq 0}$, $\{n!nV_n\}_{n\geq 1}$, $\{n!V_n\}_{n\geq 0}$, and $\{V_n - P_n\}_{n\geq 2}$. In addition, we discuss the monotonicity of some sequences involving $\{P_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$.

Key words. Catalan-Larcombe-French sequence, Fennessey-Larcombe-French sequence, log-convexity, log-concavity, log-balancedness, monotonicity, three-term recurrence.

1 Introduction

Throughout this paper, let $\{P_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ denote the Catalan-Larcombe-French sequence and the Fennessey-Larcombe-French sequence,

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respectively. For $n \geq 1$, they satisfy the following recurrence relations [5]:

$$(n+1)^2 P_{n+1} = 8(3n^2+3n+1)P_n - 128n^2 P_{n-1}, \tag{1.1}$$

$$n(n+1)^2 V_{n+1} = 8n(3n^2+5n+1)V_n - 128(n-1)(n+1)^2 V_{n-1}, \tag{1.2}$$

where $P_0 = V_0 = 1$ and $P_1 = V_1 = 8$. Some initial values of $\{P_n\}$ and $\{V_n\}$ are as follows:

n	0	1	2	3	4	5	6	7
P_n	1	8	80	896	10816	137728	1823744	24862720
V_n	1	8	144	2432	40000	649728	10486784	168681472

These numbers are closely related to the elliptic integrals, the theory of modular forms, etc. [9,16] and there are many publications to discuss their properties; see [4–11,16]. In this paper, we discuss the log-behavior of some sequences related to the Catalan-Larcombe-French sequence $\{P_n\}_{n\geq 0}$ and the Fennessey-Larcombe-French sequence $\{V_n\}_{n\geq 0}$.

Recall that, for a given sequence $\{z_n\}_{n\geq 0}$ of positive real numbers, $\{z_n\}_{n\geq 0}$ is said to be *log-concave* (or *log-convex*) if $z_n^2 \geq z_{n-1}z_{n+1}$ (or $z_n^2 \leq z_{n-1}z_{n+1}$) for all $n \geq 1$ and $\{z_n\}_{n\geq 0}$ is said to be *log-balanced* if $\{z_n\}_{n\geq 0}$ is log-convex and $\{z_n/n!\}_{n\geq 0}$ is log-concave. Log-concavity and log-convexity are instrumental in obtaining the growth rate of a sequence and they are also sources of inequalities. It is clear that a sequence $\{z_n\}_{n\geq 0}$ is log-convex (log-concave) if and only if its quotient sequence $\{z_{n+1}/z_n\}_{n\geq 0}$ is nondecreasing (nonincreasing). For a log-balanced sequence, it is log-convex, but its quotient sequence does not grow too fast. For more properties of log-balanced sequences, see [2]. Zhao [17] proved that the Catalan-Larcombe-French $\{P_n\}_{n\geq 0}$ is log-balanced, and Yang and Zhao [15] showed that the Fennessey-Larcombe-French sequence $\{V_n\}_{n\geq 1}$ is log-concave. In the next section, we mainly discuss the log-balancedness of the sequences $\{P_n^2\}_{n\geq 0}$, $\{n!V_n\}_{n\geq 1}$ and $\{n!nV_n\}$, the log-convexity of $\{(n-1)!V_n\}_{n\geq 1}$ and the log-concavity of $\{V_n - P_n\}_{n\geq 2}$. In addition, we discuss the monotonicity of some sequences involving $\{P_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$.

2 The log-behavior of some sequences related to $\{P_n\}$ and $\{V_n\}$

In this section, we state and prove the main results of this paper.

Theorem 2.1 *For the Catalan-Larcombe-French sequence $\{P_n\}_{n\geq 0}$, the sequence $\{P_n^2\}_{n\geq 0}$ is log-balanced.*

Proof. For $n \geq 0$, let $x_n = \frac{P_{n+1}}{P_n}$ and $w_n = \frac{x_n^2}{n+1}$. In order to prove the log-concavity of $\{P_n^2/n!\}_{n\geq 0}$, we show that $\{w_n\}_{n\geq 0}$ is decreasing. It is obvious that

$$w_n - w_{n+1} = \frac{(n+2)x_n^2 - (n+1)x_{n+1}^2}{(n+1)(n+2)}.$$

For $n \geq 1$, let $\lambda_n = \frac{16n}{n+1}$. We note that (see [18])

$$\lambda_{n-1} < x_n < \lambda_n, \quad n \geq 4.$$

Then we obtain

$$\begin{aligned}(n+2)x_n^2 - (n+1)x_{n+1}^2 &> \frac{256(n+2)(n-1)^2}{n^2} - \frac{256(n+1)^3}{(n+2)^2} \\ &= \frac{256(n^4 - 2n^3 - 11n^2 - 4n + 8)}{n^2(n+2)^2} \\ &> 0, \quad (n \geq 5).\end{aligned}$$

This means that $w_n - w_{n+1} > 0$ for $n \geq 5$. We can verify that $w_n > w_{n+1}$ for $0 \leq n \leq 4$. Then $\{w_n\}_{n\geq 0}$ is decreasing. On the other hand, $\{P_n^2\}$ is log-convex. Hence $\{P_n^2\}_{n\geq 0}$ is log-balanced. ∎

Now we prove a lemma.

Lemma 2.1 *For the Fennessey-Larcombe-French sequence* $\{V_n\}$, *let* $y_n = V_{n+1}/V_n$ *for* $n \geq 1$. *Then* $16 < y_n < \frac{16(n+2)}{n+1}$ *for every* $n \geq 1$.

Proof. First of all, we have from (1.2) that

$$y_n = 8\left[\frac{3n+2}{n+1} - \frac{1}{(n+1)^2}\right] - \frac{128(n-1)}{ny_{n-1}}, \quad n \geq 1. \tag{2.1}$$

We next prove the conclusion by mathematical induction. Since

$$y_1 = 18, \quad y_2 = \frac{152}{9}, \quad y_3 = \frac{625}{38},$$

we have

$$16 < y_n < \frac{16(n+2)}{n+1}, \quad 1 \leq n \leq 3.$$

We now assume that $16 < y_n < \frac{16(n+2)}{n+1}$ for $n \geq 4$. By (2.1), we have

$$\begin{aligned} y_{n+1} - 16 &= 8 - \frac{8}{n+2} - \frac{8}{(n+2)^2} - \frac{128n}{(n+1)y_n} \\ &= \frac{8[(n+1)y_n(n^2+3n+1) - 16n(n+2)^2]}{(n+1)(n+2)^2 y_n} \\ &\geq \frac{128}{(n+1)(n+2)^2 y_n} \\ &> 0 \end{aligned}$$

and

$$\begin{aligned} y_{n+1} - \frac{16(n+3)}{n+2} &= \frac{8n-8}{n+2} - \frac{8}{(n+2)^2} - \frac{128n}{(n+1)y_n} \\ &< \frac{8n-8}{n+2} - \frac{8}{(n+2)^2} - \frac{8n}{n+2} \\ &< 0. \end{aligned}$$

Hence $16 < y_n < \frac{16(n+2)}{n+1}$ for every $n \geq 1$. ∎

Theorem 2.2 *For the Fennessey-Larcombe-French sequence $\{V_n\}_{n\geq 0}$, the sequences $\{n!nV_n\}_{n\geq 1}$ and $\{n!V_n\}_{n\geq 1}$ are log-balanced, and the sequence $\{(n-1)!V_n\}_{n\geq 1}$ is log-convex.*

Proof. The log-concavity of $\{V_n\}_{n\geq 1}$ and $\{nV_n\}_{n\geq 1}$ has been verified in [15]. In order to prove the log-balancedness of $\{n!nV_n\}_{n\geq 1}$ and $\{n!V_n\}_{n\geq 1}$, it is sufficient to prove that $\{n!nV_n\}_{n\geq 1}$ and $\{n!V_n\}_{n\geq 1}$ are log-convex. In fact, since $\{1/n\}_{n\geq 1}$ is log-convex, we only show that $\{n!nV_n\}_{n\geq 1}$ is log-convex.

For $n \geq 1$, let $U_n = nV_n$. It follows from (1.2) that

$$U_{n+1} = \left(24 + \frac{8}{n+1} + \frac{8}{n}\right)U_n - \frac{128(n+1)}{n}U_{n-1}, \quad n \geq 2.$$

For $n \geq 1$, put $T_n = n!nV_n$. Then we have

$$T_{n+1} = \left[24(n+1) + 8 + \frac{8(n+1)}{8n}\right]T_n - 128(n+1)^2T_{n-1}, \quad n \geq 2. \tag{2.2}$$

For $n \geq 1$, set $r_n = T_{n+1}/T_n$. By (2.2), we can derive

$$r_n = 24(n+1) + 8 + \frac{8(n+1)}{n} - \frac{128(n+1)^2}{r_{n-1}}, \quad n \geq 2. \tag{2.3}$$

By means of Lemma 2.1, we have

$$r_n = \frac{(n+1)^2V_{n+1}}{nV_n} > 16(n+1). \tag{2.4}$$

In order to prove the log-convexity of $\{T_n\}_{n\geq 1}$, we next prove by induction that $\{r_n\}_{n\geq 1}$ is increasing. In fact, it is obvious that $r_1 < r_2$. For $n \geq 2$, assume now that $r_{n-1} < r_n$. It follows from (2.3) and (2.4) that

$$\begin{aligned} r_{n+1} - r_n &= 24 - \frac{8}{n(n+1)} + 128(n+1)^2\left(\frac{1}{r_{n-1}} - \frac{1}{r_n}\right) \\ &\quad -\frac{128(2n+3)}{r_n} \\ &> 24 - \frac{8}{n(n+1)} - \frac{8(2n+3)}{n+1} \\ &= \frac{8n^2-8}{n(n+1)} \\ &> 0. \end{aligned}$$

Then $\{r_n\}_{n\geq 1}$ is increasing. Naturally, $\{T_n\}_{n\geq 1} = \{n!nV_n\}_{n\geq 1}$ is log-convex. On the other hand, since the sequences $\{1/n\}_{n\geq 1}$ and $\{n!nV_n\}_{n\geq 1}$ are log-convex, the sequences $\{n!V_n\}_{n\geq 1}$ and $\{(n-1)!V_n\}_{n\geq 1}$ are log-convex. It follows from the definition of the log-balancedness that the sequences $\{n!nV_n\}_{n\geq 1}$ and $\{n!V_n\}_{n\geq 1}$ are log-balanced. ■

For the Catalan-Larcombe-French sequence $\{P_n\}_{n\geq 0}$ and the Fennessey-Larcombe-French sequence $\{V_n\}_{n\geq 0}$, Jarvis, Larcombe, and French [5] prove that

$$V_n > P_n, \quad n \geq 2.$$

Now we discuss the log-behavior of some sequences related to $V_n - P_n$.

Theorem 2.3 *For the Catalan-Larcombe-French sequence $\{P_n\}_{n\geq 0}$ and the Fennessey-Larcombe-French sequence $\{V_n\}_{n\geq 0}$, the sequence $\{V_n - P_n\}_{n\geq 2}$ is log-concave and the sequence $\{n!(V_n - P_n)\}_{n\geq 4}$ is log-balanced.*

Proof. There is a formula between the Catalan-Larcombe-French sequence $\{P_n\}_{n\geq 0}$ and the Fennessey-Larcombe-French sequence $\{V_n\}_{n\geq 0}$ (see [5]):

$$V_n = (2n+1)P_n - 16nP_{n-1}, \quad n \geq 1.$$

Then we have

$$V_n - P_n = 2n(P_n - 8P_{n-1}), \quad n \geq 1.$$

For $n \geq 2$, let

$$x_n = \frac{P_{n+1}}{P_n}, \quad s_n = \frac{(n+1)(P_{n+1} - 8P_n)}{n(P_n - 8P_{n-1})},$$
$$\text{and} \quad t_n = \frac{(n+1)(V_{n+1} - P_{n+1})}{V_n - P_n}.$$

Then we get

$$s_n = \frac{(n+1)(x_n - 8)x_{n-1}}{n(x_{n-1} - 8)}, \quad t_n = (n+1)s_n.$$

From (1.1) we derive

$$x_n = \frac{8(3n^2 + 3n + 1)}{(n+1)^2} - \frac{128n^2}{(n+1)^2 x_{n-1}}. \tag{2.5}$$

By applying (2.5), we have

$$\begin{aligned} s_n &= \frac{8[(2n+1)x_{n-1} - 16n]}{(n+1)(x_{n-1} - 8)}, \\ t_n &= 16n + \frac{8x_{n-1}}{x_{n-1} - 8}. \end{aligned}$$

Through computation, we obtain

$$\begin{aligned} \frac{s_{n+1}}{8} - \frac{s_n}{8} &= \frac{x_{n-1}x_n - (8n+24)x_n + 8nx_{n-1} + 128}{(n+1)(n+2)(x_{n-1} - 8)(x_n - 8)} \\ &= \frac{x_n(x_{n-1} - 16) - 8(x_n - 16) - 8n(x_n - x_{n-1})}{(n+1)(n+2)(x_{n-1} - 8)(x_n - 8)}, \\ \frac{t_{n+1}}{8} - \frac{t_n}{8} &= 2 + \frac{x_n}{x_n - 8} - \frac{x_{n-1}}{x_{n-1} - 8} \\ &= 3 + \frac{8}{x_n - 8} - \frac{x_{n-1}}{x_{n-1} - 8}. \end{aligned}$$

It follows from (2.5) that $\{x_n\}$ is bounded. On the other hand, $\{x_n\}$ is increasing (see [17]). Then $\lim_{n \to +\infty} x_n$ exists. From (2.5) we derive that $\lim_{n \to +\infty} x_n = 16$. Then we have

$$8 < x_n < 16 \quad (n \geq 1) \quad \text{and} \quad 12 < x_n < 16 \quad (n \geq 3).$$

Therefore, we obtain

$$\begin{aligned}\frac{s_{n+1}}{8}-\frac{s_n}{8} &< \frac{(x_n-8)(x_n-16)-8n(x_n-x_{n-1})}{(n+1)(n+2)(x_{n-1}-8)(x_n-8)} \\ &< 0, \quad (n\geq 2), \\ \frac{t_{n+1}}{8}-\frac{t_n}{8} &= \frac{2(x_{n-1}-12)}{x_{n-1}-8}+\frac{8}{x_n-8} \\ &> 0, \quad (n\geq 4).\end{aligned}$$

Then $\{s_n\}_{n\geq 2}$ is decreasing and $\{t_n\}_{n\geq 4}$ is increasing. Hence the sequence $\{V_n-P_n\}_{n\geq 2}$ is log-concave and the sequence $\{n!(V_n-P_n)\}_{n\geq 4}$ is log-convex. Then $\{n!(V_n-P_n)\}_{n\geq 4}$ is log-balanced. ■

Recently, Sun presented a series of conjectures on monotonicity of combinatorial sequences of the types $\{\sqrt[n]{z_n}\}$ and $\{\sqrt[n+1]{z_{n+1}}/\sqrt[n]{z_n}\}$ in [12], where $\{z_n\}$ is a sequence of positive integers. Many conjectures of Sun [12] have been confirmed. See [1,3,13,14,17,18]. In the final of this section, we discuss the monotonicity of some sequences involving the Catalan-Larcombe-French sequence $\{P_n\}_{n\geq 0}$ and the Fennessey-Larcombe-French sequence $\{V_n\}_{n\geq 0}$. The following results given by Wang and Zhu in [13] will be useful:

Lemma 2.2 *Let $\{z_n\}_{n\geq 0}$ be a sequence of positive numbers. Assume that $\{z_n\}_{n\geq N}$ is log-convex (log-concave) and $\sqrt[N]{z_N}<\sqrt[N+1]{z_{N+1}}$ ($\sqrt[N]{z_N}>\sqrt[N+1]{z_{N+1}}$) for some $N\geq 1$. Then $\{\sqrt[n]{z_n}\}_{n\geq N}$ is strictly increasing (decreasing).*

Lemma 2.3 *Let $\{z_n\}_{n\geq 0}$ be a sequence of positive numbers. Assume that $\{z_n\}_{n\geq 0}$ is log-concave and $z_0\geq 1$. Then the sequence $\{\sqrt[n]{z_n}\}_{n\geq 1}$ is decreasing.*

Theorem 2.4 *For $n\geq 1$, the sequences $\{\sqrt[n]{n!nV_n}\}$, $\{\sqrt[n]{n!V_n}\}$ and $\{\sqrt[n]{(n-1)!V_n}\}$ are strictly increasing, and the sequence $\{\sqrt[n]{V_{n+2}-P_{n+2}}\}$ is strictly decreasing.*

Proof. We note that

$$V_1 < \sqrt{4V_2}, \quad V_1 < \sqrt{2V_2}, \quad \text{and} \quad V_1 < \sqrt{V_2}.$$

It follows from Theorem 2.2 and Lemma 2.2 that the sequences $\{\sqrt[n]{n!nV_n}\}_{n\geq 1}$, $\{\sqrt[n]{n!V_n}\}_{n\geq 1}$ and $\{\sqrt[n]{(n-1)!V_n}\}_{n\geq 1}$ are strictly increasing.

For $n \geq 0$, let $z_n = V_{n+2} - P_{n+2}$. Clearly, $z_0 > 1$. It follows from Theorem 2.3 and Lemma 2.3 that the sequence $\{\sqrt[n]{V_{n+2} - P_{n+2}}\}_{n\geq 1}$ is strictly decreasing. ■

In fact, we can also give the approximate values of $\{\sqrt[n]{n!nV_n}\}_{n\geq 1}$, $\{\sqrt[n]{n!V_n}\}_{n\geq 1}$ and $\{\sqrt[n]{(n-1)!V_n}\}_{n\geq 1}$ when $n \to +\infty$. Now we give the approximate value of $\{\sqrt[n]{n!V_n}\}_{n\geq 1}$ when $n \to +\infty$. For $n \geq 1$, let $y_n = V_{n+1}/V_n$. It follows from (2.1) that

$$\lim_{n\to+\infty} y_n = 16.$$

Noting that

$$\begin{aligned} \sqrt[n]{V_n} &= \sqrt[n]{y_{n-1}y_{n-2}\cdots y_1 V_1} \\ &= e^{\frac{1}{n}\ln(y_{n-1}y_{n-2}\cdots y_1 V_1)}, \end{aligned}$$

we have

$$\lim_{n\to+\infty} \sqrt[n]{V_n} = 16.$$

On the other hand,

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}, \quad (n \to +\infty),$$

we obtain

$$\sqrt[n]{n!V_n} \sim \frac{16n}{e}, \quad (n \to +\infty).$$

Remark: In this section, for the Catalan-Larcombe-French sequence $\{P_n\}_{n\geq 0}$ and the Fennessey-Larcombe-French sequence $\{V_n\}_{n\geq 0}$, we give some results for $\{P_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$. In fact, we mainly discuss the log-behavior of some sequences related to $\{P_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$. Now we cannot give enumerative context for the Catalan-Larcombe-French sequence and the Fennessey-Larcombe-French sequence, and this might be our future work.

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