The Herscovici's conjecture for $C_m \square C_n$ *

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Abstract. Given a distribution of pebbles on the vertices of a connected graph G, a pebbling move on G consists of taking two pebbles off one vertex and placing one on an adjacent vertex. The *t-pebbling number* $\pi_t(G)$ is the smallest positive integer such that for every distribution of $\pi_t(G)$ pebbles and every vertex v, t pebbles can be moved to v. For $t = 1$, Graham conjectured that $\pi_1(G \square H) \leq \pi_1(G)\pi_1(H)$ for any connected graphs G and H, where $G \square H$ denotes the Cartesian product of G and H. Herscovici further conjectured that $\pi_{st}(G \square H) \leq \pi_s(G)\pi_t(H)$ for any positive integers s and t. Lourdusamy [A. Lourdusamy, t-pebbling the product of graphs, Acta Ciencia Indica, XXXII(1)(2006), 171–176] also conjectured that $\pi_t(C_m \square C_n) \leq \pi_1(C_m)\pi_t(C_n)$ for cycles C_m and C_n. In this paper, we show that $\pi_{st}(C_m \square C_n) \leq \pi_s(C_m)\pi_t(C_n)$, which confirms this conjecture due to Lourdusamy.

Keywords: pebbling number, Graham's conjecture, Herscovici's conjecture.

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1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For a graph G, let D be a distribution of pebbles on the vertices of G, or a *distribution* on G. For any vertex v of G, $D(v)$ denotes the number of pebbles on v in D. For $S \subseteq V(G)$, we denote $D(S) = \sum_{v \in S} D(v)$ and $p = \sum_{v \in V(G)} D(v)$. A *pebbling move* consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. For $v \in V(G)$, *the pebbling number of* v *in* G is the smallest number m such that from every distribution of m pebbles on G, we can move a pebble to v by

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a sequence of pebbling moves. This m is denoted by $\pi(G, v)$. The *pebbling number of* G, denoted by $\pi(G)$, is the smallest number m such that from every distribution of m pebbles on G, it is possible to move a pebble to any vertex by a sequence of pebbling moves. Similarly, the *t-pebbling number of* v *in* G is the smallest number m such that from every distribution of m pebbles on G, we can move t pebbles to v by a sequence of pebbling moves. This m is denoted by $\pi_t(G, v)$. The *t-pebbling number of* G, denoted by $\pi_t(G)$, is the smallest number m such that from every distribution of m pebbles on G, it is possible to move t pebbles to any vertex by a sequence of pebbling moves. Clearly, $\pi_t(G) = \max\{\pi_t(G, v) | v \in V(G)\}$, $\pi(G) = \pi_1(G)$, and $\pi_t(P_n) = t2^{n-1}$, where P_n is the path on n vertices.

We say that a graph G has the *2t-pebbling property* if for any distribution with more than $2\pi_t(G) - q$ pebbles, where q is the number of vertices with at least one pebble, it is possible, using pebbling moves, to get $2t$ pebbles to any vertex. Lourdusamy et al. [7–10] showed that the even cycle, the star graph, the n-cube, the complete graph, the complete r-partite graph, the fan graph and the wheel graph have the $2t$-pebbling property. Gao and Yin [2] showed that the tree graph and C_5 have the $2t$-pebbling property.

The Cartesian product of graphs G and H is denoted by $G\Box H$. The following well-known conjecture first appeared in [1].

Conjecture 1.1 (Graham [1]). *$\pi(G\Box H) \leq \pi(G)\pi(H)$ for any connected graphs G and H.*

Many articles (see, e.g., [1,5,11,12]) have given evidences supporting Conjecture 1.1. Snevily and Foster [12] proved $\pi(C_m\Box C_n) \leq \pi(C_m)\pi(C_n)$ when $m \geq 11$ or $n \geq 11$ except for the cases $C_4\Box C_{11}$ and $C_6\Box C_{11}$. Herscovici [5] improved this result as follows:

Theorem 1.1 ([5]). *$\pi(C_m\Box C_n) \leq \pi(C_m)\pi(C_n)$ for cycles C_m and C_n.*

Lourdusamy [7] extended Graham's conjecture (Conjecture 1.1) as follows:

Conjecture 1.2 (Lourdusamy [7]). *$\pi_t(G\Box H) \leq \pi(G)\pi_t(H)$ for any connected graphs G and H.*

Lourdusamy et al. [7-10] showed that Conjecture 1.2 holds when G is a fan graph, or a wheel graph, or a complete graph, or a complete multi-partite graph, or a path, or a star and H is a graph having the $2t$-pebbling property. Lourdusamy in [7] also further posed the following conjecture.

Conjecture 1.3 (Lourdusamy [7]). *$\pi_t(C_m\Box C_n) \leq \pi(C_m)\pi_t(C_n)$ for cycles C_m and C_n.*

Herscovici in [6] further extended Conjecture 1.2 as follows:

Conjecture 1.4 (Herscovici [6]). *$\pi_{st}(G\Box H) \leq \pi_s(G)\pi_t(H)$ for any connected graphs G and H.*

Conjecture 1.4 is a symmetric version of Conjecture 1.2. Gao and Yin in [2] proved that if G is a tree and H has the $2t$-pebbling property, then

Conjecture 1.4 holds. That is the following theorem:

Theorem 1.2 ([2]). *Let T be a tree and H be a graph having the $2t$-pebbling property. For all positive integers s and t, and all vertices $x \in V(T)$ and $y \in V(H)$, we have $\pi_{st}(T\square H,(x,y)) \le \pi_s(T,x)\pi_t(H)$. In particular, if P_n is the path on n vertices, then $\pi_{st}(P_n\square H) \le s2^{n-1}\pi_t(H)$.*

Hao et al. [4] also proved that Conjecture 1.4 is true when G is a thorn graph of a complete graph and H is a graph having the $2t$-pebbling property. Gao and Yin [3] proved that $\pi_t(C_5\square C_5) = 16t+7$. Clearly, $\pi_{st}(C_5\square C_5) \le \pi_s(C_5)\pi_t(C_5)$ by $\pi_t(C_5) = 4t+1$ ([5]) and $16st+7 \le (4s+1)(4t+1)$.

In this paper, we prove that Conjecture 1.4 is true for the cartesian product of two cycles, which confirms Conjecture 1.3 completely, that is the following theorem:

Theorem 1.3. $\pi_{st}(C_m\square C_n) \le \pi_s(C_m)\pi_t(C_n)$ *for cycles* C_m *and* C_n.

2. Proof of Theorem 1.3

We now present some lemmas that will be used in the proof of Theorem 1.3.

Lemma 2.1. *([5])* $\pi_t(C_{2k+1}) = \frac{2^{k+2}-(-1)^k}{3} + 2^k(t-1)$ *and* $\pi_t(C_{2k}) = t2^k$. *In particular,* $\pi_t(C_5) = 4t+1$.

Lemma 2.2. *If* $t \ge 2$, *then* $\pi_t(C_{2k+1}) > 2^{k+1}$.

Proof. We have that

$$\begin{aligned}
\pi_t(C_{2k+1}) - 2^{k+1} &= \tfrac{2^{k+2}-(-1)^k}{3} + 2^k(t-1) - 2^{k+1}\\
&\ge \frac{2^{k+2}-(-1)^k+3(2^k-2^{k+1})}{3}\\
&= \frac{2^k-(-1)^k}{3}\\
&> 0,
\end{aligned}$$

then $\pi_t(C_{2k+1}) > 2^{k+1}$. □

Lemma 2.3. *A cycle has the $2t$-pebbling property.*

Proof. Lourdusamy [7] proved that C_{2n} has the $2t$-pebbling property. Now, we prove that C_{2n+1} has the $2t$-pebbling property. Let $C_{2n+1} = x_0x_1x_2\ldots x_{n-1}x_nx_{n+1}\ldots x_{2n-1}x_{2n}x_0$. We also let $q(x_i)$ be 1 if x_i is occupied, and 0 otherwise.

C_3 has the $2t$-pebbling property as C_3 is isomorphic to K_3 and K_3 has the $2t$-pebbling property. Gao and Yin [2] showed that C_5 has the $2t$-pebbling property. We can assume that $n \ge 3$. Let D be a distribution on C_{2n+1} with $2\pi_t(C_{2n+1}) - q + 1$ pebbles, where q is the number of vertices with at least one pebble. Without loss of generality, let x_0 be the root vertex and $p(x_0) = \ell$ for $0 \le \ell \le 2t-1$. Then we have that

$$\begin{aligned}
&D(C_{2n+1} - x_0)\\
&\geq 2\pi_t(C_{2n+1}) - q + 1 - \ell\\
&\geq 2(\tfrac{2^{n+2}-(-1)^n}{3} + 2^n(t-1)) + 1 - q - \ell\\
&= 2 \times \tfrac{2^{n+2}-(-1)^n}{3} + 2^n(2t - \ell - 1) + (\ell - 1)2^n + 1 - q - \ell.
\end{aligned}$$

Note that $\frac{2^{n+2}-(-1)^n}{3} \geq 2^n + 2^{n-2} + 1$ for $n \geq 3$. We get

$$D(C_{2n+1} - x_0) \geq \frac{2^{n+2} - (-1)^n}{3} + 2^n(2t - \ell - 1) + \ell 2^n + 2^{n-2} + 2 - q - \ell.$$

If $\ell \geq 1$, then $\ell 2^n + 2^{n-2} + 2 - q - \ell \geq \ell(2^n - 1) + 2^{n-2} + 2 - (2n+1) \geq 0$, that is $D(C_{2n+1} - x_0) \geq \frac{2^{n+2}-(-1)^n}{3} + 2^n(2t - \ell - 1)$, hence we can move an additional $2t - \ell$ pebbles to x_0.

We may assume that $\ell = 0$, that is $q(x_0) = 0$. Similarly, we have that

$$p \geq \frac{2^{n+2} - (-1)^n}{3} + 2^n(2t - 1) + 2^{n-2} + 2 - q.$$

If $q \leq 2n - 2$, then $2^{n-2} + 2 - q \geq 2^{n-2} + 4 - 2n \geq 0$. Therefore we can move $2t$ pebbles to x_0. If $n \geq 5$, then $2^{n-2} + 2 - q \geq 2^{n-2} + 2 - 2n \geq 0$. We also can move $2t$ pebbles to x_0. Now assume that $q \geq 2n - 1$ and $n \leq 4$.

If $q = 2n$, then $q(x_i) = 1$ for $x_i \in \{x_1, x_2, \ldots, x_{2n}\}$. So we can move one pebble to x_0 at a cost of at most $n - 1 + 2 = n + 1$ pebbles, and the remaining

$$\begin{aligned}
&\tfrac{2^{n+2}-(-1)^n}{3} + 2^n(2t-1) + 2^{n-2} + 2 - q - (n+1)\\
&= \tfrac{2^{n+2}-(-1)^n}{3} + 2^n(2t-2) + 2^n + 2^{n-2} + 1 - 3n\\
&\geq \tfrac{2^{n+2}-(-1)^n}{3} + 2^n(2t-2)
\end{aligned}$$

pebbles on $V(C_{2n+1}) - x_0$ are sufficient to put an additional $2t - 1$ pebbles on x_0.

If $q = 2n - 1, n = 3$, then we can move one pebble to x_0 at a cost of at most 5 pebbles, and the remaining

$$\begin{aligned}
\tfrac{2^5-(-1)^3}{3} + 2^3(2t-1) + 2^{3-2} + 2 - 5 - 5 &= \tfrac{2^5-(-1)^3}{3} + 2^3(2t-2) + 2\\
&\geq \tfrac{2^5-(-1)^3}{3} + 2^3(2t-2)
\end{aligned}$$

pebbles on $V(C_7) - x_0$ are sufficient to put an additional $2t - 1$ pebbles on x_0.

If $q = 2n - 1, n = 4$, then we can move one pebble to x_0 at a cost of at most 7 pebbles, and the remaining

$$\begin{aligned}\frac{2^6-(-1)^4}{3} + 2^4(2t-1) + 2^{4-2} + 2 - 7 - 7 &= \frac{2^6-(-1)^4}{3} + 2^4(2t-2) + 8 \\ &\geq \frac{2^6-(-1)^4}{3} + 2^4(2t-2)\end{aligned}$$

pebbles on $V(C_9) - x_0$ are sufficient to put an additional $2t - 1$ pebbles on x_0. Therefore, we can move $2t$ pebbles to any specified target vertex. □

Let $C_n = x_0x_1x_2 \ldots x_{n-2}x_{n-1}x_0$ and let $C_m = y_0y_1y_2 \ldots y_{m-2}y_{m-1}y_0$.

Lemma 2.4. *Suppose that D is a distribution of pebbles on $C_m \square C_n$ with $p = \pi_s(C_m)\pi_t(C_n)$ pebbles. Let $x \in V(C_n)$ and $y \in V(C_m)$.*

(1) If $t \geq 2$, then we can move s pebbles to (y_0, x_0) using $2^{\lfloor \frac{n}{2} \rfloor}\pi_s(C_m)$ pebbles;

(2) If $s \geq 2$, then we can move t pebbles to (y_0, x_0) using $2^{\lfloor \frac{m}{2} \rfloor}\pi_t(C_n)$ pebbles.

Proof. (1) We write A and B for the subgraphs induced by the vertex sets $\{x_0, x_1, \ldots, x_{\lfloor \frac{n}{2} \rfloor}\}$ and $\{x_{\lfloor \frac{n+1}{2} \rfloor}, \ldots, x_{n-2}, x_{n-1}, x_0\}$, respectively. Assume that $n = 2k + 1$. Then we have that $D(C_m \square A) \geq 2^k\pi_s(C_m)$ or $D(C_m \square B) \geq 2^k\pi_s(C_m)$. Otherwise, if $D(C_m \square A) < 2^k\pi_s(C_m)$ and $D(C_m \square B) < 2^k\pi_s(C_m)$, then we have

$$p \leq D(C_m \square A) + D(C_m \square B) < 2^{k+1}\pi_s(C_m).$$

By Lemma 2.2, $p < \pi_t(C_{2k+1})\pi_s(C_m)$, which contradicts $p = \pi_s(C_m)\pi_t(C_{2k+1})$. Let $D(C_m \square A) \geq 2^k\pi_s(C_m)$. By Theorem 1.2 and Lemma 2.3, we can move s pebbles to (y_0, x_0) using $2^k\pi_s(C_m)$ pebbles.

Assume that $n = 2k$. If $D(C_m \square A) < 2^k\pi_s(C_m)$ and $D(C_m \square B) < 2^k\pi_s(C_m)$, then we have

$$p \leq D(C_m \square A) + D(C_m \square B) < 2^{k+1}\pi_s(C_m) \leq t2^k\pi_s(C_m),$$

which contradicts $p = \pi_s(C_m)\pi_t(C_{2k}) = t2^k\pi_s(C_m)$. If $D(C_m \square A) \geq 2^k\pi_s(C_m)$ or $D(C_m \square B) \geq 2^k\pi_s(C_m)$, then we can move s pebbles to (y_0, x_0) using $2^k\pi_s(C_m)$ pebbles.

The argument of (2) is similar. □

We first prove the case $s = 1$ of Theorem 1.3. That is the following Theorem 2.1.

Theorem 2.1. *$\pi_t(C_m \square C_n) \leq \pi(C_m)\pi_t(C_n)$ for cycles C_m and C_n.*

Proof. Suppose that D is a distribution of pebbles on the vertices of $C_m \square C_n$ and $p = \pi(C_m)\pi_t(C_n)$. We use induction on t. By Theorem 1.1, the result is true for $t = 1$. Now, assume that the result is true for $t \leq \ell$ ($\ell \geq 1$). We will prove that the result is true for $t = \ell + 1$. Without loss of generality, we assume that the target vertex is (y_0, x_0).

Assume that $n = 2k+1$. Then by Lemma 2.4 (1), we can move one pebble to (y_0, x_0) using $2^k\pi(C_m)$ pebbles, there are now

$$\begin{aligned}
&\pi(C_m)\pi_{\ell+1}(C_{2k+1}) - 2^k\pi(C_m)\\
&= \pi(C_m)(\tfrac{2^{k+2}-(-1)^k}{3} + 2^k(\ell+1-1) - 2^k)\\
&= \pi(C_m)(\tfrac{2^{k+2}-(-1)^k}{3} + 2^k(\ell-1))\\
&= \pi(C_m)\pi_\ell(C_{2k+1})
\end{aligned}$$

pebbles on $C_m \square C_{2k+1} - (y_0, x_0)$. By the induction hypothesis, the remaining $\pi(C_m)\pi_\ell(C_{2k+1})$ pebbles on $C_m \square C_{2k+1} - (y_0, x_0)$ are sufficient to move an additional ℓ pebbles to (y_0, x_0).

Assume that $n = 2k$. Then we can move one pebble to (y_0, x_0) using $2^k\pi(C_m)$ pebbles by Lemma 2.4 (1), By the induction hypothesis, the remaining

$$\pi(C_m)\pi_{\ell+1}(C_{2k}) - 2^k\pi(C_m) = \pi(C_m)((\ell+1)2^k - 2^k) = \pi(C_m)\pi_\ell(C_{2k})$$

pebbles on $C_m \square C_{2k} - (y_0, x_0)$ are sufficient to move an additional ℓ pebbles to (y_0, x_0). The proof of Theorem 2.1 is completed. □

Proof of Theorem 1.3. Suppose that D is a distribution of pebbles on $C_m \square C_n$ with $p = \pi_s(C_m)\pi_t(C_n)$ pebbles. We use induction on s. By Theorem 2.1, the result is true for $s = 1$. Now, assume that the result is true for $s \leq \ell$ ($\ell \geq 1$). We will prove that the result is true for $s = \ell + 1$. Without loss of generality, we assume that the target vertex is (y_0, x_0).

Assume that $m = 2k+1$. Then we can move t pebbles to (y_0, x_0) using $2^k\pi_t(C_n)$ pebbles by Lemma 2.4 (2). By the induction hypothesis, the remaining

$$\begin{aligned}
&\pi_{\ell+1}(C_{2k+1})\pi_t(C_n) - 2^k\pi_t(C_n)\\
&= (\tfrac{2^{k+2}-(-1)^k}{3} + 2^k(\ell+1-1) - 2^k)\pi_t(C_n)\\
&= (\tfrac{2^{k+2}-(-1)^k}{3} + 2^k(\ell-1))\pi_t(C_n)\\
&= \pi_\ell(C_{2k+1})\pi_t(C_n)
\end{aligned}$$

pebbles on $C_{2k+1} \square C_n - (y_0, x_0)$ are sufficient to move an additional $\ell \cdot t$ pebbles to (y_0, x_0). Therefore, a total of $t + \ell \cdot t = (\ell+1)t$ pebbles can be moved to (y_0, x_0).

Assume that $m = 2k$. Then we can move t pebbles to (y_0, x_0) using $2^k\pi_t(C_n)$ pebbles by Lemma 2.4 (2). By the induction hypothesis, the remaining

$$\begin{aligned}
\pi_{\ell+1}(C_{2k})\pi_t(C_n) - 2^k\pi_t(C_n) &= ((\ell+1)2^k - 2^k)\pi_t(C_n)\\
&= \ell \cdot 2^k\pi_t(C_n)\\
&= \pi_\ell(C_{2k})\pi_t(C_n)
\end{aligned}$$

pebbles on $C_{2k}\Box C_n-(y_0,x_0)$ are sufficient to move an additional $\ell\cdot t$ pebbles to (y_0,x_0). Therefore, a total of $t+\ell\cdot t=(\ell+1)t$ pebbles can be moved to (y_0,x_0). The proof of Theorem 1.3 is completed. □

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