A Characterization for a graphic sequence to be potentially $K_{1^3,4}$-graphic*

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Abstract: A graphic sequence $\pi = (d_1, d_2, ..., d_n)$ is said to be potentially $K_{1^3,4}$-graphic if there is a realization of π containing $K_{1^3,4}$ as a subgraph, where $K_{1^3,4}$ is the $1 \times 1 \times 1 \times 4$ complete 4-partite graph. In this paper, we characterize the graphic sequences potentially $K_{1^3,4}$ and the result is simple. In addition, we apply this characterization to compute the values of $\sigma(K_{1^3,4}, n)$.

Keywords: Graph, Degree sequence, Potentially $K_{1^3,4}$-graphic .

1. Introduction

The set of all sequences $\pi = (d_1, d_2, ..., d_n)$ of non-negative, non-increasing integers with $d_1 \leq n - 1$ is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be graphic if it is the degree sequence of a simple graph G on n vertices, and the graph G is called a realization of π. If each term of a graphic sequence π is nonzero, then π is said to be positive graphic. The set of all positive graphic sequences in NS_n is denoted by GS_n. Given a graph H, a graphic sequence π is said to be potentially H-graphic, if there is a realization of π containing H as a subgraph. For $\pi \in NS_n$, denote $\sigma(\pi) = d_1 + d_2 + ... + d_n$. In this paper, we consider a simple characterization on potentially $K_{1^3,4}$-graphic sequence. In the following, the symbol x^y in a sequences means y consecutive terms, each equal to x, and the symbol $\sigma(H, n)$ is the smallest even integer that every positive sequence $\pi \in GS_n$ with $\sigma(\pi) \geq \sigma(H, n)$ is potentially H-graphic.

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Theorem 1.1 is a sufficient condition about potentially $K_{1^2,s}$-graphic sequences for $s \geq 2$.

Theorem 1.1 [1] Let $s \geq 2, n \geq 3s+1$ and $\pi = (d_1, d_2, ..., d_n) \in GS_n$. If $d_2 \geq s+1$ and $d_{s+2} \geq 2$, then π is potentially $K_{1^2,s}$-graphic.

Theorem 1.2 investigates the condition that yields potentially $K_{1^3,s}$-graphic sequences for $s = 1, 2, 3$.

Theorem 1.2 Let $\pi = (d_1, d_2, ..., d_n) \in GS_n$.

(1) [2] If $n \geq 9, d_4 \geq 3$ and $d_7 \geq 2$, then π is potentially $K_{1^3,1}$-graphic;

(2) [3] If $n \geq 11, d_3 \geq 4, d_5 \geq 3$ and $d_8 \geq 2$, then π is potentially $K_{1^3,2}$-graphic sequence;

(3) [4] If $n \geq 11, d_3 \geq 5, d_6 \geq 3$ and $d_9 \geq 2$, then π is potentially $K_{1^3,3}$-graphic sequence.

Theorem 1.3 is a sufficient condition about potentially $K_{1^3,s}$-graphic sequences for $s \geq 2$.

Theorem 1.3 [5] Let $s \geq 2, n \geq 7s-3$ and $\pi = (d_1, d_2, ..., d_n) \in GS_n$. If $d_3 \geq s+2, d_{s+3} \geq 3$ and $d_{3s+2} \geq 2$, then π is potentially $K_{1^3,s}$-graphic.

In this paper, we give a simple characterization on potentially $K_{1^3,4}$-graphic sequence, that is Theorem 1.4.

Theorem 1.4 Let $n \geq 7$ and $\pi = (d_1, d_2, ..., d_n) \in GS_n$ be a positive sequence. Then π is potentially $K_{1^3,4}$-graphic if and only if π satisfies the following conditions:

(1)$d_3 \geq 6, d_7 \geq 3$;

(2)$\pi \notin S$, the set S consists of the following sequences:

$(n-1, 7^3, 3^6, 1^{n-10})(n \geq 10)$, $(n-1, 6^2, 5, 3^4, 1^{n-8})(n \geq 8)$,
$(n-1, 7, 6^2, 3^5, 1^{n-9})(n \geq 9)$, $(n-1, 6^2, 3^6, 1^{n-9})(n \geq 9)$,
$(n-1, 6^3, 4, 3^3, 1^{n-8})(n \geq 8)$, $(n-1, 6^2, 3^5, 1^{n-8})(n \geq 8)$,
$(n-1, 6^3, 3^4, 2, 1^{n-9})(n \geq 9)$, $(n-2, 6^3, 3^4, 1^{n-8})(n \geq 8)$,

$n = 8$: $(6^6, 5^2), (6^5, 4^3), (6^5, 4, 3^2), (6^4, 5^4), (6^4, 5, 3^3), (6^4, 4^4), (6^4, 4^2, 3^2), (6^4, 4, 3^2, 2), (6^3, 5, 4, 3^3), (6^3, 5, 3^3, 2), (6^3, 4, 3^4), (6^3, 3^4, 2), (6^8)$,

$n = 9$: $(7^4, 4, 3^4), (7^3, 6, 3^5), (7^2, 6^2, 3^4, 2), (7^2, 6, 3^6), (7, 6^3, 3^5), (7, 6^3, 3^3, 2^2), (7, 6^2, 4, 3^5), (7, 6^2, 3^5, 2), (6^8, 4), (6^8, 2), (6^5, 3^3, 1), (6^4, 4^5), (6^4, 4, 3^4), (6^4, 4, 3^3, 1), (6^4, 3^4, 2), (6^4, 3^3, 2, 1), (6^3, 5, 3^5), (6^3, 5, 3^4, 1), (6^3, 4^6), (6^3, 4^2, 3^4), (6^3, 4, 3^4, 2), (6^3, 3^6), (6^3, 3^5, 1), (6^3, 3^4, 2^2), (6^9)$,

$n = 10$: $(8^2, 7^2, 3^6), (8, 7^3, 3^5, 2), (7^4, 3^6), (7^4, 3^5, 1), (7^4, 3^4, 2^2), (7, 6^2, 3^7), (7, 6^2, 3^6, 1), (6^4, 3^5, 1), (6^4, 3^4, 1^2), (6^3, 4^7), (6^3, 4, 3^6), (6^3, 4, 3^5, 1), (6^3, 3^6, 2), (6^3, 3^5, 2, 1)$,

$n = 11$: $(9, 8^3, 3^7), (8^4, 3^6, 2), (6^3, 3^7, 1), (6^3, 3^6, 1^2)$,

$n = 12$: $(9^4, 3^8)$.

As an application of Theorem 1.4, it can be used to find the values of $\sigma(K_{1^3,4}, n)$.

2. Proof of Theorem 1.4

To prove Theorem 1.4, some known results are needed.

Suppose $\pi = (d_1, d_2, ..., d_n) \in NS_n$, $1 \le k \le n$, and

$$\pi''_k = \begin{cases} (d_1 - 1, ..., d_{k-1} - 1, d_{k+1} - 1, ..., d_{d_k+1} - 1, d_{d_k+2}, ..., d_n), \\ if\ d_k \ge k, \\ (d_1 - 1, ..., d_{d_k} - 1, d_{d_k+1}, ..., d_{k-1}, d_{k+1}, ..., d_n), \\ if\ d_k < k. \end{cases}$$

Let $\pi'_k = (d'_1, d'_2, ..., d'_{n-1}), d'_1 \ge ... \ge d'_{n-1}$ is a rearrangement in non-increasing order of the $n-1$ terms of π''_k. π'_k is called the residual sequence obtained by laying off d_k from π. Obviously, π'_k is obtained from π by deleting d_k and decreasing d_k largest degrees from $d_1, d_2, ..., d_{k-1}, d_{k+1}, ..., d_n$ each by one unity.

Theorem 2.1 [6] Let $\pi = (d_1, d_2, ..., d_n) \in NS_n$ and $1 \le k \le n$. Then π is graphic if and only if π'_k is graphic.

Theorem 2.2 [7] If $\pi = (d_1, d_2, ..., d_n) \in NS_n$ has a realization G containing H as a subgraph, then there is a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

Let $\pi = (d_1, d_2, ..., d_n) \in NS_n$, we construct

$$\rho_1 = (d_2 - 1, d_3 - 1, ..., d_7 - 1, d_8^{(1)}, ..., d_n^{(1)})$$

obtained from π by removing d_1, decreasing the first d_1 remaining terms each by one unity, and then reordering the last $n - 7$ terms to be non-increasing. Suppose

$$\rho_2 = (d_3 - 2, d_4 - 2, ..., d_7 - 2, d_8^{(2)}, ..., d_n^{(2)})$$

obtained from ρ_1 by removing $d_2 - 1$, decreasing the first $d_2 - 1$ remaining terms each by one unity, and then reordering the last $n - 7$ terms to be non-increasing. Similarly, removing $d_3 - 2$ from ρ_2, decreasing the first $d_3 - 2$ remaining terms each by one unity, and then reordering the last $n - 7$ terms to be non-increasing, define

$$\rho_3 = (d_4 - 3, d_5 - 3, ..., d_7 - 3, d_8^{(3)}, ..., d_n^{(3)}).$$

Let $\pi = (d_1, d_2, ..., d_n) \in GS_n$. If π has a realization G with the vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ such that $d_G(v_i) = d_i$ for $1 \le i \le n$ and G contains $K_{1^r,s}$ such that

$$\{v_1\}, \{v_2\}, ..., \{v_r\} \text{ and } \{v_{r+1}, ..., v_{r+s}\}$$

are the $(r + 1)$-partite sets of the vertex set of $K_{1^r,s}$, then π is said to be potentially $A_{1^r,s}$-graphic.

Theorem 2.3 [8] Let $\pi \in NS_n, \pi$ is potentially $A_{1^r,s}$-graphic if and only if ρ_r is graphic.

For $r = 3$ and $s = 4$, π is potentially $A_{1^3,4}$-graphic if and only if ρ_3 is graphic.

Theorem 2.4 [4] Let $\pi = (3^x, 2^y, 1^z)$, where $x + y + z = n \geq 1$ and $\sigma(\pi)$ is even. Then $\pi \in GS_n$ if and only if $\pi \notin A$, where $A = \{(2), (2^2), (3, 1), (3^2), (3, 2, 1), (3^2, 2), (3^3, 1), (3^2, 1^2)\}$.

Theorem 2.5 [4] Let $\pi = (4^x, 3^y, 2^z, 1^m)$, where $x + y + z + m = n \geq 5, x \geq 1$ and $\sigma(\pi)$ is even. Then $\pi \in GS_n$ if and only if $\pi \notin B$, where $B = \{(4, 3^2, 1^2), (4, 3, 1^3), (4^2, 2, 1^2), (4^2, 3, 2, 1), (4^3, 1^2), (4^3, 2^2), (4^3, 3, 1), (4^4, 2), (4^2, 3, 1^3), (4^2, 1^4), (4^3, 2, 1^2), (4^4, 1^2), (4^3, 1^4)\}$.

Theorem 2.6 [9] Let $\pi = (5^x, 4^y, 3^z, 2^m, 1^n) \in NS_n$, where $x + y + z + m + n \geq 6, x \geq 1$ and $\sigma(\pi)$ is even. Then $\pi \in GS_n$ if and only if $\pi \notin C$, C is the following sequences:
$\{(5^5, 3), (5^5, 2, 1), (5^5, 1^3), (5^5, 1), (5^4, 4, 2), (5^4, 4, 1^2), (5^4, 3^2), (5^4, 3, 2, 1), (5^4, 3, 1^3), (5^4, 3, 1), (5^4, 2^3), (5^4, 2^2, 1^2), (5^4, 2^2), (5^4, 2, 1^4), (5^4, 2, 1^2), (5^4, 1^6), (5^4, 1^4), (5^4, 1^2), (5^3, 4^2, 1), (5^3, 4, 3, 2), (5^3, 4, 3, 1^2), (5^3, 4, 2^2, 1), (5^3, 4, 2, 1^3), (5^3, 4, 2, 1), (5^3, 4, 1^5), (5^3, 4, 1^3), (5^3, 3^2, 1), (5^3, 3, 2^2), (5^3, 3, 2, 1^2), (5^3, 3, 1^4), (5^3, 3, 1^2), (5^3, 2^3, 1), (5^3, 2^2, 1^3), (5^3, 2^2, 1), (5^3, 2, 1^5), (5^3, 2, 1^3), (5^3, 1^7), (5^3, 1^5), (5^3, 1^3), (5^2, 4^2, 3, 1), (5^2, 4^2, 2^2), (5^2, 4^2, 2, 1^2), (5^2, 4^2, 1^4), (5^2, 4^2, 1^2), (5^2, 4, 3, 2, 1), (5^2, 4, 3, 1^3), (5^2, 4, 2^3), (5^2, 4, 2^2, 1^2), (5^2, 4, 2, 1^4), (5^2, 4, 2, 1^2), (5^2, 4, 1^4), (5^2, 3^3, 1), (5^2, 3^2, 1^2), (5^2, 3, 2^2, 1), (5^2, 3, 2, 1^3), (5^2, 3, 1^5), (5^2, 3, 1^3), (5^2, 2^2, 1^2), (5^2, 2, 1^4), (5^2, 1^6), (5^2, 1^4), (5, 4^3, 2, 1), (5, 4^3, 1^3), (5, 4^2, 3, 1^2), (5, 4^2, 2^2, 1), (5, 4^2, 2, 1^3), (5, 4^2, 1^5), (5, 4^2, 1^3), (5, 4, 3, 2, 1^2), (5, 4, 3, 1^4), (5, 4, 2, 1^3), (5, 4, 1^5), (5, 3^2, 1^3), (5, 3, 1^4)\}$.

The proof of Theorem 1.4

The first step is to prove the necessary condition of Theorem 1.4. Assume that π is potentially $K_{1^3,4}$-graphic. (1) is obvious. The corresponding ρ_3 of S is $(3, 2, 1), (2^2), (2), (3^3, 1), (3^2, 2), (3^2), (3, 1), (3^2, 1^2), (4, 3^2, 1^2), (4, 3, 1^3), (4, 2, 1^2), (4^2, 3, 1^3), (4^2, 1^4), (4, 2^3), (4^3, 1^4), (5, 3^3, 2), (5, 3, 2^3), (5, 2, 1^3), (6, 3^4), (6^2, 3^4), (6, 4, 3^4), (6, 3^4, 2)$ or $(6, 2, 1^4)$, none of them is graphic.

To prove the sufficient condition, we use induction on n. Suppose that $n = 7$ and $\pi \in GS_n$ satisfies Theorem 1.4, then π is $(6^7), (6^5, 5^2), (6^4, 5^2, 4), (6^4, 4^3), (6^3, 5^4), (6^3, 5^3, 3), (6^3, 5^2, 4^2), (6^3, 5, 4^2, 3), (6^3, 4^4), (6^3, 4^2, 3^2)$ or $(6^3, 3^4)$. These sequences are all potentially $K_{1^3,4}$-graphic. Now assume that the sufficient condition holds for $n - 1 (n \geq 8)$ and prove that π is potentially $K_{1^3,4}$-graphic for n. We consider the following cases.

Case 1. $d_n \geq 6$

As $\pi \neq (6^8), (6^9)$, then the residual sequence $\pi'_n = (d'_1, d'_2, ..., d'_{n-1})$ obtained by laying off d_n from π satisfies $d'_3 \geq 6$ and $d'_7 \geq 3$. If $\pi'_n \notin S$, by the induction hypothesis π'_n is potentially $K_{1^3,4}$-graphic and so is π. If $\pi'_n \in S$, then π'_n is $(6^6, 5^2), (6^4, 5^4), (6^8)$ or (6^9). So the corresponding π is $(7^4, 6^5), (7^2, 6^7), (7^6, 6^3)$ or $(7^6, 6^4)$. It is easy to compute ρ_3 is $(5, 4^2, 3^3), (5^2, 3^4), (5, 4^4, 3)$ or $(5^3, 4^3, 3)$. So π is potentially $K_{1^3,4}$-graphic.

Case 2. $d_n = 5$

If $\pi'_n = (d'_1, d'_2, ..., d'_{n-1})$ satisfies Theorem 1.4, then π'_n is potentially $K_{1^3,4}$-graphic, and so is π.

If π'_n doesn't satisfy (1), then $d_3 = 6$ and $d_8 = 5$. Therefore, π must be $(d_1, d_2, 6, d_4, d_5, d_6, d_7, 5^{n-7})$.

If $d_1 + d_2 \le n + 4$, then $\rho_3 = (d_4 - 3, d_5 - 3, d_6 - 3, d_7 - 3, 5^{n+5-d_1-d_2}, 4^{d_1+d_2-12})$. According to Theorem 2.6, if ρ_3 is not $(5, 3, 2^3)$ or $(5, 3^3, 2)$, then ρ_3 is graphic. So π is potentially $K_{1^3,4}$-graphic. If $\rho_3 = (5, 3, 2^3), (5, 3^3, 2)$, the corresponding $\pi \in \{(6^4, 5^4), (6^6, 5^2)\} \subset S$.

If $d_1 + d_2 \ge n + 5$, then $\rho_3 = (4^x, 3^y, 2^z)(y + z \ge 4)$, which is graphic by Theorem 2.5 and Theorem 2.4, so π is potentially $K_{1^3,4}$-graphic.

Assume that π'_n doesn't satisfy (2), then π'_n is $(6^6, 5^2), (6^4, 5^4), (6^8)$ or (6^9). Hence the corresponding π is $(7^5, 6, 5^3), (7^4, 6^3, 5^2), (7^3, 6^5, 5), (7^4, 6, 5^4), (7^3, 6^3, 5^3), (7^2, 6^5, 5^2), (7, 6^7, 5), (7^5, 6^3, 5)$ or $(7^5, 6^4, 5)$. It is easy to compute ρ_3 is $(4^3, 3^2, 2), (4^2, 2^4), (4^2, 3^4), (4^2, 3^2, 2^2), (4, 3^4, 2), (5^2, 3^4), (4^4, 3^2)$ or $(5^2, 4^3, 3^2)$, all of them are graphic sequences.

Case 3. $d_n = 4$

If $\pi'_n = (d'_1, d'_2, ..., d'_{n-1})$ satisfies Theorem 1.4, then π'_n is potentially $K_{1^3,4}$-graphic, and so is π.

If π'_n doesn't satisfy (1), then $d_3 = 6$ and $4 \le d_7 \le 5$. The general form of π must be $(d_1, d_2, 6, d_4, d_5, d_6, 4, 4^{n-7})$ or $(d_1, d_2, 6, d_4, d_5, d_6, 5, 5^m, 4^{n-m-7})(n - m - 7 > 0)$.

If $d_7 = 4$ and $d_1 + d_2 \le n + 4$, then $\rho_3 = (d_4 - 3, d_5 - 3, d_6 - 3, d_7 - 3, 4^{n+5-d_1-d_2}, 3^{d_1+d_2-12})$. According to Theorem 2.5, if ρ_3 is not $(4, 3^2, 1^2), (4, 3, 1^3), (4^2, 3, 1^3), (4^2, 1^4)$ or $(4^3, 1^4)$, then ρ_3 is graphic. So π is potentially $K_{1^3,4}$-graphic. If $\rho_3 = (4, 3^2, 1^2), (4, 3, 1^3), (4^2, 3, 1^3), (4^2, 1^4), (4^3, 1^4)$, the corresponding $\pi \in \{(6^5, 4^3), (6^4, 4^4), (6^4, 4^5), (6^3, 4^6), (6^3, 4^7)\} \subset S$.

If $d_1 + d_2 \ge n + 5$, then $\rho_3 = (3^x, 2^y, 1^z)(y \ge 1, x + y + z \ge 5)$, which is graphic by Theorem 2.4, so π is potentially $K_{1^3,4}$-graphic.

If $d_7 = 5$ and $d_1 + d_2 \le m + 11$, then $\rho_3 = (d_4 - 3, d_5 - 3, d_6 - 3, d_7 - 3, 5^{m+12-d_1-d_2}, 4^{n-m+d_1+d_2-19})$, which is graphic by Theorem 2.6, so π is potentially $K_{1^3,4}$-graphic. If $d_1 + d_2 \ge m + 12$, then $\rho_3 = (4^x, 3^y, 2^z)(x + y + z \ge 5, z \ge 1)$, which is graphic by Theorem 2.5 and Theorem 2.4.

Assume that π'_n doesn't satisfy (2), then π'_n is $(6^8), (6^6, 5^2), (6^5, 4^3), (6^4, 5^4), (6^4, 4^4), (6^8, 4), (6^4, 4^5), (6^3, 4^6)$ or (6^9). As $\pi \ne (6^8, 4)$, the corresponding π is $(7^4, 6^4, 4), (7^3, 6^4, 5, 4), (7^2, 6^6, 4), (7^4, 6^2, 5^2, 4), (7^4, 6, 4^4), (7^4, 5^4, 4), (7^3, 6^2, 5^3, 4), (7^2, 6^4, 5^2, 4), (7, 6^6, 5, 4), (7^4, 4^5), (7^4, 6^4, 4^2), (7^4, 4^6), (7^3, 5, 4^6)$ or $(7^4, 6^5, 4)$. It is easy to compute ρ_3 is $(4^2, 3^4), (3^6), (4, 6^4, 2), (4, 3^2, 2, 1^2), (4, 3^2, 2^3), (4, 3, 2, 1^3), (3^4, 2^2), (4, 3^4, 2), (4^3, 3^4), (4, 3^3, 1^3), (3, 2, 1^3)$ or $(5, 4^3, 3^3)$, all of them are graphic sequences.

Case 4. $d_n = 3$

If $\pi'_n = (d'_1, d'_2, ..., d'_{n-1})$ satisfies Theorem 1.4, then π'_n is potentially $K_{1^3,4}$-graphic, and so is π.

If π'_n doesn't satisfy (1), then $d_3 = 6$ and $3 \leq d_6 \leq 5$. Therefore, π must be $(d_1, d_2, 6, d_4, d_5, d_6, 3, 3^{n-7})$, $(d_1, d_2, 6, d_4, d_5, d_6, 4, 4^m, 3^{n-7-m})$ or $(d_1, d_2, 6, d_4, d_5, d_6, 5, 5^m, 4^a, 3^{n-7-m-a})(n-m-7 > 0, n-7-m-a > 0)$.

If $d_7 = 3$ and $d_1 + d_2 \leq n + 4$, then $\rho_3 = (d_4 - 3, d_5 - 3, d_6 - 3, d_7 - 3, 3^{n+5-d_1-d_2}, 2^{d_1+d_2-12})$. According to Theorem 2.4, if ρ_3 is not $(3,1), (3^2), (3,2,1), (3^2,2), (3^3,1)$ or $(3^2,1^2)$, then ρ_3 is graphic. So π is potentially $K_{1^3,4}$-graphic. If $\rho_3 = (3,1), (3^2), (3,2,1), (3^2,2), (3^3,1), (3^2,1^2)$, the corresponding $\pi \in \{(7^2,6,3^6), (6^3,4,3^4), (6^3,3^6), (6^4,3^4), (6^3,5,4,3^3), (7,6^2,4,3^5), (7,6^2,3^7), (6^3,5,3^5), (6^4,5,3^3), (7,6^3,3^5), (6^3,4,3^6), (6^5,4,3^2), (6^4,4,3^4), (6^3,4^2,3^4), (6^4,4^2,3^2)\} \subset S$. If $d_1 + d_2 \geq n + 5$, then $\rho_3 = (d_4 - 3, d_5 - 3, d_6 - 3, d_7 - 3, 2^x, 1^y)(x + y \geq 1)$. According to Theorem 2.4, if ρ_3 is not $(2), (2^2)$ or $(3,2,1)$, then ρ_3 is graphic. So π is potentially $K_{1^3,4}$-graphic. If $\rho_3 = (2), (2^2), (3,2,1)$, the corresponding $\pi \in \{(7,6^2,3^5), (7,6^2,5,3^4), (8,6^2,3^6), (7,6^3,4,3^3), (8,7,6^2,3^5)\} \subset S$.

If $d_7 = 4$ and $d_1 + d_2 \leq m + 11$, then $\rho_3 = (d_4 - 3, d_5 - 3, d_6 - 3, d_7 - 3, 4^{m+12-d_1-d_2}, 3^{n-m+d_1+d_2-19})$, which is graphic by Theorem 2.5, so π is potentially $K_{1^3,4}$-graphic. If $d_1 + d_2 \geq m + 12$, then $\rho_3 = (3^x, 2^y, 1^z)(z \geq 1, x + y + z \geq 5)$, which is graphic by Theorem 2.4. Therefore, π is potentially $K_{1^3,4}$-graphic.

If $d_7 = 5$ and $d_1 + d_2 \leq m + 11$, then $\rho_3 = (d_4 - 3, d_5 - 3, d_6 - 3, d_7 - 3, 5^{m+12-d_1-d_2}, 4^{d_1+d_2-13}, 3^{n-7-m-a})$, which is graphic by Theorem 2.6, so π is potentially $K_{1^3,4}$-graphic. If $d_1 + d_2 \geq m + 12$, then $\rho_3 = (4^x, 3^y, 2^z, 1^w)(z \geq 2, x + y + z + w \geq 5)$, which is graphic by Theorem 2.5 and Theorem 2.4, so π is potentially $K_{1^3,4}$-graphic.

Assume that π'_n doesn't satisfy (2), then π'_n is $(6^8), (6^6,5^2), (6^5,4,3^2), (6^4,5^4), (6^4,5,3^3), (6^4,4^4), (6^4,4^2,3^2), (6^3,5,4,3^3), (6^3,4,3^4), (7,6^3,4,3^3), (7,6^2,5,3^4), (7,6^2,3^5), (6^4,3^4), (7^4,4,3^4), (7^3,6,3^5), (7^2,6,3^6), (7,6^3,3^5), (7,6^2,4,3^5), (6^8,4), (6^4,4^5), (6^4,4,3^4), (6^3,5,3^5), (6^3,4^6), (6^3,4^2,3^4), (6^3,3^6), (6^9), (8,7,6^2,3^5), (8,6^2,3^6), (8^2,7^2,3^6), (7^4,3^6), (7,6^2,3^7), (6^3,4^7), (6^3,4,3^6), (6^3,3^6,2), (9,8^3,3^7), (9^4,3^8)$ or $(9,7^3,3^6)$.

Since $\pi \neq (7^4,4,3^4), (8^2,7^2,3^6), (7^4,3^6), (9,7^3,3^6), (9,8^3,3^7), (9^4,3^8), (8,7,6^2,3^5)$, the corresponding π is $(7^3,6^5,3), (7^3,6^3,5^2,3), (7^2,6^5,5,3), (7,6^7,3), (7^3,6^2,4,3^3), (7^3,6,5^4,3), (7^2,6^3,5^3,3), (7,6^5,5^2,3), (6^7,5,3), (7^3,6,5,3^4), (7^2,6^3,3^4), (7^3,6,4^4,3), (7^3,6,4^2,3^3), (7^3,5,4,3^4), (7^2,6^2,4,3^4), (7^3,4,3^5), (8,7^2,6,4,3^4), (8,7^2,5,3^5), (8,7^2,3^6), (8^3,7,4,3^5), (8^3,6,3^6), (8^2,7,3^7), (8,7^2,6,3^6), (8,7^2,4,3^6), (7^3,6^5,4,3), (7^3,6,4^5,3), (7^3,6,4,3^5), (7^3,5,3^6), (7^2,6^2,3^6), (7^3,4^6,3), (7^3,4^2,3^5), (7^3,3^7), (7^3,6^6,3), (9,8,7,6,3^6), (9,7^2,3^7), (9^2,8,7,3^7), (8^3,7,3^7), (8,7^2,3^8), (7^3,4^7,3), (7^3,4,3^7), (10,9^2,8,3^8), (9^4,3^8), (10,8^2,7,3^7)$ or $(10^3,9,3^9)$. The corresponding ρ_3 is $(3^6), (3^4,2^2), (5,3^5), (3^2,2,1^2), (3^2,2^4), (4,3^4,2), (3,2^2,1), (3^2,2^2), (3,2^3,1), (3,2^2,1^3), (3,2,1^3), (2^2,1^2), (2,1^2), (3,1^3), (1^2), (4,1^4), (4,3^6), (3^3,2,1^3), (2^4), (3^2,2,1^4), (2^3,1^2), (2^3), (5,4,3^5), (4,2^2,1^2), (3^4,1^4), (5,1^5)$ or $(6,1^6)$, all of them are graphic sequences.

Case 5. $d_n = 2$

If $\pi'_n = (d'_1, d'_2, ..., d'_{n-1})$ satisfies Theorem 1.4, then π'_n is potentially $K_{1^3,4}$-graphic, and so is π.

If π'_n doesn't satisfy (1), then $d_2 = d_3 = 6$ and $3 \le d_5 \le 5$. The general item of π must be $(d_1, 6^2, d_4, d_5, d_6, 3, 3^m, 2^{n-7-m})$, $(d_1, 6^2, d_4, d_5, d_6, 4, 4^m, 3^a, 2^{n-7-m-a})$ or $(d_1, 6^2, d_4, d_5, d_6, 5, 5^m, 4^a, 3^b, 2^{n-7-m-a-b})(n-m-7 > 0, n-7-m-a > 0, n-7-m-a-b > 0)$.

If $d_7 = 3$ and $d_1 \le 5+m$, then $\rho_3 = (d_4-3, d_5-3, d_6-3, d_7-3, 3^{m+6-d_1}, 2^{d_1+n-m-13})$. According to Theorem 2.4, if ρ_3 is not $(2), (2^2), (3, 2, 1)$ or $(3, 2^2)$, then ρ_3 is graphic. So π is potentially $K_{1^3,4}$-graphic. If $\rho_3 = (2), (2^2), (3, 2, 1), (3, 2^2)$, the corresponding $\pi \in \{(6^3, 3^4, 2), (6^4, 4, 3^2, 2), (8, 6^3, 3^4, 2), (6^3, 4, 3^4, 2), (6^3, 5, 3^3, 2), (7, 6^2, 3^5, 2), (6^3, 3^4, 2^2), (6^4, 3^4, 2), (6^3, 3^6, 2)\} \subset S$. If $d_1 \ge m+6$, then $\rho_3 = (2^x, 1^y)(y \ge 1)$, which is graphic by Theorem 2.4, so π is potentially $K_{1^3,4}$-graphic.

If $d_7 = 4$ and $d_1 \le m+5$, then $\rho_3 = (d_4-3, d_5-3, d_6-3, d_7-3, 4^{m+6-d_1}, 3^{a+d_1-6}, 2^{n-m-a-7})$, which is graphic by Theorem 2.5, so π is potentially $K_{1^3,4}$-graphic. If $d_1 \ge m+6$, then $\rho_3 = (3^x, 2^y, 1^z)(y \ge 4)$, which is graphic by Theorem 2.4, so π is potentially $K_{1^3,4}$-graphic.

If $d_7 = 5$ and $d_1 \le m+5$, then $\rho_3 = (d_4-3, d_5-3, d_6-3, d_7-3, 5^{m+6-d_1}, 4^{a+d_1-6}, 3^b, 2^{n-7-m-a-b})$, which is graphic by Theorem 2.5, so π is potentially $K_{1^3,4}$-graphic. If $d_1 \ge m+6$, then $\rho_3 = (4^x, 3^y, 2^z, 1^w)(z \ge 3)$, which is graphic by Theorem 2.5 and Theorem 2.4, so π is potentially $K_{1^3,4}$-graphic.

Assume that π'_n doesn't satisfy (2), then π'_n is $(6^6, 5^2), (6^5, 4^3), (6^5, 4, 3^2), (6^4, 5^4), (6^4, 5, 3^3), (6^4, 4^4), (6^4, 4^2, 3^2), (6^4, 4, 3^2, 2), (6^3, 5, 4, 3^3), (6^3, 5, 3^3, 2), (6^3, 4, 3^4), (6^3, 3^4, 2), (6^8), (7^4, 4, 3^4), (7^3, 6, 3^5), (7^2, 6^2, 3^4, 2), (7^2, 6, 3^6), (7, 6^3, 3^5), (7, 6^3, 3^3, 2^2), (7, 6^2, 4, 3^5), (7, 6^2, 3^5, 2), (6^8, 4), (6^8, 2), (6^4, 4^5), (6^4, 4, 3^4), (6^4, 3^4, 2), (6^3, 5, 3^5), (6^3, 4^6), (6^3, 4^2, 3^4), (6^3, 4, 3^4, 2), (6^3, 3^6), (6^3, 3^4, 2^2), (6^9), (8^2, 7^2, 3^6), (8, 7^3, 3^5, 2), (7^4, 3^6), (7^4, 3^4, 2^2), (7, 6^2, 3^7), (6^3, 4^7), (6^3, 4, 3^6), (6^3, 3^6, 2), (9, 8^3, 3^7), (8^4, 3^6, 2), (9^4, 3^8), (9, 7^3, 3^6), (8, 7, 6^2, 3^5), (7, 6^3, 4, 3^3), (8, 6^3, 3^4, 2), (7, 6^2, 5, 3^4), (8, 6^2, 3^6), (7, 6^2, 3^5)$ or $(6^4, 3^4)$.

Since $\pi \ne (7, 6^3, 3^3, 2^2), (7^2, 6^2, 3^4, 2), (6^8, 2)$, the corresponding π is $(7^2, 6^4, 5^2, 2), (7, 6^6, 5, 2), (7^2, 6^3, 4^3, 2), (7^2, 6^3, 4, 3^2, 2), (7^2, 6^2, 5^4, 2), (7, 6^4, 5^3, 2), (6^6, 5^2, 2), (7^2, 6^2, 5, 3^3, 2), (7, 6^4, 3^3, 2), (7^2, 6^2, 4^4, 2), (7^2, 6^2, 4^2, 3^2, 2), (7^2, 6^2, 4, 3^2, 2^2), (7^2, 6, 5, 4, 3^3, 2), (7, 6^3, 4, 3^3, 2), (7^2, 6, 5, 3^3, 2^2), (7^2, 6, 4, 3^4, 2), (7^2, 6, 3^4, 2^2), (7^2, 6^6, 2), (8^2, 7^2, 4, 3^4, 2), (8^2, 7, 6, 3^5, 2), (8, 7^3, 3^5, 2), (8^2, 6^2, 3^4, 2^2), (7^4, 3^4, 2^2), (8, 7^2, 6, 3^4, 2^2), (8^2, 6, 3^6, 2), (8, 7^2, 3^6, 2), (8, 7, 6^2, 3^5, 2), (7^3, 6, 3^5, 2), (8, 7, 6^2, 3^3, 2^3), (7^3, 6, 3^3, 2^3), (8, 7, 6, 4, 3^5, 2), (7^3, 4, 3^5, 2), (8, 7, 6, 3^5, 2^2), (7^3, 3^5, 2^2), (7^2, 6^6, 4, 2), (7^2, 6^6, 2^2), (7^2, 6^2, 4^5, 2), (7^2, 6^2, 4, 3^4, 2), (7^2, 6^2, 3^4, 2^2), (7^2, 6, 5, 3^5, 2), (7, 6^3, 3^5, 2), (7^2, 6, 4^6, 2), (7^2, 6, 4^2, 3^4, 2), (7^2, 6, 4, 3^4, 2^2), (7^2, 6, 3^6, 2), (7^2, 6, 3^4, 2^3), (7^2, 6^7, 2), (9^2, 7^2, 3^6, 2), (8^4, 3^6, 2), (9, 8^2, 7, 3^6, 2), (9, 8, 7^2, 3^5, 2^2), (8^3, 7, 3^5, 2^2), (8^2, 7^2, 3^6, 2), (8^2, 7^2, 3^4, 2^3), (8, 7, 6, 3^7, 2), (7^3, 3^7, 2), (7^2, 6, 4^7, 2), (7^2, 6, 4, 3^6, 2), (7^2, 6, 3^6, 2^2), (10, 9,$

$8^2,3^7,2),(9^3,8,3^7,2),(9^2,8^2,3^6,2^2),(10^2,9^2,3^8,2),(10,8,7^2,3^6,2),(9,8,6^2,3^5,2),(8,7,6^2,4,3^3,2),(7^3,6,4,3^3,2),(9,7,6^2,3^4,2^2),(8,7,6,5,3^4,2),(7^3,5,3^4,2),(9,7,6,3^6,2),(8,7,6,3^5,2)$ or $(7^3,3^5,2)$. It is easy to compute ρ_3 is $(3^4,2^2),(4,3^4,2),(3^2,2^2,1^2),(3^2,2,1^2),(3^2,2^4),(4,3^2,2^3),(5,3^3,2^2),(3,2^2,1),(3^2,2^2),(3,2^2,1^3),(3,2,1^3),(3,1^3),(2^2,1^2),(1^2),(4,1^4),(4,2,1^2),(4^2,3^4,2),(4^2,3^4,2^2),(3^3,2,1^3),(3,2^3,1),(4,3^2,2^4),(2^4),(3^2,2,1^4),(2^3,1^2),(2^3),(5^2,3^4,2),(2,1^2),(4,2^2,1^2),(4,3^4,2^2),(4,3^2,2,1^4),(5,1^5)$ or $(6,1^6)$, all of them are graphic sequences.

Case 6. $d_n = 1$

If $\pi'_n = (d'_1,d'_2,...,d'_{n-1})$ satisfies Theorem 1.4, then π'_n is potentially $K_{1^3,4}$-graphic, and so is π.

If π'_n doesn't satisfy (1), then $d_1 = d_3 = 6$ and $3 \le d_4 \le 5$. Therefore, π must be $(6^3,d_4,d_5,d_6,3,3^m,2^p,1^a),(6^3,d_4,d_5,d_6,4,4^m,3^p,2^q,1^a)$ or $(6^3,d_4,d_5,d_6,5,5^m,4^p,3^q,2^r,1^a)(a>0)$.

If $d_7 = 3$, then $\rho_3 = (d_4-3,d_5-3,d_6-3,d_7-3,3^m,2^p,1^a)$. According to Theorem 2.4, if ρ_3 is not $(3,1),(3,2,1),(3^3,1)$ or $(3^2,1^2)$, then ρ_3 is graphic. So π is potentially $K_{1^3,4}$-graphic. If $\rho_3 = (3,1),(3,2,1),(3^3,1),(3^2,1^2)$, the corresponding $\pi \in \{(6^3,3^5,1),(6^3,3^5,2,1),(6^3,5,3^4,1),(6^3,3^7,1),(6^5,3^3,1),(6^4,3^5,1),(6^3,4,3^5,1),(6^3,3^6,1^2),(6^4,3^4,1^2)\} \subset S$.

If $d_7 = 4$, then $\rho_3 = (d_4-3,d_5-3,d_6-3,d_7-3,4^m,3^p,2^q,1^a)$, which is graphic by Theorem 2.5 and Theorem 2.4, so π is potentially $K_{1^3,4}$-graphic.

If $d_7 = 5$, then $\rho_3 = (d_4-3,d_5-3,d_6-3,d_7-3,5^m,4^p,3^q,2^r,1^a)$, which is graphic by Theorem 2.6, Theorem 2.5 and Theorem 2.4, so π is potentially $K_{1^3,4}$-graphic.

Assume that π'_n doesn't satisfy (2), then π'_n is $(6^6,5^2),(6^5,4^3),(6^5,4,3^2),(6^4,5^4),(6^4,5,3^3),(6^4,4^4),(6^4,4^2,3^2),(6^4,4,3^2,2),(6^3,5,4,3^3),(6^3,5,3^3,2),(6^3,4,3^4),(6^3,3^4,2),(6^8),(7^4,4,3^4),(7^3,6,3^5),(7^2,6^2,3^4,2),(7^2,6,3^6),(7,6^3,3^5),(7,6^3,3^3,2^2),(7,6^2,4,3^5),(7,6^2,3^5,2),(6^8,4),(6^8,2),(6^5,3^3,1),(6^4,4^5),(6^4,4,3^4),(6^4,4,3^3,1),(6^4,3^4,2),(6^4,3^3,2,1),(6^3,5,3^5),(6^3,5,3^4,1),(6^3,4^6),(6^3,4^2,3^4),(6^3,4,3^4,2),(6^3,3^6),(6^3,3^5,1),(6^3,3^4,2^2),(6^9),(8^2,7^2,3^6),(8,7^3,3^5,2),(7^4,3^6),(7^4,3^5,1),(7^4,3^4,2^2),(7,6^2,3^7),(7,6^2,3^6,1),(6^4,3^5,1),(6^4,3^4,1^2),(6^3,4^7),(6^3,4,3^6),(6^3,4,3^5,1),(6^3,3^6,2),(6^3,3^5,2,1),(9,8^3,3^7),(8^4,3^6,2),(6^3,3^7,1),(6^3,3^6,1)$ or $(9^4,3^8)$.

Since $\pi \ne (6^4,4,3^3,1),(6^4,3^3,2,1),(7^4,3^5,1),(7,6^2,3^6,1),(6^4,3^4,1^2),(n-1,7^3,3^6,1^{n-10}),(n-1,7,6^2,3^5,1^{n-9}),(n-1,6^3,4,3^3,1^{n-8}),(n-1,6^3,3^4,2,1^{n-9}),(n-1,6^2,5,3^4,1^{n-8}),(n-1,6^2,3^6,1^{n-9}),(n-1,6^2,3^5,1^{n-8}),(n-2,6^3,3^4,1^{n-8})$, the corresponding π is $(7,6^5,5^2,1),(6^7,5,1),(7,6^4,4^3,1),(7,6^4,4,3^2,1),(7,6^3,5^4,1),(6^5,5^3,1),(7,6^3,5,3^3,1),(6^5,3^3,1),(7,6^3,4^4,1),(7,6^3,4^2,3^2,1),(7,6^3,4,3^2,2,1),(7,6^2,5,4,3^3,1),(7,6^2,5,3^3,2,1),(7,6^2,4,3^4,1),(7,6^2,3^4,2,1),(7,6^7,1),(8,7^3,4,3^4,1),(8,7^2,6,3^5,1),(7^4,3^5,1),(8,7,6^2,3^4,2,1),(7^3,6,3^4,2,1),(8,7,6,3^6,1),(7^3,3^6,1),(8,6^3,3^5,1),(7^2,6^2,3^5,1),(8,6^3,3^3,2^2,1),(7^2,6^2,3^3,2^2,1),(8,6^2,4,3^5,1),(7^2,6,4,3^5,1),(8,6^2,3^5,2,1),(7,6^7,4,1),(7,6^7,2,1),(7,6^4,3^3,1^2),(7,6^3,4^5,1),(7,6^3,4,3^4,1),(7,6^3,4,3^3,$

$1^2),(7,6^3,3^4,2,1),(7,6^3,3^3,2,1^2),(7,6^2,5,3^5,1),(6^4,3^5,1),(7,6^2,5,3^4,1^2),$ $(7,6^2,4^6,1),(7,6^2,4^2,3^4,1),(7,6^2,4,3^4,2,1),(7,6^2,3^5,1^2),(7,6^2,3^4,2^2,1),$ $(7,6^8,1),(9,8,7^2,3^6,1),(8^3,7,3^6,1),(9,7^3,3^5,2,1),(8^2,7^2,3^5,2,1),(8,7^3,$ $3^6,1),(8,7^3,3^5,1^2),(8,7^3,3^4,2^2,1),(8,6^2,3^7,1),(7^2,6,3^7,1),(8,6^2,3^6,1^2),$ $(7^2,6,3^6,1^2),(7,6^3,3^5,1^2),(7,6^3,3^4,1^3),(7,6^2,4^7,1),(7,6^2,4,3^6,1),(7,6^2,$ $4,3^5,1^2),(7,6^2,3^6,2,1),(7,6^2,3^5,2,1^2),(10,8^3,3^7,1),(9^2,8^2,3^7,1),(9,8^3,$ $3^6,2,1),(7,6^2,3^7,1^2),(7,6^2,3^6,1^2),(10,9^3,3^8,1)$ or $(7^2,6,3^5,2,1)$. It is easy to compute the corresponding ρ_3 is $(3^4,2^2),(4,3^4,2),(6,3^4,2),(3^2,2^2,1^2),$ $(3^2,2,1^2),(3^2,2^4),(4,3^2,2^3),(2,1^2),(3,2^2,1),(3^2,2^2),(3,2^2,1^3),(3,2,1^3),(3,$ $1^3),(2^2,1^2),(1^2),(4,1^4),(4,2,1^2),(4^2,3^4,2),(4^2,3^4,2^2),(3^3,2,1^3),(3,2^3,1),$ $(4,3^2,2^4),(2^4),(3^2,2,1^4),(2^3),(2^3,1^2),(5^2,3^4,2),(4,2^2,1^2),(4,3^2,2,1^4),(5,$ $1^5),(6,1^6)$ or $(4,3^4,2^2)$, all of them are graphic sequences.

3. Application of Theorem 1.4

Yin and Lai computed the values of $\sigma(K_{1^3,4},n)$ independently when $n \geq 50$ in [10] and $n \geq 48$ in [11].

We now give an application of Theorem 1.4. It is simple to use Theorem 1.4 to compute the values of $\sigma(K_{1^3,4},n)$.

Theorem 3.1 $\sigma(K_{1^3,4},7) = 38, \sigma(K_{1^3,4},8) = 50, \sigma(K_{1^3,4},9) = 56,$ $\sigma(K_{1^3,4},10) = 50, \sigma(K_{1^3,4},11) = 56, \sigma(K_{1^3,4},12) = 62$ and for $n \geq 13$,

$$\sigma(K_{1^3,4},n) = \begin{cases} 7n-10, & n \textit{ is even}, \\ 7n-11, & n \textit{ is odd}. \end{cases}$$

Proof. For $7 \leq n \leq 12$, $(6^2,5^4,4),(6^8),(6^9),(9,7^3,3^6),(9,8^3,3^7)$ and $(9^4,3^8)$ are not potentially $K_{1^3,4}$ by Theorem 1.4.

Since $\sigma(\pi)$ is even, $\sigma(K_{1^3,4},7) \geq 6\times 2+5\times 4+4+2 = 38, \sigma(K_{1^3,4},8)$ $\geq 6\times 8+2 = 50, \sigma(K_{1^3,4},9) \geq 6\times 9+2 = 56, \sigma(K_{1^3,4},10) \geq 9+7\times 3+3\times 6+$ $2 = 50, \sigma(K_{1^3,4},11) \geq 9+8\times 3+3\times 7+2 = 56, \sigma(K_{1^3,4},12) \geq 9\times 4+3\times 8+$ $2 = 62$. By Theorem 1.4, $\sigma(K_{1^3,4},7) = 38, \sigma(K_{1^3,4},8) = 50, \sigma(K_{1^3,4},9) =$ $56, \sigma(K_{1^3,4},10) = 50, \sigma(K_{1^3,4},11) = 56$ and $\sigma(K_{1^3,4},12) = 62$.

For $n \geq 13$, let

$$\pi = \begin{cases} ((n-1)^2,5^{n-2}), & n \textit{ is even}, \\ ((n-1)^2,5^{n-3},4), & n \textit{ is odd}. \end{cases}$$

By Theorem 1.4, π is not potentially $K_{1^3,4}$-graphic, which has degree sum

$$\sigma(\pi) = \begin{cases} 7n-12, & n \textit{ is even}, \\ 7n-13, & n \textit{ is odd}. \end{cases}$$

Thus, $\sigma(K_{1^3,4}, n) \geq \sigma(\pi) + 2$, which establishes the lower bound.

Let $n \geq 13$, and $\pi = (d_1, d_2, ..., d_n) \in GS_n$ be a positive sequence satisfy Theorem 3.1. Then the following will prove that π is potentially $K_{1^3,4}$-graphic.

(1)If $n \geq 13$ and

$$\sigma(K_{1^3,4}, n) = \begin{cases} 7n-10, & n \ is \ even, \\ 7n-11, & n \ is \ odd \ , \end{cases}$$

then π is not the following sequences:

$(n-1, 7^3, 3^6, 1^{n-10})(n \geq 10)$, $(n-1, 6^2, 5, 3^4, 1^{n-8})(n \geq 8)$,
$(n-1, 7, 6^2, 3^5, 1^{n-9})(n \geq 9)$, $(n-1, 6^2, 3^6, 1^{n-9})(n \geq 9)$,
$(n-1, 6^3, 4, 3^3, 1^{n-8})(n \geq 8)$, $(n-1, 6^2, 3^5, 1^{n-8})(n \geq 8)$,
$(n-1, 6^3, 3^4, 2, 1^{n-9})(n \geq 9)$, $(n-2, 6^3, 3^4, 1^{n-8})(n \geq 8)$.

(2)Now we check the condition that $d_3 \geq 6$. To the contrary, assume that $d_3 \leq 5$. Then

$$\sigma(\pi) = d_1+d_2+...+d_n \leq \begin{cases} 2(n-1)+5(n-2) = 7n-12 < 7n-10, \\ \quad n \ is \ even, \\ 2(n-1)+5(n-3)+4 = 7n-13 < 7n-11, \\ \quad n \ is \ odd \ . \end{cases}$$

(3)Assume that $d_7 \leq 2$. Then $\sigma(\pi) = \sum_{i=1}^{6} d_i + \sum_{i=7}^{n} d_i \leq 6(6-1) + \sum_{i=7}^{n} min(6, d_i) + \sum_{i=7}^{n} d_i = 30 + 2\sum_{i=7}^{n} d_i \leq 5n < 7n-13 < 7n-12$, so there is the condition that $d_7 \geq 3$.

So π is potentially $K_{1^3,4}$-graphic by Theorem 1.4.

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