A New View of Bipartite Ramsey Numbers

Zhenming Bi, Gary Chartrand and Ping Zhang
Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008-5248, USA
ping.zhang@wmich.edu

Abstract

For a bipartite graph G and a positive integer s, the s-bipartite Ramsey number $BR_s(G)$ of G is the minimum integer t with $t \geq s$ for which every red-blue coloring of $K_{s,t}$ results in a monochromatic G. A formula is established for $BR_s(K_{r,r})$ when $s \in \{2r-1, 2r\}$ when $r \geq 2$. The s-bipartite Ramsey numbers are studied for $K_{3,3}$ and various values of s. In particular, it is shown that $BR_s(K_{3,3}) = 41$ when $s \in \{5, 6\}$, $BR_s(K_{3,3}) = 29$ when $s \in \{7, 8\}$ and $17 \leq BR_{10}(K_{3,3}) \leq 23$.

Key Words: Ramsey number, bipartite Ramsey number, s-bipartite Ramsey number.
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1 Introduction to s-Bipartite Ramsey Numbers

A well-known recreational problem with a graph theory connection is the following:

> *Suppose that every two people at a party are either acquainted or are strangers. What is the smallest number of people who must be present at the party to be guaranteed that there are three among them who are either mutual acquaintances or mutual strangers?*

It is known that the answer to this question is 6 and that this is directly related to the concept of Ramsey numbers of graphs.

In a *red-blue coloring* of a graph G, every edge of G is colored red or blue. For two graphs F and H, the *Ramsey number* $R(F, H)$ of F and H is the smallest positive integer n such that every red-blue coloring of the complete graph K_n of order n results in either a subgraph isomorphic to F all of whose edges are colored red (a *red* F) or a subgraph isomorphic to H all of whose edges are colored blue (a *blue* H). A graph (subgraph) all of whose edges are colored the same is called a *monochromatic graph* (*subgraph*). If $F \cong H$, then we write $R(F)$ for $R(F, H)$. We refer to the book [4] for graph theory notation and terminology not described in this paper.

A group of n people at a party can be represented by a red-blue coloring of the complete graph K_n of order n, where the vertices of K_n are the people at the party and an edge uv is colored red, say, if u and v are acquaintances and colored blue if they are strangers. Therefore, the answer to the party problem is the Ramsey number $R(K_3, K_3)$, which is known to be 6.

In 1975 Beineke and Schwenk [2] introduced a bipartite version of Ramsey numbers. For two bipartite graphs F and H, the *bipartite Ramsey number* $BR(F, H)$ of F and H is the smallest positive integer r such that every red-blue coloring of the r-regular complete bipartite graph $K_{r,r}$ results in either a red F or a blue H. Consequently, if $BR(F, H) = r$ for bipartite graphs F and H, then every red-blue coloring of $K_{r,r}$ results in a red F or a blue H, while there exists a red-blue coloring of $K_{r-1,r-1}$ for which there is neither a red F nor a blue H. Beineke and Schwenk [2] showed that $BR(F, H)$ exists for every two bipartite graphs F and H. If $F \cong H$, then here too we write $BR(F)$ for $BR(F, H)$. It was shown in [2] that for the 4-cycle C_4 that $BR(C_4) = 5$.

To summarize then, if $BR(F, H) = r$ for bipartite graphs F and H, then every red-blue coloring of $K_{r,r}$ results in a red F or a blue H, while there exists a red-blue coloring of $K_{r-1,r-1}$ for which there is neither a red F nor a blue H. In [1], the question arose as to what might occur for red-blue colorings of the intermediate graph $K_{r-1,r}$. This led to the concept of the 2-Ramsey number. For bipartite graphs F and H, the *2-Ramsey number* $R_2(F, H)$ of F and H is the smallest positive integer n such that every red-blue coloring of the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ of order n results in a red F or a blue H. Thus, either $R_2(F, H) = 2BR(F, H)$ or $R_2(F, H) = 2BR(F, H) - 1$.

Let's return to the bipartite Ramsey number $BR(C_4, C_4) = 5$. This implies that $R_2(C_4, C_4)$ is either 9 or 10. Since there is a red-blue coloring of $K_{4,5}$ that results in neither a red C_4 nor a blue C_4 (see Figure 1, where each solid edge is colored red and each dashed edge is colored blue), it follows that $R_2(C_4, C_4) \geq 10$ and so $R_2(C_4, C_4) = 10$.

We now turn to a different recreational problem.

> *There are eight girls at a party. What is the minimum number of boys who must be invited to the party to guarantee that there exists a group of six people, three girls and three boys, such that either* (1) *every one of the three girls is acquainted with every one of the three boys or* (2) *every one of the three girls is a stranger of every one of the three boys?*

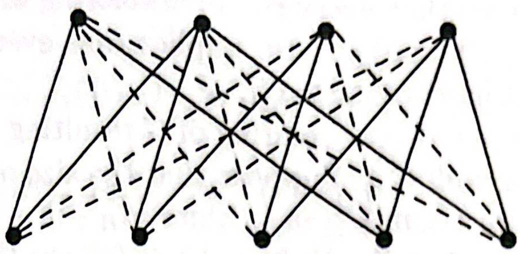

Figure 1: A red-blue coloring of $K_{4,5}$

Prior to providing an answer to this question, we introduce a new concept suggested by this problem. Let F and H be two bipartite graphs. For a positive integer s, the *s-bipartite Ramsey number* $BR_s(F,H)$ of F and H is the smallest integer t with $t \geq s$ such that every red-blue coloring of $K_{s,t}$ results in a red F or a blue H. If $F \cong H$, then we write $BR_s(F,H)$ as $BR_s(F)$. Consequently, if $BR_s(F) = t$, then $t \geq s$ and every red-blue coloring of $K_{s,t}$ results in a monochromatic F; while if $t > s$, then there is a red-blue coloring of $K_{s,t-1}$ for which there is no monochromatic F. To illustrate this concept, we determine the s-bipartite Ramsey number $BR_s(C_4)$ of C_4 for each integer $s \geq 2$, beginning with $s = 2$.

Proposition 1.1 *The 2-bipartite Ramsey number $BR_2(C_4)$ does not exist.*

Proof. For an arbitrary integer $t \geq 2$, the red-blue coloring of $K_{2,t}$ in which the red and blue subgraphs are $K_{1,t}$ has no monochromatic C_4. ■

The s-bipartite Ramsey number $BR_s(C_4)$ does exist, however, for every integer $s \geq 3$. To establish these values, it is convenient to make use of the so-called *Zarankiewicz number* $Z_{s,t}(m,n)$, defined in [6] as the maximum size of a subgraph of $K_{m,n}$ not containing $K_{s,t}$. This number was named after the Polish mathematician Kazimierz Zarankiewicz, who proposed the *Zarankiewicz Problem* in 1951 of determining these numbers [3, 8]. In particular, $Z_{2,2}(3,7) = 10$ (see [6]). More generally, Cǔlík obtained the following result (see [5]).

Theorem 1.2 *For integers s,t,m,n with $1 \leq s \leq m$ and $n > (t-1)\binom{m}{s}$,*

$$Z_{s,t}(m,n) = (s-1)n + (t-1)\binom{m}{s}.$$

With the aid of Theorem 1.2, we now determine the 3-bipartite and 4-bipartite Ramsey numbers of C_4.

Proposition 1.3 $BR_3(C_4) = BR_4(C_4) = 7$.

Proof. First, we show that every red-blue coloring of $G = K_{3,7}$ produces a monochromatic C_4. This then also implies that every red-blue coloring of $K_{4,7}$ produces a monochromatic C_4.

Let there be given a red-blue coloring of G resulting in the red subgraph G_R and the blue subgraph G_B. Suppose that the size of G_R is m_R and the size of G_B is m_B where say $m_R \geq m_B$. Since $m_R + m_B = 21$, it follows that $m_R \geq 11$. Since $m_R \geq 11 > Z_{2,2}(3,7) = 10$, it follows that G_R contains C_4 as a subgraph, and so $K_{3,7}$ contains a red C_4.

Next, we show that there is a red-blue coloring of $K_{4,6}$ that produces no monochromatic C_4. Let the partite sets of $K_{4,6}$ be $U = \{u_1, u_2, u_3, u_4\}$ and $W = \{w_1, w_2, \ldots, w_6\}$. Also, let $S_1 = \{1,2\}$, $S_2 = \{1,3\}$, $S_3 = \{1,4\}$, $S_4 = \{2,3\}$, $S_5 = \{2,4\}$, $S_6 = \{3,4\}$ be the 2-element subsets of $\{1,2,3,4\}$. Now, assign the color red to the edge u_iw_j if and only if $i \in S_j$ for $1 \leq i \leq 4$ and $1 \leq j \leq 6$. Therefore, for every two integers a and b with $1 \leq a < b \leq 4$, the vertices u_a and u_b are not joined to any pair of vertices of W by edges of the same color. Thus, $K_{4,6}$ contains no monochromatic C_4.

This also implies that there are red-blue colorings of $K_{3,6}$ and $K_{4,5}$ that produce no monochromatic C_4. Therefore, $BR_3(C_4) = BR_4(C_4) = 7$. ■

Since $BR_5(C_4) = BR(C_4) = 5$, we have the following.

Proposition 1.4 *For each integer $s \geq 2$,*

$$BR_s(C_4) = \begin{cases} \text{does not exist} & \text{if } s = 2 \\ 7 & \text{if } s = 3, 4 \\ s & \text{if } s \geq 5. \end{cases}$$

2 The s-Bipartite Ramsey Number $BR_s(K_{3,3})$

Since $C_4 = K_{2,2}$, it follows that $BR_4(K_{2,2}) = 7$ by Proposition 1.3. Therefore, for any group of 11 people at a party, four girls and seven boys, there are two girls and two boys such that either (1) both girls are acquainted with both boys or (2) both girls are strangers of both boys. On the other hand, this need not occur in a group of four girls and six boys. This observation returns us to the problem described earlier whose solution is the value of the 8-bipartite Ramsey number $BR_8(K_{3,3})$, which will be studied in Section 2. Next, we present a result dealing with the more general regular complete bipartite graphs and determine a formula for $BR_s(K_{r,r})$ for an arbitrary integer $r \geq 2$ when $s = 2r - 1$ or $s = 2r$. For two disjoint sets X and Y of vertices of a graph G, let $[X, Y]$ denote the set of edges joining a vertex of X and a vertex of Y.

Theorem 2.1 *For each integer $r \geq 2$,*

$$BR_{2r-1}(K_{r,r}) = (2r-2)\binom{2r-1}{r}+1 \quad (1)$$

$$BR_{2r}(K_{r,r}) = (r-1)\binom{2r}{r}+1. \quad (2)$$

Furthermore, $BR_{2r-1}(K_{r,r}) = BR_{2r}(K_{r,r})$.

Proof. Let $r \geq 2$ be a fixed integer. First, we verify (1). Let $t = (2r-2)\binom{2r-1}{r}$ and let $G = K_{2r-1,t}$, where $U = \{u_1, u_2, \ldots, u_{2r-1}\}$ and $W = \{w_1, w_2, \ldots, w_t\}$ are the partite sets of G. We show that there is a red-blue coloring of G that avoids a monochromatic $K_{r,r}$. Let $S_1, S_2, \ldots, S_{\binom{2r-1}{r}}$ denote the $\binom{2r-1}{r}$ r-element subsets of U. Let $W_1, W_2, \ldots, W_{\binom{2r-1}{r}}$ and $W_1', W_2', \ldots, W'_{\binom{2r-1}{r}}$ be partitions of W into $\binom{2r-1}{r}$ subsets such that $|W_i| = |W_i'| = r-1$ for $1 \leq i \leq \binom{2r-1}{r}$. For each integer i with $1 \leq i \leq \binom{2r-1}{r}$,

- ⋆ join every vertex in W_i to every vertex of S_i by a red edge and join every vertex of W_i to every vertex in $U - S_i$ by a blue edge and
- ⋆ join every vertex in W_i' to every vertex of S_i by a blue edge and join every vertex of W_i' to every vertex in $U - S_i$ by a red edge.

Therefore, for each set S_i ($1 \leq i \leq \binom{2r-1}{r}$), there does not exist an r-element subset W^* of W such that the edges in $[W^*, S_i]$ have the same color. Thus, there is no monochromatic $K_{r,r}$ in G and so

$$BR_{2r-1}(K_{r,r}) \geq (2r-2)\binom{2r-1}{r}+1.$$

Next, let $H = K_{2r-1,t+1}$, where $U = \{u_1, u_2, \ldots, u_{2r-1}\}$ and $W = \{w_1, w_2, \ldots, w_{t+1}\}$ are the partite sets of H. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Suppose that the size of H_R is m_R and the size of H_B is m_B where say $m_R \geq m_B$. Since

$$m_R + m_B = (2r-1)\left[(2r-2)\binom{2r-1}{r}+1\right],$$

it follows that $m_R \geq (r-1)(2r-1)\binom{2r-1}{r} + r$. Since $r \geq 2$, we have

$$\begin{aligned}(r-1)(2r-1)\binom{2r-1}{r}+r &> (r-1)(2r-1)\binom{2r-1}{r}+1 \\ &= Z_{r,r}\left(2r-1, (2r-2)\binom{2r-1}{r}+1\right).\end{aligned}$$

This implies that H_R contains $K_{r,r}$ as a subgraph. Thus, H contains a red $K_{r,r}$ and so $BR_{2r-1}(K_{r,r}) \leq (2r-2)\binom{2r-1}{r} + 1$. Therefore, (1) holds.

Next, we verify (2). Let $t = (r-1)\binom{2r}{r}$ and let $G = K_{2r,t}$, where $U = \{u_1, u_2, \ldots, u_{2r}\}$ and $W = \{w_1, w_2, \ldots, w_t\}$ are the partite sets of G. We show that there is a red-blue coloring of G that avoids a monochromatic $K_{r,r}$. Let $S_1, S_2, \ldots, S_{\binom{2r}{r}}$ denote the $\binom{2r}{r}$ r-element subsets of U. Partition W into $\binom{2r}{r}$ subsets $W_1, W_2, \ldots, W_{\binom{2r}{r}}$ where $|W_i| = r-1$ for $1 \leq i \leq \binom{2r}{r}$. For each integer i with $1 \leq i \leq \binom{2r}{r}$, join every vertex of W_i to every vertex of S_i by a red edge and join every vertex of W_i to every vertex in $T_i = U - S_i$ by a blue edge. (Observe that $T_1, T_2, \ldots, T_{\binom{2r}{r}}$ are the $\binom{2r}{r}$ r-element subsets of U.) For each set S_i $(1 \leq i \leq \binom{2r}{r})$, there does not exist an r-element subset W' of W such that each vertex of W' is joined to every vertex of S_i by edges of the same color. Hence, there is no monochromatic $K_{r,r}$ and so $BR_{2r}(K_{r,r}) \geq (r-1)\binom{2r}{r} + 1$.

Next, let $H = K_{2r,t+1}$ and let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Suppose that the size of H_R is m_R and the size of H_B is m_B where say $m_R \geq m_B$. Since

$$m_R + m_B = 2r[(r-1)\binom{2r}{r} + 1],$$

it follows that $m_R \geq r(r-1)\binom{2r}{r} + r$. Then

$$m_R > r(r-1)\binom{2r}{r} + r - 1 = Z_{r,r}(2r, (r-1)\binom{2r}{r} + 1).$$

Thus, H_R contains $K_{r,r}$ as a subgraph and so

$$BR_{2r}(K_{r,r}) \leq (r-1)\binom{2r}{r} + 1.$$

Therefore, (2) holds.

As observed,

$$(2r-2)\binom{2r-1}{r} + 1 = (r-1)\binom{2r}{r} + 1;$$

therefore, $BR_{2r-1}(K_{r,r}) = BR_{2r}(K_{r,r})$. ∎

Of course, K_5 and $K_{3,3}$ are the two graphs that play a pivotal role in Kuratowski's characterization [7] of planar graphs. The Ramsey number $R(K_5)$ is not known, while it is stated in [2] that $BR(K_{3,3}) = 17$. In this section, we study $BR_s(K_{3,3})$ for various values of s. The following is then a consequence of Theorem 2.1.

Corollary 2.2 $BR_5(K_{3,3}) = BR_6(K_{3,3}) = 41$.

On the other hand, $s = 5$ is the smallest value of s for which $BR_s(K_{3,3})$ exists.

Proposition 2.3 *For each integer* $s = 2, 3, 4$, *the number* $BR_s(K_{3,3})$ *does not exist.*

Proof. For $2 \leq s \leq 4$ and an arbitrary large integer t, the red-blue coloring of $K_{s,t}$, in which the red subgraph is $K_{2,t}$ and and blue subgraph are $K_{s-2,t}$ (where $K_{0,t}$ is an empty graph), produces no monochromatic $K_{3,3}$. ∎

We now provide an answer to the party problem mentioned earlier by determining $BR_s(K_{3,3})$ not only for $s = 8$ but for $s = 7$ as well. For a complete bipartite graph $H = K_{s,t}$ with $s \leq t$, let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_t\}$ be the partite sets of H. For a subset $S \subseteq U$, a vertex $w \in W$ *is joined to* S if w is adjacent to every vertex in S. A 3-element subset of a set S is referred to as a *triple* of S. It is well known for integers r and n with $1 \leq r \leq n-1$ that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ or equivalently $\binom{n}{r} - \binom{n-1}{r} = \binom{n-1}{r-1}$. In particular, if $r = 3$ and $n \geq 4$, then

$$\binom{n}{3} - \binom{n-1}{3} = \binom{n-1}{2} \text{ is an increasing function of } n. \quad (3)$$

We are now prepared to determine $BR_s(K_{3,3})$ for $s = 7, 8$.

Theorem 2.4 $BR_7(K_{3,3}) = BR_8(K_{3,3}) = 29$.

Proof. We first show that $BR_8(K_{3,3}) = 29$. Let there be given a red-blue coloring of $H = K_{8,29}$ resulting in the red subgraph H_R and the blue subgraph H_B. Suppose that the size of H_R is m_R and the size of H_B is m_B where say $m_R \geq m_B$. Since $m_R + m_B = 232$, it follows that $m_R \geq 116$. We claim that H_R contains $K_{3,3}$ as a subgraph. Assume, to the contrary, that this is not the case. Let U and W be the partite sets of H where $|U| = 8$ and $|W| = 29$. There are $\binom{8}{3} = 56$ triples of U. Since H_R does not contain $K_{3,3}$, it follows that for each triple T of U, at most two vertices of W are adjacent to the vertices of T. Hence, the size of H_R is at most $2 \cdot 56 = 112$, contradicting the fact that $m_R \geq 116$. Therefore, H_R contains $K_{3,3}$ as a subgraph and so $BR_8(K_{3,3}) \leq 29$.

Next, we show that there is a red-blue coloring of $G = K_{8,28}$ that avoids a monochromatic $K_{3,3}$. Let $U = \{u_1, u_2, \ldots, u_8\}$ and $W = \{w_1, w_2, \ldots, w_{28}\}$ be the partite sets of G. Consider the following fourteen 4-element subsets $S_1, S_2, \ldots, S_{14}$ of U, where $\{u_a, u_b, u_c, u_d\}$ is denoted by *abcd*. Note that each triple of U appears exactly once in these fourteen 4-element subsets of U.

1234	1256	1278	1357	1368	1458	1467
2358	2367	2457	2468	3456	3478	5678

Let $T_i = U - S_i$ for $1 \le i \le 14$. We claim that each triple of U appears exactly once in the fourteen 4-element subsets $T_1, T_2, \ldots, T_{14}$ of U. Assume, to the contrary, that there is a triple $\{u_a, u_b, u_c\}$ that appears in two of the sets $T_1, T_2, \ldots, T_{14}$, say these two sets are $T' = \{u_a, u_b, u_c, u_d\}$ and $T'' = \{u_a, u_b, u_c, u_e\}$. Then $S' = U - T'$ and $S'' = U - T''$ are two of the sets $S_1, S_2, \ldots, S_{14}$. Since $U^* = U - \{u_a, u_b, u_c, u_d, u_e\}$ is a triple of U and U^* is a subset of both S' and S'', this is impossible. Therefore, as claimed, each triple of U appears exactly once in these fourteen 4-element subsets $T_1, T_2, \ldots, T_{14}$ of U. Therefore, each 3-element subset of U appears exactly twice in the 28 subsets $S_1, S_2, \ldots, S_{14}, T_1, T_2, \ldots, T_{14}$ of U. For each integer i with $1 \le i \le 14$, join w_i and w_{14+i} to the four vertices in S_i by red edges and to the remaining four vertices in $U - S_i = T_i$ by blue edges. Let G_R and G_B be the resulting red and blue subgraphs of G. Since each triple of U belongs to exactly two of the sets $N_{G_R}(w_i)$ for $1 \le i \le 28$ and exactly two of the sets $N_{G_B}(w_i)$ for $1 \le i \le 28$, neither G_R nor G_B contains $K_{3,3}$ as a subgraph. Hence, this red-blue coloring of G avoids a monochromatic $K_{3,3}$. Therefore, $BR_8(K_{3,3}) \ge 29$ and so $BR_8(K_{3,3}) = 29$.

Next, we show that $BR_7(K_{3,3}) = 29$. Since $BR_8(K_{3,3}) = 29$, there is a red-blue coloring of $K_{8,28}$ that avoids a monochromatic $K_{3,3}$. Because $K_{7,28}$ is a subgraph of $K_{8,28}$, it follows that there is a red-blue coloring of $K_{7,28}$ that avoids a monochromatic $K_{3,3}$. Thus, $BR_7(K_{3,3}) \ge 29$.

To show $BR_7(K_{3,3}) \le 29$, let $H = K_{7,29}$ whose partite sets are U and W with $|U| = 7$ and $|W| = 29$. Since the size of H is 203, each red-blue coloring of H produces a subgraph of H containing at least 102 edges of the same color. Among all red-blue colorings of H whose red subgraph contains at least 102 edges, let c be one such that the vertices of W are joined to a minimum number of triples of U (with repetitions) by red edges. Let H_R be the red subgraph resulting from c. Then the average degree of the vertices of W in H_R exceeds 3.5. We show that H_R contains $K_{3,3}$ as a subgraph by considering two cases.

Case 1. *Each vertex in W has degree* 3 *or* 4 *in H_R*. Suppose that H_R has x vertices of degree 4 and so $29 - x$ vertices of degree 3. Thus, the size of H_R is $4x + 3(29 - x) = x + 87$. Since H_R contains at least 102 edges, it follows that $x \ge 15$. For each $w \in W$, if $\deg_{H_R} w = 4$, then w is joined to four triples in U; while if $\deg_{H_R} w = 3$, then w is joined to one triple in U. Hence, the vertices of W are joined to $4x + (29 - x) = 29 + 3x$ triples of U in H_R. Since $15 \le x \le 29$, the minimum number occurs when $x = 15$, in which case, the vertices of W are joined to 74 triples of U in H_R. Since there are $\binom{7}{3} = 35$ triples in U, at least three vertices of W are joined to the same triple of U in H_R and so H_R contains $K_{3,3}$ as a subgraph in this case.

Case 2. *There exists at least one vertex in W not having degree* 3 *or* 4 *in H_R*.

Subcase 2.1. *Some vertex $w \in W$ has degree at least* 5 *in H_R*. Since the average degree of the vertices of W in H_R is 3.5, there is a vertex $w' \in W$ having degree 3 or less in H_R. Replace the edges of H_R incident with w by $\deg_{H_R} w - 1$ red edges incident with w and replace the edges of H_R incident with w' by $\deg_{H_R} w' + 1$ red edges incident with w', resulting in a new red-blue coloring c' of H, whose red subgraph is H'_R. Suppose that $\deg_{H_R} w = d \geq 5$ and $\deg_{H_R} w' = d' \leq 3$. In H_R, the vertex w is joined to $\binom{d}{3}$ triples of U; while in H'_R, the vertex w is joined to $\binom{d-1}{3}$ triples of U. On the other hand, if $d' = 3$, then w' is joined to $\binom{3}{3} = 1$ triple of U in H_R and w' is joined to $\binom{4}{3} = 4$ triples of U in H'_R; if $d' = 2$, then w' is joined to no triple of U in H_R and w' is joined to $\binom{3}{3} = 1$ triple of U in H'_R; and if $d' \leq 1$, then w' is joined to no triple of U in both H_R and H'_R. Since $d \geq 5$, it then follows by (3) that the vertices of W are joined to at least $[\binom{5}{3} - \binom{4}{3}] + [\binom{3}{3} - \binom{4}{3}] = 3$ *fewer* triples of U in H'_R than in H_R, which is impossible.

Subcase 2.2. *Some vertex $w \in W$ has degree at most* 2 *in H_R*. Then there is a vertex $w' \in W$ having degree 4 or more in H_R. Replace the edges of H_R incident with w by $\deg_{H_R} w + 1$ red edges incident with w and replace the edges of H_R incident with w' by $\deg_{H_R} w' - 1$ red edges incident with w', resulting in a new red-blue coloring c^* of H, whose red subgraph is H^*_R. Suppose that $\deg_{H_R} w = d \leq 2$ and $\deg_{H_R} w' = d' \geq 4$. In H_R, the vertex w is joined to $\binom{d}{3} = \binom{2}{3} = 0$ triples of U; while in H^*_R, the vertex w is joined to $\binom{d+1}{3} \leq \binom{3}{3} = 1$ triple of U. On the other hand, w' is joined to $\binom{d'}{3}$ triples of U in H_R and w' is joined to $\binom{d'-1}{3}$ triples of U in H^*_R. Since $d \leq 2$ and $d' \geq 4$, it then follows by (3) that the vertices of W are joined to at least $[\binom{2}{3} - \binom{3}{3}] + [\binom{4}{3} - \binom{3}{3}] = 2$ *fewer* triples in H^*_R than in H_R, which is impossible.

Since Case 2 is impossible, only Case 1 can occur for the red-blue coloring c of H. Thus, every red-blue coloring of H produces a monochromatic $K_{3,3}$. Therefore, $BR_7(K_{3,3}) \leq 29$ and so $BR_7(K_{3,3}) = 29$. ■

Returning to our party problem, if there are eight girls at a party, then we must invite 29 boys to the party to be certain that there are three girls and three boys where all three girls are acquaintances of all three boys or all three girls are strangers of all three boys.

While $BR_s(K_{3,3})$ is unknown for those integers s with $9 \leq s < BR(K_{3,3})$, the results presented above might suggest that $BR_9(K_{3,3}) = BR_{10}(K_{3,3})$. We present some bounds for $BR_{10}(K_{3,3})$.

Theorem 2.5 $17 \leq BR_{10}(K_{3,3}) \leq 23$.

Proof. First, we show that there exists a red-blue coloring of $K_{10,16}$ that avoids a monochromatic $K_{3,3}$. For $G = K_{10,16}$, let $U = \{u_1, u_2, \ldots, u_{10}\}$ and $W = \{w_1, w_2, \ldots, w_{16}\}$ be the partite sets of G. Consider the following sixteen 5-element subsets $S_1, S_2, \ldots, S_{16}$ of U, where $\{u_a, u_b, u_c, u_d, u_e\}$ is denoted by $abcde$.

12345	12367	12489	12560	12780	13480	13789	13569
14570	14679	46780	25789	45689	26790	35680	34579

For each integer i with $1 \leq i \leq 16$, let $\overline{S}_i = U - S_i$. These sixteen 5-element subsets $\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_{16}$ are listed below.

67890	45890	35670	34789	34569	25679	24560	24780
23689	23580	12359	13460	12370	13458	12479	12680

It can be verified that each 3-element subset of U belongs to

(i) at most two of these 16 sets $S_1, S_2, \ldots, S_{16}$ and

(ii) at most two of these 16 sets $\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_{16}$.

We now define a red-blue coloring of G by joining each vertex w_i $(1 \leq i \leq 16)$ to the five vertices in S_i by red edges and to the remaining five vertices in $\overline{S}_i$ by blue edges. Let G_R and G_B be the resulting red and blue subgraphs of G. Thus, $\deg_{G_R} w_i = \deg_{G_B} w_i = 5$ for $1 \leq i \leq 16$. By (i) and (ii), there is no $K_{3,3}$ in G_R and no $K_{3,3}$ in G_B. Therefore, $BR_{10}(K_{3,3}) \geq 17$.

Next, we show that $BR_{10}(K_{3,3}) \leq 23$. Let $H = K_{10,23}$ where the partite sets are U and W with $|U| = 10$ and $|W| = 23$. We show that every red-blue coloring of H results in a monochromatic $K_{3,3}$. Assume, to the contrary, that there exists a red-blue coloring c of H that avoids a monochromatic $K_{3,3}$. Let H_R and H_B denote the red and blue subgraphs, respectively, of H obtained from the coloring c. Let m_R and m_B denote the size of H_R and H_B. Thus, $m_R + m_B = 230$. We may assume that $m_R \geq m_B$ and so $m_R \geq 115$. Thus, the average degree of the vertices of W in H_R is at least $115/23 = 5$.

We claim that the vertices of W in H_R are joined to a minimum number of triples of U (with repetitions) when H_R is a 5-regular graph. Note that if H_R is 5-regular, then the vertices of W are joined to $23 \cdot \binom{5}{3} = 23 \cdot 10 = 230$ triples of U in H_R. Assume, to the contrary, that H_R is not 5-regular. If $\delta(H_R) \geq 5$ and H_R is not 5-regular, then the vertices of W are joined to more than 230 triples of U in H_R. Thus, we may assume that $\delta(H_R) \leq 4$. Since the average degree of the vertices of W in H_R is at least 5, there are $w, w' \in W$ such that $\deg_{H_R} w \leq 4$ and $\deg_{H_R} w' \geq 6$. Replace the edges of H_R incident with w by $\deg_{H_R} w + 1$ red edges incident with w and replace the edges of H_R incident with w' by $\deg_{H_R} w' - 1$ red edges incident with w', resulting in a new red-blue coloring c^* of H, whose red subgraph is H_R^*. Since $\deg_{H_R} w \leq 4$ and $\deg_{H_R} w' \geq 6$, with the aid of (3), it can be shown that in H_R^*, the vertices of W are joined to *fewer* triples in H_R^* than in H_R, which is impossible. Hence, H_R is a 5-regular graph, as claimed. The total number of triples in U is $\binom{10}{3} = 120$.

Let $U = \{1, 2, \ldots, 9, 0\}$ and let $a, b \in U$. Consider the 23 5-tuples of U corresponding to the neighborhoods of the 23 vertices of W. If ab belongs to 6 of these 5-tuples, then some element in $U - \{a, b\}$ must belong to 3 of these six 5-tuples, implying that H_R contains $K_{3,3}$, which is impossible. Thus, ab belongs to at most five 5-tuples. There are $\binom{10}{2} = 45$ distinct pairs of vertices in U. Since each pair of vertices of U belongs to at most five 5-tuples, the pairs of vertices of U must occur at most $5 \cdot 45 = 225$ times among the 5-tuples of U. However, since the 23 5-tuples contain $23 \cdot \binom{5}{2} = 23 \cdot 10 = 230$ pairs of vertices of U, it follows that some pair of vertices of U belongs to at least 6 of these 5-tuples, which contradicts the fact that every pair of vertices of U belongs to at most five 5-tuples. Therefore, $BR_{10}(K_{3,3}) \leq 23$. ■

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