A note on the complexity of the total domatic partition problem in graphs*

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Abstract

In this paper, we study the total domatic partition problem for bipartite graphs, split graphs, and graphs with balanced adjacency matrices. We show that the total domatic partition problem is NP-complete for bipartite graphs and split graphs, and show that the total domatic partition problem is polynomial-time solvable for graphs with balanced adjacency matrices. Furthermore, we show that the total domatic partition problem is linear-time solvable for any chordal bipartite graph G if a Γ-free form of the adjacency matrix of G is given.

Keywords: Graph algorithm; Total domatic partition; Balanced adjacency matrix; Hypergraph;

1 Introduction

Let $G = (V, E)$ be a finite, simple, undirected graph. Unless stated otherwise, it is understood that $|V| = n$ and $|E| = m$. We also use $V(G)$ and $E(G)$ to denote vertex set and edge set of G, respectively. For any vertex $v \in V$, the *open neighborhood* of v in G is $N_G(v) = \{u \in V | (u, v) \in E\}$ and the *closed neighborhood* of v in G is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a

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vertex v in G is $deg_G(v) = |N_G(v)|$. The *minimum degree* of a vertex of G is denoted by $\delta(G)$.

A vertex v of a graph $G = (V, E)$ *dominates* a vertex w if $v \in N_G[w]$. A vertex v of G *totally dominates* a vertex w if $v \in N_G(w)$. A subset $D \subseteq V$ *dominates* (respectively, *totally dominates*) a vertex v if v is dominated (respectively, totally dominated) by some vertex in D, i.e., $|D \cap N_G[v]| \geq 1$ (respectively, $|D \cap N_G(v)| \geq 1$). A *dominating* (respectively, *total dominating*) *set* of a graph $G = (V, E)$ is a subset D of V such that D dominates (respectively, totally dominates) every vertex v in V.

For a positive integer k, a *k-total* (respectively, *k-tuple*) *dominating set* of a graph $G = (V, E)$ is a subset D of V such that $|D \cap N_G(v)| \geq k$ (respectively, $|D \cap N_G[v]| \geq k$) for every $v \in V$. Clearly, G does not have any k-total (respectively, k-tuple) dominating set if $k > \delta(G)$ (respectively, $k > \delta(G) + 1$).

Two sets A and B are *disjoint* if $A \cap B = \emptyset$. A collection $P = \{S_1, S_2, \ldots, S_\ell\}$ forms a *partition* of a set S if $S = S_1 \cup S_2 \cup \cdots \cup S_\ell$, and $S_i \cap S_j = \emptyset$ for any two distinct sets S_i and S_j in P.

A partition $P = \{V_1, V_2, \ldots, V_\ell\}$ of V is a *domatic* (respectively, *total domatic*) *partition* of a graph $G = (V, E)$ if V_i is a dominating (respectively, total dominating) set of G for $i = 1, 2, \ldots, \ell$. The *domatic* (respectively, *total domatic*) *partition problem* is to find a domatic (respectively, total domatic) partition of G of maximum size. The *domatic* (respectively, *total domatic*) *number* of G, denoted by $d(G)$ (respectively, $d_t(G)$), is the size of a maximum domatic (respectively, total domatic) partition of G. For any positive integer k, the k-domatic (respectively, total k-domatic) partition problem is to find a domatic (respectively, total domatic) partition P of G such that $|P| = k$.

The domatic number of a graph was introduced by Cockayne and Hedetniemi [9]. The concept of the total domatic number was introduced by the same authors and Dawes [7]. They proved that $d(G) \leq \delta(G) + 1$ and $d_t(G) \leq \delta(G)$ for any graph G. The domatic partition problem has been widely studied from the algorithmic point of view [4, 5, 8, 11, 14, 15, 18, 19, 21]. The total domatic number of a graph has been investigated in [2, 3, 5, 7, 11, 22, 23, 24, 25]. However, few results exist in the literature about the algorithmic complexity of the total domatic partition problem in graphs. From the algorithmic point of view, the total k-domatic partition problem on the 2-section graph of the order-interval hypergraph of a finite poset is NP-complete for any fixed positive integer $k \geq 3$ [5], and the total 2-domatic partition problem on bipartite graphs is NP-complete [13]. The results motivate us to consider the algorithmic complexity of the total domatic partition problem for other classes of graphs.

In this paper, we concentrate on bipartite graph, split graphs, and graphs with balanced adjacency matrices. For any fixed integer $k \geq 3$

(respectively, $k \geq 2$), we show that the total k-domatic partition problem is NP-complete, even when restricted to bipartite graphs (respectively, split graphs). For any positive integer $k \leq \delta(G)$, we show that the k-total domatic partition problem is polynomial-time solvable for any graph G with a balanced adjacency matrix. Furthermore, we show that the total k-domatic partition problem is linear-time solvable for any chordal bipartite graph G if a Γ-free form of the adjacency matrix of G is given.

2 The NP-completeness results

An *independent set* of a graph $G = (V, E)$ is a subset S of V such that no two vertices of S are adjacent. A *clique* of G is a subset of pairwise adjacent vertices of V. A graph $G = (V, E)$ is a *split graph* if V can be partitioned into an independent set I and a clique Q. Split graphs form a subclass of *chordal graphs* [6]. A *bipartite* graph is a graph G whose vertices can be divided into two disjoint sets A and B such that every edge in G connects a vertex in A to one in B. A split (respectively, bipartite) graph is usually written as $G = (I, Q, E)$ (respectively, $G = (A, B, E)$).

In this section, we present NP-completeness results for bipartite graphs and split graphs. Before presenting the NP-completeness results, we restate the k-domatic partition problem and the total k-domatic partition problem as decision problems.

(1) **The k-domatic partition problem:**

Instance: A graph $G = (V, E)$ and a positive integer $k \leq \delta(G) + 1$.

Question: Is $d(G) \geq k$?

(2) **The total k-domatic partition problem:**

Instance: A graph $G = (V, E)$ and a positive $k \leq \delta(G)$.

Question: Is $d_t(G) \geq k$?

Theorem 1. *For any fixed integer $k \geq 3$, the total k-domatic partition problem on bipartite graphs is NP-complete.*

Proof. It is obvious that the total k-domatic partition problem is a member of NP. The total 2-domatic partition problem on bipartite graphs is NP-complete [13]. In the following, we show the NP-completeness of the total k-domatic partition problem on bipartite graphs for any fixed positive integer $k \geq 3$ by a polynomial-time reduction from the total $(k-1)$-domatic partition problem on bipartite graphs.

Let $G = (A, B, E)$ be a bipartite graph. We create two new vertices x and y. Let $A' = A \cup \{x\}$ and let $B' = B \cup \{y\}$. We connect x to y and all the vertices in B, and then connect y to all the vertices in A. Let H be the resulting graph. Clearly, the graph H is a bipartite graph with two disjoint sets of vertices A' and B'.

Let $d_t(G) = \ell$ and let $\mathcal{D} = \{D_1, D_2, \ldots, D_\ell\}$ be a total domatic partition of G. Clearly, $D_1, D_2, \ldots, D_\ell$ are also disjoint total dominating sets of H. Let $D_{\ell+1} = \{x, y\}$. By the construction of H, the set $D_{\ell+1}$ is a total dominating set of H. Then, $D_1, D_2, \ldots, D_{\ell+1}$ are disjoint total dominating sets of H. We have $d_t(H) \geq d_t(G) + 1$.

Let $d_t(H) = \kappa$ and let $\mathcal{S} = \{S_1, S_2, \ldots, S_\kappa\}$ be a total domatic partition of H. We consider the following two cases.

Case 1: The vertices x and y are in the same set of $\mathcal{S}$. Let S_i be the set containing the vertices x and y. Note that $V(H) = V \cup \{x, y\}$. Let $S_j \in \mathcal{S} \setminus \{S_i\}$ and let $S'_j = S_j \cup S_i \setminus \{x, y\}$. Clearly, the collection $P = (\mathcal{S} \setminus \{S_i, S_j\}) \cup \{S'_j\}$ is a total domatic partition of G. Therefore, $d_t(G) \geq d_t(H) - 1$.

Case 2: The vertices x and y are in different sets of $\mathcal{S}$. Let S_i and S_j be distinct sets of $\mathcal{S}$ such that $x \in S_i$ and $y \in S_j$. Then, every vertex in B (respectively, A) is totally dominated by the vertex x (respectively, y), and every vertex in A (respectively, B) is totally dominated by some vertex in $S_i \cap B$ (respectively, $S_j \cap A$). Let $S'_i = (S_i \setminus \{x\}) \cup (S_j \setminus \{y\})$. Then, S'_i is a total dominating set of G and the collection $P = (\mathcal{S} \setminus \{S_i, S_j\}) \cup \{S'_i\}$ is a total domatic partition of G. Therefore, $d_t(G) \geq d_t(H) - 1$.

Following the discussion above, we know that $d_t(H) \leq d_t(G) + 1$ and $d_t(H) \geq d_t(G)+1$. Therefore, $d_t(H) = d_t(G)+1$. It implies that $d_t(H) \geq k$ for any fixed integer $k \geq 3$ if and only if $d_t(G) \geq k - 1$. □

Theorem 2. *For any fixed integer $k \geq 2$, the total k-domatic partition problem on split graphs is NP-complete.*

Proof. The total k-domatic partition problem is clearly in NP. The k-domatic partition problem on chordal graphs is NP-complete for any fixed positive integer $k \geq 3$[14]. Notice that the problem of determining whether a 2-tuple dominating set of a chordal graph can be partitioned into two disjoint dominating sets is NP-complete [11]. Since the set of all vertices of a connected graph is a 2-tuple dominating set, the 2-domatic partition problem on chordal graphs is NP-complete. Therefore, the k-domatic partition problem on chordal graphs is NP-complete for any fixed positive integer $k \geq 2$.

In the following, we show the NP-completeness of the total k-domatic partition problem on split graphs for any fixed positive integer $k \geq 2$ by a polynomial-time reduction from the k-domatic partition problem on chordal graphs.

Let $G = (V, E)$ be a chordal graph. We construct a graph H by the following steps.

(1) For each vertex $v \in V$, we create a new vertex v' and connect the vertex v' to all vertices in $N_G[v]$.

(2) We add edges to G to form a subgraph $G'' = (V'', E'')$ such that $V'' = V$ and V'' is a clique of H.

Let $V' = \{v' \mid v \in V\}$ and let $E' = \{(u, v') \mid v' \in V', v \in V, u \in N_G[v]\}$. Since V'' is a clique of H and $V'' = V$, $E'' = \{(u, v) \mid u \neq v \text{ and } u, v \in V\}$ Then, $V(H) = V' \cup V''$ and $E(H) = E' \cup E''$. Clearly, the construction of H can be done in polynomial time. By the construction of H, V' is an independent set and V'' is a clique. Therefore, H is a split graph.

Let $d(G) = \ell$ and let $\mathcal{D} = \{D_1, D_2, \ldots, D_\ell\}$ be a domatic partition of G. For $i = 1, 2, \ldots, \ell$, let $D_i' = D_i \cup \{v' \mid v \in D_i\}$. Then, D_i' is a total dominating set of H and the set $\mathcal{D}' = \{D_1', D_2', \ldots, D_\ell'\}$ is a total domatic partition of $V(H)$. We have $d_t(H) \geq d(G)$.

Let $d_t(H) = \kappa$ and let $\mathcal{S} = \{S_1, S_2, \ldots, S_\kappa\}$ be a total domatic partition of H. Let S_i be a set of $\mathcal{S}$ and let v be a vertex of G. By the construction of H, the set V' is an independent set of H and $N_H(v') = N_G[v]$. There exists a vertex $x \in S_i$ such that $x \in N_G[v]$ and v' is totally dominated by x. Therefore, $S_i \setminus V'$ is a dominating set of G. We have $d_t(H) \leq d(G)$.

Since $d_t(H) \geq d(G)$ and $d_t(H) \leq d(G)$, we have $d_t(H) = d(G)$. It implies that $d(G) \geq k$ for any fixed integer $k \geq 2$ if and only $d_t(H) \geq k$.

3 Graphs with balanced adjacency matrices

Suppose that $G = (V, E)$ is a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. The *adjacency matrix* of G is the (0,1)-matrix whose entry (i, j) is 1 if $(v_i, v_j) \in E$ and 0 otherwise. The *edge-vertex incidence matrix* of G is the (0,1)-matrix whose entry (i, j) is 1 if v_j is an endvertex of e_i and 0 otherwise. A (0,1)-matrix is *balanced* if it does not contain the edge-vertex incidence matrix of an odd cycle as a submatrix. A (0,1)-matrix is *totally balanced* if it does not contain the edge-vertex incidence matrix of a cycle as a submatrix.

A *hypergraph* H is an ordered pair $(V, \mathcal{E})$ where V is a set of vertices and $\mathcal{E}$ is a set of subsets of V. Each member of $\mathcal{E}$ is called a *hyperedge* of H. Let $V = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{E} = \{E_1, E_2, \ldots, E_m\}$. The *hyperedge-vertex incidence matrix* of H is the (0,1)-matrix whose entry (i, j) is 1 if E_i contains the vertex v_j and 0 otherwise. A hypergraph H is a *balanced* (respectively, *totally balanced*) hypergraph if the hyperedge-vertex incidence matrix of H is balanced (respectively, *totally balanced*).

A *transversal* of a hypergraph $H = (V, \mathcal{E})$ is a subset D of V such that $|D \cap E_i| \geq 1$ for every hyperedge $E_i \in \mathcal{E}$. Let k be a positive integer.

A *k-fold transversal* of a hypergraph $H = (V, \mathcal{E})$ is a subset S of V such $|S \cap E_i| \geq k$ for every hyperedge $E_i \in \mathcal{E}$.

Dahlhaus et al. [11] gave an algorithm to partition a k-fold transversal of a balanced hypergraph into k pairwise disjoint transversals in polynomial time. They obtained the following result.

Theorem 3 (Dahlhaus et al. [11]). *A k-fold transversal of a balanced hypergraph can be partitioned into k pairwise disjoint transversals in polynomial time.*

Lemma 1. *Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and let k be a positive integer no more than $\delta(G)$. Let $E_i = N_G(v_i)$ for $1 \leq i \leq n$ and let $H = (V, \mathcal{E})$ be a hypergraph such that $\mathcal{E} = \{E_1, E_2, \ldots, E_n\}$. Then, the following statements are true.*

(1) A subset S of V is a k-total dominating set of G if and only if S is a k-fold transversal of H.

(2) A $(0,1)$-matrix M is the adjacency matrix of G if and only if M is the hyperedge-vertex incidence matrix of H.

Proof. (1) Let S be a k-total dominating set of G. Then, $|S \cap N_G(v_i)| = |S \cap E_i| \geq k$ for $1 \leq i \leq n$. The set S is a k-fold transversal of H. Conversely, let S be a k-fold transversal of H. Then, $|S \cap E_i| = |S \cap N_G(v_i)| \geq k$ for $1 \leq i \leq n$. The set S is a k-total dominating set of G. Following the discussion above, the statement is true.

(2) Let M be the adjacency matrix of G. Let m_{ij} be the entry (i, j) in the matrix M for $1 \leq i, j \leq n$. If E_i contains the vertex v_j, then $v_j \in N_G(v_i)$. We know that $(v_i, v_j) \in E$ and $m_{ij} = 1$. If E_i does not contain the vertex v_j, then $v_j \notin N_G(v_i)$. The vertex v_i is not adjacent to v_j and $m_{ij} = 0$. Therefore, m_{ij} is 1 if E_i contains the vertex v_j and 0 otherwise. The matrix M is also the hyperedge-vertex incidence matrix of H.

Conversely, let M be the hyperedge-vertex incidence matrix of H. Also, let m_{ij} be the entry (i, j) in the matrix M for $1 \leq i, j \leq n$. If $(v_i, v_j) \in E$, then $v_j \in N_G(v_i)$. Since $E_i = N_G(v_i)$, E_i contains the vertex v_j and thus $m_{ij} = 1$. If $(v_i, v_j) \notin E$, then $v_j \notin N_G(v_i)$. We know that E_i does not contain the vertex v_j and $m_{ij} = 0$. Therefore, m_{ij} is 1 if $(v_i, v_j) \in E$ and 0 otherwise. The matrix M is also the adjacency matrix of G. Following the discussion above, the statement is true. ∎

Theorem 4. *For any positive integer $k \leq \delta(G)$, a k-total dominating set of a graph G with a balanced adjacency matrix can be partitioned into k pairwise disjoint total dominating sets in polynomial time.*

Proof. Suppose that $G = (V, E)$ is a graph with the balanced adjacency matrix M and $V = \{v_1, v_2, \ldots, v_n\}$. Let k be a positive integer and let S be a k-total dominating set of G. Clearly, $k \leq \delta(G)$. Otherwise, G does not have any k-total dominating set.

Notice that the matrix M contains n rows and n columns. We construct the set $E_i = N_G(v_i)$ for $1 \leq i \leq n$. Let $\mathcal{E} = \{E_1, E_2, \ldots, E_n\}$ and let $H = (V, \mathcal{E})$ be a hypergraph. By Statement (2) of Lemma 1, the matrix M is also the hyperedge-vertex incidence matrix of H. Since M is balanced, H is balanced.

Dahlhaus et al. [11] gave an algorithm to partition a k-fold transversal of a balanced hypergraph into k pairwise disjoint transversals. By Statement (1) of Lemma 1, the set S is a k-fold transversal of H and any transversal of H is a total dominating set of G. We can use their algorithm to partition the set S into k pairwise disjoint transversals $S_1, S_2, \ldots, S_k$ of H and this can be done in polynomial time by Theorem 3. Following the discussion above, the theorem is true.

Theorem 5. *The total domatic partition problem can be solved in polynomial time for graphs with balanced adjacency matrices.*

Proof. Let $G = (V, E)$ be a graph with a balanced adjacency matrix. Since $\delta(G)$ is the minimum degree of G, $|N_G(v) \cap V| \geq \delta(G)$ for every $v \in V$. The vertex set V is a $\delta(G)$-total dominating set of G. By Theorem 4, V can be partitioned into $\delta(G)$ pairwise disjoint total dominating sets $V_1, V_2, \ldots, V_{\delta(G)}$ in polynomial time. Notice that $d_t(H) \leq \delta(H)$ for any graph H [7]. The partition $P = \{V_1, \ldots, V_{\delta(G)}\}$ is a total domatic partition of G of maximum size. Hence, the total domatic partition problem is polynomial-time solvable for graphs with balanced adjacency matrices. ❐

4 Graphs with totally balanced adjacency matrices

In this section, we consider the total k-domatic partition problem on graphs with totally balanced adjacency matrices.

A $(0, 1)$-matrix is a Γ-*free* matrix if it does not contain the submatrix

$$\Gamma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

A Γ-*free ordering* of a (0,1)-matrix is a permutation of the rows and columns of the original matrix such that the resulting matrix is Γ-free. If a $(0, 1)$-matrix M can be permuted to be a Γ-free matrix M' by a Γ-free ordering of M, then M' is a Γ-*free form* of M.

For any (0, 1)-matrix M with p rows, n columns, and m nonzero entries, Lubiw [16] gave an algorithm to find a Γ-free form of M in $O(m \log^2(n+p))$ time if such a Γ-free form exists. The time bound was improved to $O(m \log(n+p))$ by Paige and Tarjan [17] and $O((n+p)^2)$ by Spinrad [20].

Algorithm HM.

(1) Input a Γ-free form A' of the hyperedge-vertex incidence matrix A of a totally balanced hypergraph H and a k-fold transversal S of H.

(2) Prune matrix A' and call the resulting matrix $\bar{A}$.

(3) Assign colors to the non-zero entries of $\bar{A}$ such that (a) the non-zero entries of each column are assigned the same color and (b) all the k colors appear in each row.

(4) Partition the columns according to the colors of the columns such that S_α is the set of columns with the color α, for $1 \leq \alpha \leq k$.

(5) Output transversal partition $S_1, S_2, \ldots, S_k$ of S.

Theorem 6 (Anstee and Farber[1], Hoffman et al.[12], Lubiw[16]). *A* $(0,1)$*-matrix* M *is totally balanced if and only if there is a* Γ*-free ordering of* M.

Let $H = (V, \mathcal{E})$ be a totally balanced hypergraph with $V = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{E} = \{E_1, E_2, \ldots, E_p\}$ and let A be the hyperedge-vertex incidence matrix of H. Let $\sum_{i=1}^{p} |E_i| = m$. Then, A has m nonzero entries. Following Theorem 6, we know that any totally balanced matrix has a Γ-free form. The hyperedge-vertex incidence matrix A of H is totally balanced, so A has a Γ-free form. For example, consider the following two matrices

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \text{ and } A' = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The matrix A' is a Γ-free form of the hyperedge-vertex incidence matrix A of the totally balanced hypergraph $H = (V, \mathcal{E})$, where $V = \{v_1, v_2, \ldots, v_7\}$ and $\mathcal{E} = \{E_1 = \{v_1, v_2, v_3\}, E_2 = \{v_1, v_3, v_5, v_6\}, E_3 = \{v_4, v_5, v_7\}, E_4 = \{v_4, v_5, v_6, v_7\}\}$.

Dahlhaus et al. [11] gave a parallel algorithm, called Algorithm HM, to partition a k-fold transversal S of a totally balanced hypergraph H into k pairwise disjoint transversals in $O(\log n)$ time with $O(n^2)$ processors. Furthermore, their parallel algorithm can be improved to partition a k-fold transversal S of H into k pairwise disjoint transversals in $O(\log n)$ time with $O((n+m)/\log n)$ processors if a Γ-free form A' of the hyperedge-vertex incidence matrix A of H is given and implemented by the *linked list data structure* and the *list ranking approach* of Cole and Vishkin [10] is applied to the parallel algorithm. For $k=2$, the following matrix $\bar{A}$ is a pruned matrix obtained by performing Step (2) of Algorithm HM on a Γ-free form A' as we mentioned earlier in this section.

$$\bar{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Theorem 7. *Let $H=(V,\mathcal{E})$ be a totally balanced hypergraph with the hyperedge-vertex incidence matrix A. Assume that $V=\{v_1,v_2,\ldots,v_n\}$, $\mathcal{E}=\{E_1,E_2,\ldots,E_p\}$, and the matrix A contains m nonzero entries. Given a Γ-free form A' of the hyperedge-vertex incidence matrix A of H, a k-fold transversal S of H can be partitioned into k pairwise disjoint transversals $S_1,S_2,\ldots,S_k$ in $O(n+p+m)$ time.*

Proof. The theorem can be proved by showing that Algorithm HM can be implemented in $O(n+p+m)$ sequential time. It is not difficult to see that steps (1), (2), (4), and (5) of Algorithm HM can be implemented in $O(n+p+m)$ time. Step (3) of Algorithm HM can also be implemented in $O(n+p+m)$ time. This can be verified using the notations defined in [11]. We leave the details to interested readers. Following the discussion above, the theorem is true. □

Corollary 1. *For any positive integer $k \le \delta(G)$, the total k-domatic partition problem can be solved in linear time for any graph G with the totally balanced adjacency matrix A if a Γ-free form of A is given.*

Proof. Suppose that $G=(V,E)$ with $|V|=n$ and $|E|=m$. Then, the matrix A contains n rows, n columns, and $2m$ nonzero entries. Let $V=\{v_1,v_2,\ldots,v_n\}$ and let S be a k-total dominating set of G. Clearly, $k \le \delta(G)$. Otherwise, G does not have any k-total dominating set.

We construct the set $E_i=N_G(v_i)$ for $1 \le i \le n$. Let $\mathcal{E}=\{E_1,\ldots,E_n\}$ and let $H=(V,\mathcal{E})$ be a hypergraph. By Statement (2) of Lemma 1, the matrix A is also the hyperedge-vertex incidence matrix of H. Since A is totally balanced, H is totally balanced. By Statement (1) of Lemma 1, the set S is a k-fold transversal of H and any transversal of H is a total dominating set of G. By Theorem 7, we can partition the set S into k pairwise disjoint dominating sets $S_1,S_2,\ldots,S_k$ in $O(n+m)$ time. Following the discussion above, we know that the total k-domatic partition problem is linear-time solvable for any graph G with a totally balanced adjacency matrix A if a Γ-free form of A is given. □

Corollary 2. *The total domatic partition problem is linear-time solvable for any chordal bipartite graph G if a Γ-free form of the adjacency matrix of G is given.*

Proof. For any chordal bipartite graph G, the adjacency matrix of G is totally balanced [6]. By Corollary 1 and using the arguments similar to those for proving Theorem 5, the corollary holds. □

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