Adjacent vertex distinguishing total colorings of outer 1-planar graphs *

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Abstract

An adjacent vertex distinguishing total coloring of a graph G is a proper total coloring of G such that no pair of adjacent vertices are incident to the same set of colors. The minimum number of colors required for an adjacent vertex distinguishing total coloring of G is denoted by $\chi''_a(G)$. In this paper, we prove that if G is an outer 1-planar graph with at least two vertices, then $\chi''_a(G) \leq \max\{\Delta+2, 8\}$. Moreover, we also prove that when $\Delta \geq 7$, $\chi''_a(G) = \Delta + 2$ if and only if G contains two adjacent vertices of maximum degree.

Keywords: Adjacent vertex distinguishing total coloring, Outer 1-planar graph, Maximum degree

1 Introduction

Let G be a simple and finite graph. We use $V(G), E(G), \Delta(G)$ and $\delta(G)$ (or simply V, E, Δ and δ) to denote the vertex set, edge set, maximum degree and minimum degree of G, respectively. For a vertex $v \in V(G)$, we use $N(v)$ to denote the set of neighbors of v in G. Let $N[v] = N(v) \cup \{v\}$. We use $d(v) = |N(v)|$ to denote the degree of v in G. A k-, k^--and k^+-vertex is a vertex of degree k, at most k and at least k, respectively. A k-neighbor (resp. k^--neighbor or k^+-neighbor) of v is a k-vertex (resp. k^--vertex or k^+-vertex) adjacent to v.

A proper total-k-coloring of a graph G is a mapping $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ such that any two adjacent or incident elements in $V(G) \cup E(G)$ receive different colors. Denote $C_\phi(v) = \{\phi(v)\} \cup \{\phi(uv) | uv \in E(G)\}$ the set of colors assigned to a vertex v and those edges incident to v.

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An adjacent vertex distinguishing total coloring (or total-k-avd-coloring for short) ϕ of G is a proper total-k-coloring such that $C_\phi(u) \neq C_\phi(v)$ whenever $uv \in E(G)$. The adjacent vertex distinguishing total chromatic number $\chi''_a(G)$ is the smallest integer k such that G admits a total-k-avd-coloring. Note that $\chi''_a(G) \geq \Delta+1$, and if G contains two adjacent vertices of maximum degree, then $\chi''_a(G) \geq \Delta + 2$. Based on the results of some basic families of graphs, such as paths, cycles, fans, wheels, trees, complete graphs, and complete bipartite graphs, Zhang et al. [14] put forward the following conjecture.

Conjecture 1.1. *If G is a graph with at least two vertices, then $\chi''_a(G) \leq \Delta + 3$.*

Chen [1] and Wang [9], independently confirmed Conjecture 1.1 for graphs with $\Delta = 3$. Later, Hulgan [5] presented a more concise proof on this result. Recently Lu et al. [6] verified Conjecture 1.1 for all graphs with maximum degree 4. Wang et al. characterized completely the adjacent vertex distinguishing total chromatic number of outer planar graphs [12], K_4-minor free graphs [11] and planar graphs with $\Delta \geq 14$ [10]. Huang and Wang [4] proved the conjecture for planar graphs with $\Delta \geq 11$. Conjecture 1.1 is also confirmed for planar graphs with maximum degree $\Delta = 10$ [2] and maximum degree $\Delta \geq 8$ containing no adjacent 4-cycles [8]. A graph G is called 2-degenerate if every subgraph of G contains a vertex of degree at most 2. Miao et al. [7] showed that for all 2-degenerate graph G, $\chi''_a(G) \leq \max\{\Delta+2, 6\}$ holds, and when $\Delta \geq 5$, $\chi''_a(G) = \Delta+2$ if and only if G contains two adjacent Δ-vertices.

A graph is outer 1-planar if each block has an embedding in the plane in such a way that the vertices lie on a fixed circle and the edges lie inside the disk of this circle with each of them crossing at most one another. The definition of outer 1-planarity implies that outer 1-planar graphs are all planar and is a natural extension of the family of outer planar graphs. The notion of outer 1-planar graphs were introduced by Eggleton [3], who called them outerplanar graphs with edge crossing number one, and were also investigated under the notion of pseudo-outerplanar graphs by Zhang et al. [13].

Let $\chi(G)$ and $\chi'(G)$ denote the vertex chromatic number and the edge chromatic index of a graph G, respectively. Then there is a relation $\chi''_a(G) \leq \chi(G) + \chi'(G)$ for any graph G. It is known in Zhang et al. [13] that every outer 1-planar graph with maximum degree $\Delta \geq 4$ has $\chi'(G) = \Delta(G)$. Combining this fact with the Four-color Theorem, we get the following upper bound: If G is an outer 1-planar graph with $\Delta \geq 4$, then $\chi''_a(G) \leq \Delta + 4$.

In this paper, we improve this upper bound for outer 1-planar graphs. We also characterize the outer 1-planar graphs having $\chi''_a(G) \leq \Delta + 2$ for

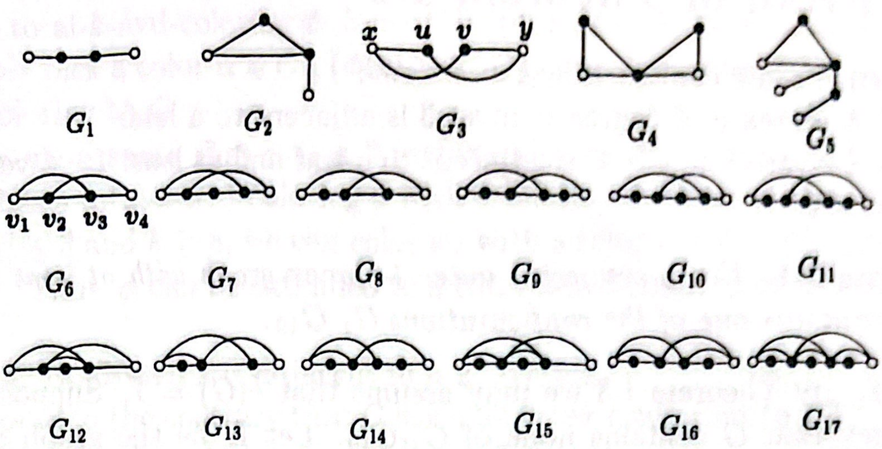

Fig. 1: Configurations in Theorem 1.3

$\Delta \geq 7$.

Theorem 1.1. *Let G be an outer 1-planar graph. Then*

(1) $\chi''_a(G) \leq 8$ if $\Delta \leq 6$;

(2) $\chi''_a(G) \leq \Delta+2$ for $\Delta \geq 7$, and $\chi''_a(G) = \Delta+2$ if and only if G contains two adjacent Δ-vertices.

For a graph G, let $k(G) = \max\{\Delta + 2, 8\}$ if G contains two adjacent Δ-vertices and $k(G) = \max\{\Delta + 1, 8\}$ otherwise. Then $k(G) \geq 8$. Thus, Theorem 1.1 is equivalent to the following theorem.

Theorem 1.2. *Let G be an outer 1-planar graph. Then $\chi''_a(G) \leq k(G)$.*

To prove our main result, we heavily rely on the following structural theorem due to [13].

Theorem 1.3. *If G is a pseudo-outerplanar diagram with $\delta \geq 2$, then G contains one of the following configurations G_1-G_{17}. Here the solid vertices have no edges of G incident with them other than those shown. Moreover,*

(a) if G contains some configuration among G_6-G_{17}, then the drawing of this configuration in the figure is a part of the diagram of G with its bending edges corresponding to the chords;

(b) if G contains the configuration G_3 and $xy \notin E(G)$, where x and y are the vertices of G_3 as described in the figure, then we can properly add an edge xy to G so that the resulting diagram is still outerplanar.

2 Proof of Theorem 1.2

We define some configurations as follows:
(G_{18}) A vertex v of degree at most 3 is adjacent to a leaf.
(G_{19}) A t-vertex v, $t \geq 4$, is adjacent to a leaf and at least $t-4$ vertices of degree ≤ 2.

Lemma 2.1. *Every connected outer 1-planar graph with at least two vertices contains one of the configurations G_1-G_{19}.*

Proof. By Theorem 1.3 we may assume that $\delta(G) = 1$. Suppose to the contrary that G contains none of G_1-G_{19}. Let H be the graph obtained by removing all leaves of G. Then H is a connected outer 1-planar graph. Since G has no G_{18}, there is no vertex of degree at most three adjacent to a leaf. In addition, since G has no G_{19}, every vertex v of degree at least 4 is adjacent to at most $d(v)-4$ leaves; that is, it has at least four neighbors that are not leaves. So for every $v \in V(G)$, $d_H(v) \geq 2$ and $d_H(v) = d_G(v)$ if $2 \leq d_G(v) \leq 3$. By Theorem 1.3, H contains one of G_1 -G_{17}. If H contains G_1 or G_3, then G_1 or G_3 must be a configuration of G by the excluding of G_{18}. If H contains one of G_2 and G_4-G_{17}, by the excluding of G_{19}, a solid 3^+-vertex in the configuration cannot be adjacent to any leaf. So G also contains this configuration. This contradicts the assumption on G. □

Proof of Theorem 1.2. Given an outer 1-planar graph G, assume that $v \in V(G)$ with $d(v) \leq 2$. Let $C = \{1, 2, \ldots, k\}$, where $k \geq 8$. Suppose that ϕ is a total-k-avd-coloring of G with v uncolored. Since v has at most two adjacent vertices and two incident edges, we always can color v in the last stage when all its incident or adjacent elements have been colored. In other words, we may omit the coloring for such vertices in the following discussion.

We shall prove Theorem 1.2 by contradiction. Let G be a counterexample such that $|E(G)|$ as small as possible. Clearly, G is connected. By the result of [6], we may assume that $\Delta \geq 5$. Denote $k = k(G)$ and $C = \{1, 2, \ldots, k\}$ the set of colors.

Claim 1. For every subgraph H of G with $|E(H)| < |E(G)|$, H has a total-k-avd-coloring.

Since any subgraph of G is also an outer 1-planar graph, then by the choice of G, for any subgraph H of G with $|E(H)| < |E(G)|$, H has a total-$k(H)$-avd-coloring. Since $k(H) \leq k(G) = k$, H is total-k-avd-colorable. □

Claim 2. No 2-vertex is adjacent to a 2^--vertex.

Suppose to the contrary that G contains a 2-vertex v adjacent to a 2^--vertex u. Since G is connected and $\Delta \geq 5$, we may assume that the other neighbor v_1 of v is a 3^+-vertex.

First, we assume that u is a leaf. Let $H = G - u$. Then by assumption, H has a total-k-avd-coloring ϕ. Since $|C\backslash\{\phi(v), \phi(vv_1)\}| \geq 6$ and $d(v_1) \geq 3$, we simply pick a color $\alpha \in C\backslash\{\phi(v), \phi(vv_1)\}$, and then extend ϕ to a total-k-avd-coloring to G.

Now we assume that u is a 2-vertex. Denote $N(u) = \{v, u_1\}$. Then $G - uv$ has a total-k-avd-coloring ϕ with u uncolored. Since $|\{\phi(v), \phi(vv_1), \phi(uu_1)\}| \leq 3$ and $k \geq 8$, we can color uv with a color $\alpha \in C\backslash\{\phi(v), \phi(vv_1), \phi(uu_1)\}$. Thus, ϕ can be extended to a total-k-avd-coloring of G, a contradiction. □

Claim 3. No 3-vertex is adjacent to a 2^--vertex.

Suppose to the contrary that G has a 3-vertex v adjacent to a 2^--vertex u. Let v_1, v_2 be the other two neighbors of v.

When u is a leaf, the proof is similar to that in Claim 2. So we assume that u is a 2-vertex. Let w be the other neighbor of u. By Claim 2 we have $d(w) \geq 3$. Then $G - uv$ has a total-k-avd-coloring ϕ with u uncolored.

Since $|C\backslash\{\phi(v), \phi(vv_1), \phi(vv_2), \phi(uw)\}| \geq 4$, we may choose a color $\alpha \in C\backslash\{\phi(v), \phi(vv_1), \phi(vv_2), \phi(uw)\}$ such that $\{\alpha\} \cup C_\phi(v) \neq C_\phi(v_i)$, $i = 1, 2$. By the choice of α, one can extend ϕ to a total-k-avd-coloring of G by coloring uv with α, a contradiction. □

Claim 4. Each 4-vertex is adjacent to at most one 2-vertex.

Suppose to the contrary that G has a 4-vertex v adjacent to at least two 2-vertices u_1 and u_2. Let w be the other neighbor of u_1. Then $G - u_1v$ has a total-k-avd-coloring ϕ with u_1 uncolored. The following proof is similar to that of Claim 3. □

Claim 5. No 4-vertex v is adjacent to a 2-vertex u with $N[u] \subseteq N[v]$.

Suppose to the contrary that G contains such a pair of adjacent vertices u, v. Let v_1, v_2 and v_3 be the other three neighbors of v. Without loss of generality, we assume $N(u) = \{v, v_3\}$. Then $G - uv$ has a total-k-avd-coloring ϕ with u uncolored. Note that $|C\backslash(C_\phi(v) \cup \{\phi(uv_3)\})| \geq 3$.

If $|C\backslash(C_\phi(v) \cup \{\phi(uv_3)\})| \geq 4$, then there must be one color $\alpha \in C\backslash(C_\phi(v) \cup \{\phi(uv_3)\})$ such that $\{\alpha\} \cup C_\phi(v) \neq C_\phi(v_i)$, $i = 1, 2, 3$.

If $|C\backslash(C_\phi(v) \cup \{\phi(uv_3)\})| = 3$, then $\phi(uv_3) \notin C_\phi(v)$. So in any extension of ϕ to G, the color sets of v and v_3 are always distinct. Therefore, we can choose one color $\alpha \in C\backslash(C_\phi(v) \cup \{\phi(uv_3)\})$ such that $\{\alpha\} \cup C_\phi(v) \neq C_\phi(v_i)$, $i = 1, 2, 3$.

By the choose of α, we color uv with α to extend ϕ a total-k-avd-coloring of G, a contradiction. □

Claim 6. G contains no configuration G_{19}.

Suppose G contains a vertex v with neighbors $v_1, v_2, \ldots, v_t$, $t \geq 4$, such that $d(v) = 1$ and $d(v_i) \leq 2$ for all $i = 2, 3, \ldots, t-3$. For $2 \leq i \leq t-3$, if v_i is a 2-vertex, we denote by $u_i \neq v$ the other neighbor of v_i. It follows from Claim 2 that $d(u_i) \geq 3$. By the induction assumption, $G - v_1$ has a total-k-avd-coloring ϕ. We may assume that $\phi(v) = 1$, $\phi(vv_i) = i$ for $i = 2, 3, \ldots, t$.

If $k = t + 1$, since $\Delta(G) \geq t$, then $k = \Delta + 1$ and $t = \Delta$. By the definition of k, G does not contain two adjacent Δ-vertices. Since v is a Δ-vertex, we can color vv_1 by $t + 1$. So assume that $k \geq t + 2$.

(a) For $t = 4$, since $k \geq 8$, there is at least one $\alpha \in \{5, 6, 7, 8\}$ such that $\alpha \cup C_\phi(v) \neq C_\phi(v_i)$, $i = 2, 3, 4$.

(b) For $t = 5$, if for any $\beta \in \{6, 7, 8\}$, there is an $i \in \{3, 4, 5\}$ such that $\beta \cup C_\phi(v) = C_\phi(v_i)$, then we color vv_1 as follows: If $d(v_2) = 1$, we first recolor vv_2 with 6, then color vv_1 with 7; if $d(v_2) = 2$, we first recolor vv_2 with a color $\alpha \in \{6, 7\} \setminus \{v_2u_2\}$, color vv_1 with a color in $\{6, 7\} \setminus \{\alpha\}$.

(c) For $t \geq 6$, suppose for any $\beta \in \{t + 1, t + 2\}$, there is an $i \in \{t, t - 1, t - 2\}$ such that $\beta \cup C_\phi(v) = C_\phi(v_i)$. If for each $i \in \{t, t - 1, t - 2\}$, $|C_\phi(v_i) \cap \{t + 1, t + 2\}| \leq 1$ holds, then we color vv_1 and vv_2 with $t + 1$ and $t + 2$ as in (b). If there is an $i \in \{t, t - 1, t - 2\}$ such that $\{t + 1, t + 2\} \subseteq C_\phi(v_i)$, since there are at least two other 2^--neighbors of v, if one of $\{1, t, t - 1\} \notin C_\phi(v_i)$, then color vv_1 and vv_2 with $t + 1$ and $t + 2$ as in (b), otherwise there is a $j \in \{2, 3, \ldots, t - 3\}$, such that $j \notin C_\phi(v_i)$. Choose an $l \in \{2, 3, \ldots, t - 3\} \setminus \{j\}$ and recolor vv_l and vv_1 with $t + 1$ and $t + 2$ as in (b). □

Claim 7. No 5-vertex v is adjacent to two 2-vertices u_1, u_2 such that $N[u_i] \subseteq N[v]$, $i = 1, 2$ and $N[u_1] \neq N[u_2]$.

Suppose G contains such a 5-vertex v. Let v_1, v_2 and v_3 be the other three neighbors of v. Without loss of generality, we assume that $N(u_i) = \{v, v_i\}$, $i = 1, 2$. Let $G' = G - \{vu_1, vu_2\}$. Then G' has a total-k-avd-coloring ϕ with u_1 and u_2 uncolored.

If $\phi(u_1v_1) \neq \phi(u_2v_2)$ or $\phi(u_1v_1) = \phi(u_2v_2) \in C_\phi(v)$, then let $A = C \backslash C_\phi(v)$. Since $k \geq 8$, then $|A| \geq 4$. Let A_2 denote the set of all 2-element subsets of A. Then $|A_2| \geq 6$. Thus there is at least one element $\{\alpha_1, \alpha_2\} \in A_2$ such that $\{\alpha_1, \alpha_2\} \cup C_\phi(v) \neq C_\phi(v_i)$, $i = 1, 2, 3$. Since $\phi(u_1v_1) \neq \phi(u_2v_2)$ or $\phi(u_1v_1) = \phi(u_2v_2) \in C_\phi(v)$, we may assume that $\alpha_i \neq \phi(u_iv_i)$, $i = 1, 2$. We color the edge u_iv_i with α_i, $i = 1, 2$.

If $\phi(u_1v_1) = \phi(u_2v_2) = \beta \notin C_\phi(v)$, then let $A = C \backslash (C_\phi(v) \cup \{\beta\})$ and A_2 be the set of all 2-element subsets of A. Since $|A| \geq 3$, then $|A_2| \geq 3$. Thus, there exists at least one element $\{\alpha_1, \alpha_2\} \in A_2$ such that $\{\alpha_1, \alpha_2\} \cup C_\phi(v) \neq C_\phi(v_3)$. Moreover, since $\beta \notin C_\phi(v)$, we have $\{\alpha_1, \alpha_2\} \cup C_\phi(v) \neq C_\phi(v_i)$, $i = 1, 2$ in any extension of ϕ to G. Again we can obtain a total-k-avd-coloring of G by coloring the edge u_iv_i with α_i, $i = 1, 2$. □

Claims 1-7 implies that G does not contain configurations G_1, G_2, G_4, G_5, G_7-G_{19}. Next, we prove that G does not contain G_6 as a configuration either.

Suppose to the contrary that G has two adjacent 3-vertices v_2 and v_3. Moreover, v_2 and v_3 have two common neighbors v_1 and v_4. By Claim 3, both v_1 and v_4 are 3^+-vertices. See Fig.1.

Let $G' = G - v_2v_3$. Then G' has a total-k-avd-coloring ϕ with v_3 uncolored. We consider the most complex case, that is, both of v_1 and v_4 are 3-vertices.

Case 1. $\phi(v_3v_4) \notin C_\phi(v_1) \cup C_\phi(v_2)$.

Then the color set of v_3 is different from that of v_1 and v_2 in any extension of ϕ to G. The same result also holds for the color sets of v_2 and v_4, since the colors of v_2v_3 and v_3v_4 are distinct in any proper total coloring. Observe that $|C \setminus (C_\phi(v_1) \cup C_\phi(v_2) \cup \{\phi(v_3v_4)\})| \geq 1$, we can color v_2v_3 with a color $\alpha \in C \setminus (C_\phi(v_1) \cup C_\phi(v_2) \cup \{\phi(v_3v_4)\})$. Then $C(v_1) \neq C(v_2)$. Furthermore, since $|\{\phi(v_1), \phi(v_1v_3), \phi(v_2), \alpha, \phi(v_4), \phi(v_3v_4)\}| \leq 6$, we may color v_3 with a color $\beta \in C \setminus \{\phi(v_1), \phi(v_1v_3), \phi(v_2), \alpha, \phi(v_4), \phi(v_3v_4)\}$ satisfying $\{\alpha, \beta\} \cup C_\phi(v_3) \neq C_\phi(v_4)$.

Case 2. $\phi(v_3v_4) \in C_\phi(v_2)$ and $\phi(v_3v_4) \notin C_\phi(v_1)$.

Then we have that $\phi(v_3v_4) = \phi(v_2)$, and in any extension of ϕ to G, the color set of v_1 is different from those of v_2 and v_3. Since $|C \setminus (C_\phi(v_2) \cup \{\phi(v_1v_3), \phi(v_3v_4)\})| \geq 4$, one can color v_2v_3 with a color $\alpha \in C \setminus (C_\phi(v_2) \cup \{\phi(v_1v_3), \phi(v_3v_4)\})$ such that $\{\alpha\} \cup C_\phi(v_2) \neq C_\phi(v_4)$. Since $|\{\phi(v_1), \phi(v_1v_3), \phi(v_2), \alpha, \phi(v_4), \phi(v_2), \phi(v_3v_4)\}| \leq 5$, we can choose a color $\beta \notin \{\phi(v_1), \phi(v_1v_3), \phi(v_2), \alpha, \phi(v_4), \phi(v_2), \phi(v_3v_4)\}$ to color v_3 such that the new color set of v_3 is different from those of v_2 and v_4.

Case 3. $\phi(v_3v_4) \in C_\phi(v_1)$ and $\phi(v_3v_4) \notin C_\phi(v_2)$.

Since in any extension of ϕ to G, the color on v_2v_3 is different from $\phi(v_3v_4)$, we can get that the color sets of v_1, v_3 and v_4 are different from that of v_2. Since $|C_\phi(v_1) \cup C_\phi(v_2) \cup \{\phi(v_3v_4)\}| \leq 6$, we color v_2v_3 with a color $\alpha \in C \setminus (C_\phi(v_1) \cup C_\phi(v_2) \cup \{\phi(v_3v_4)\})$. Further, since the number of colors on the incident or adjacent elements of v_3 is at most six, there exists at least one color β such that $\{\alpha, \beta\} \cup C_\phi(v_3) \neq C_\phi(v_4)$. By the choice of α, we also have $\{\alpha, \beta\} \cup C_\phi(v_3) \neq C_\phi(v_1)$. Now we get a total-$k$-avd-coloring of G, a contradiction.

Case 4. Both of $C_\phi(v_1)$ and $C_\phi(v_2)$ contain $\phi(v_3v_4)$.

In this case we also have $|C_\phi(v_1) \cup C_\phi(v_2) \cup \{\phi(v_3v_4)\}| \leq 6$, so we can color v_2v_3 with one color $\alpha \in C \setminus (C_\phi(v_1) \cup C_\phi(v_2) \cup \{\phi(v_3v_4)\})$ such that $\{\alpha\} \cup C_\phi(v_2) \neq C_\phi(v_4)$. And since $\alpha \notin C_\phi(v_1)$, in any extension of ϕ to G we always have that neither of the color set of v_2 and v_3 is the same as that of v_1.

Case 4.1. The number of colors on the incident or adjacent elements of v_3 is at most five. Then there exists at least one color β such that $\{\alpha, \beta\} \cup C_\phi(v_3) \neq C_\phi(v_4)$ and $\{\alpha, \beta\} \cup C_\phi(v_3) \neq C_\phi(v_2) \cup \{\alpha\}$.

Case 4.2. The number of colors on the incident or adjacent elements of v_3 is six. Then we must have that $\phi(v_1v_2) = \phi(v_3v_4)$ and there is a color β such that $\{\alpha, \beta\} \cup C_\phi(v_3) \neq C_\phi(v_4)$. We color v_3 with color β to get an extension of ϕ to G.

If $\phi(v_1v_3) \notin C_\phi(v_2)$, then $C_\phi(v_2) \cup \{\alpha\} \neq \{\alpha, \beta\} \cup C_\phi(v_3)$. So the new color sets of v_2 and of v_3 are always distinct.

If $\phi(v_1v_3) \in C_\phi(v_2)$, then we have $\phi(v_1v_3) = \phi(v_2v_4)$. However, by the choice of β, $\beta \neq \phi(v_2)$. Thus, $\{\alpha\} \cup C_\phi(v_2) \neq \{\alpha, \beta\} \cup C_\phi(v_3)$. The extension of ϕ to G is a total-k-avd-coloring of G, a contradiction. □

Finally, we assume that G contains G_3 as a configuration.

Claim 8. Both of x and y are 6^+-vertices in G.

First, by Claims 2-4, both of x and y are 5^+-vertices in G. Suppose $d(x) = 5$. Let x_1, x_2 and x_3 be the remaining neighbors of x. Let $G' = G - \{xu, xv\}$. Then G' has a total-k-avd-coloring ϕ with u and v uncolored. Let $A = C \setminus C_\phi(x)$ and A_2 be the 2-element subsets of A. Then $|A| \geq 4$ and $|A_2| \geq 6$. Thus, there must be an element $\{\alpha, \beta\} \in A_2$ such that $\{\alpha, \beta\} \cup C_\phi(x) \neq C_\phi(x_i)$, $i = 1, 2, 3$. Since yu and yv are colored with distinct colors, we can properly color xu and xv by α and β. Then ϕ can be extended to a total-k-avd-coloring of G, a contradiction. The same argument holds when $d(y) = 5$. □

Claim 9. Both of x and y have at least three 6^+-neighbors.

Similar to the argument as in the proof of Lemma 8, we can extend ϕ to G. □

Let G^* be the graph obtained from G by deleting the two 2-vertices in all G_3 configurations in G. Then G^* is also an outer 1-planar graph. Call the two 3^+-vertices x and y of G_3 *special* vertices in G^*.

Claim 10. G^* contains a *special* vertex x which has at most three 3^+-neighbors in G^*.

First, by Claim 9, we observe the following fact: No new 2^--vertex will appear in G^*.

So G^* contains no G_3 and $\delta(G^*) \geq 2$. Applying Theorem 1.3 again, G^* contains one of the configurations G_1, G_2, G_4-G_{17}. On the other hand, by the previous argument, G contains none of G_1, G_2, G_4-G_{17}. Hence, we conclude that some special vertex in G becomes a solid 3^+-vertex in G^*. It is easy to see that each solid 3^+-vertex has at most three 3^+-neighbors in G^*. □

Since no new 2^--vertex will appear in G^*, the special vertex x in Claim 10 is also adjacent to at most three 3^+-neighbors in G. Combining Claims 9 and 10 together, we have that x has exactly three 6^+-neighbors in G. Furthermore, by the structure of the configurations, each special vertex lies in at most two G_3 configurations in G. Let $d = d_G(x)$.

Case A. x lies in two G_3 configurations.

We use G_3^1 and G_3^2 to denote the two G_3 configurations containing x. Denote $V(G_3^i) = \{x, u_i, v_i, y_i\}$: $i = 1, 2$. Let x_1, x_2, x_3 be the 6^+-neighbors of x. Let $G' = G - \{xu_1, xv_1, xu_2, xv_2\}$. Then G' has a total-k-avd-coloring with u_1, v_1, u_2 and v_2 uncolored. Let $A = C \setminus C_\phi(v)$. Then $|A| = k-(d+1-4) = k-d+3 \geq 4$. Let A_4 denote the sets of all 4-element subsets of A.

If $|A| \geq 5$, then $|A_4| \geq 5$. There must be an element $\{\alpha_1, \beta_1, \alpha_2, \beta_2\} \in A_4$ such that $\{\alpha_1, \beta_1, \alpha_2, \beta_2\} \cup C_\phi(x) \neq C_\phi(x_i)$, $i = 1, 2, 3$.

If $|A| = 4$, then $k = d + 1$. The proof can be given with a similar argument as in the case $k = d+1$ of Claim 6.

Case B. x lies in only one G_3 configuration. Denote the two 2-neighbors of x in G_3 as u and v. Recall that x has exactly three 6^+-neighbors, there is another 2-neighbor w of x. Let $G' = G - \{xu, xv, xw\}$. Let $A = C \setminus C_\phi(v)$. Then $|A| = k - (d+1-3) = k - d + 2 \geq 3$. Let A_3 denote the sets of all 3-element subsets of A. The following argument is similar to the above case. □

It is easy to check that each extension of ϕ is a total-k-avd-coloring of G. This contradiction completes the proof of Theorem 1.2.

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