COMPONENTS OF THE KERNEL IN A STAIRCASE STARSHAPED POLYGON

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ABSTRACT. For a non simply connected orthogonal polygon T, assume that $T = S\backslash(A_1 \cup \ldots \cup A_n)$, where S is a simply connected orthogonal polygon and where $A_1, \ldots, A_n$ are pairwise disjoint sets, each the connected interior of an orthogonal polygon, $A_i \subseteq S, 1 \leq i \leq n$. If set T is staircase starshaped, then Ker $T = \cap\{$Ker $(S\backslash A_i) : 1 \leq i \leq n\}$. Moreover, each component of this kernel will be the intersection of the nonempty staircase convex set Ker S with a box, providing an easy proof that each of these components is staircase convex. Finally, there exist at most $(n+1)^2$ such components, and the bound $(n+1)^2$ is best possible.

1. INTRODUCTION

We begin with some definitions and comments that also appear in [1] and [2]. A set B in $\mathbb{R}^d$ is called a *box* if and only if B is a convex polytope (possibly degenerate) whose edges are parallel to the coordinate axes. A nonempty set S in $\mathbb{R}^d$ is an *orthogonal polytope* if and only if S is a connected union of finitely many boxes. An orthogonal polytope in $\mathbb{R}^2$ is an *orthogonal polygon.* Let λ be a simple polygonal path in $\mathbb{R}^d$ whose edges are parallel to the coordinate axes. That is, let λ be a simple rectilinear path in $\mathbb{R}^d$. For points x and y in S, the path λ is called an $x-y$ *path* if and only if λ lies in S and has endpoints x and y. The $x-y$ path λ is a *staircase path* (or simply a *staircase*) if and only if, as we travel along λ from x to y, no two edges of λ have opposite directions.That is, for each standard basis vector $e_i, 1 \leq i \leq d$, either each edge of λ parallel to e_i is a positive multiple of e_i or each edge of λ parallel to e_i is a negative multiple of e_i. In the plane, an edge (or subset of an edge) $[v_{i-1}, v_i]$ of path λ will be called *north, south, east,* or *west* according to the direction of vector $\overrightarrow{v_{i-1}v_i}$. Similarly, we use the terms north, south, east, west, northeast, northwest, southeast, southwest to describe the relative position of points.

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For points x and y in a set S, we say x *sees* y (x is *visible* from y) *via staircase paths* if and only if S contains an $x - y$ staircase path. A set S is *staircase convex (orthogonally convex)* if and only if, for every pair of points x, y in S, x sees y via staircase paths. Similarly, a set S is *staircase starshaped (orthogonally starshaped)* if and only if, for some point p in S, p sees each point of S via staircase paths. The set of all such points p is the *staircase kernel* of S, Ker S.

Many results in convexity that involve the usual idea of visibility via straight line segments have interesting analogues that instead use the concept of visibility via staircase paths. Some of these analogues concern the kernel of a starshaped set. For example, when set S in $\mathbb{R}^d$ is starshaped via straight line segments, it is easy to show that the associated kernel is a convex set. In a staircase analogue [4, Theorem 2], when the orthogonal polygon S is starshaped via staircase paths, then every component of Ker S is staircase convex. Furthermore, [4,Theorem 1] shows that, when S is a simply connected orthogonal polygon and S is starshaped via staircase paths, then Ker S itself will be staircase convex. Perhaps surprisingly, [3, Example 1] reveals that the planar results do not extend to $\mathbb{R}^d, d \geq 3$, although the staircase convexity of Ker S will hold for certain classes of orthogonal polytopes, (See [1] and [3].)

Here we return to the planar case and, in particular, to the case in which an orthogonal polygon T is not simply connected. Assume that T is a union of fully two-dimensional boxes. Using the bounded components $A_1, \ldots, A_n$ of $\mathbb{R}^2 \backslash T$, we write $T = S \backslash (A_1 \cup \ldots \cup A_n)$, where S is simply connected. Theorems from [2] explore in $\mathbb{R}^d$ the relationship between similar sets Ker S and Ker T, when the bounded components $A_1, \ldots, A_n$ of $R^d \backslash T$ are interiors of boxes. It turns out that, in $\mathbb{R}^2$, these results may be adapted and extended to yield results for any arbitrary orthogonal polygon. Specifically, in the plane, these results allow us to explore the relationship between such sets Ker S and Ker T, to find an easy proof that the components of Ker T are staircase convex, and to obtain an upper bound on the number of these components in terms of n.

Throughout the paper, for S a set in $\mathbb{R}^2$, $bdryS$ and clS will denote the boundary and the closure, respectively, for set S. If λ is a simple path containing points a and b, then $\lambda(a, b)$ will denote the subpath of λ from a to b, ordered from a to b.

Readers may refer to Valentine [8], to Lay [7], to Danzer, Grűnbaum, Kleegl [5], and to Eckhoff [6] for discussions concerning visibility via straight line segments and starshaped sets.

2. The Results.

We begin with a helpful lemma.

Lemma 1. *Let S be an orthogonal polygon, and let A be the connected interior of an orthogonal polygon, with $A \subseteq S$. Then Ker $(S\backslash A) \subseteq$ Ker S.*

Proof. Let $x\varepsilon$ Ker $(S\backslash A)$, and let $y\varepsilon S$, to show that there is an $x - y$ staircase in S. If $y\varepsilon S\backslash A$, the result is immediate, so assume that $y\varepsilon A$. Let $\lambda(x,y)$ represent any $x - y$ staircase. Certainly $\lambda(x,y) \cap A$ is a finite union of staircase paths. Since $x \notin A$, there exists some $z\varepsilon bdry A$ such that $\lambda(z,y)\backslash\{z\} \subseteq A$. Since $z\varepsilon S\backslash A, x$ sees z via a staircase path $\mu(x,z) \subseteq S\backslash A$. Certainly $\mu(x,z)$ and $\lambda(x,z)$ (and any $x - z$ staircase) will employ compatible edges. (That is, the directions of their edges will be the same.) Hence $\mu(x,z) \cup \lambda(z,y)$ will be an $x - y$ staircase. Moreover, $\mu(x,z) \cup \lambda(z,y) \subseteq (S\backslash A) \cup (cl A) = S$. That is, x sees y via an $x - y$ staircase in S. Hence $x\varepsilon$ Ker S, finishing the proof. □

Theorem 1. *Let S be an orthogonal polygon, and let A be the connected interior of an orthogonal polygon, with $A \subseteq S$. Let D denote the union of all horizontal and vertical lines that meet A. Then $\mathbb{R}^2\backslash D$ is a union of four closed regions $R_j, 1 \leq j \leq 4$, one at each vertex of the smallest box containing A.* For each $R_j, 1 \leq j \leq 4$, either $R_j\cap$ (Ker S) is disjoint from Ker $(S\backslash A)$ or $R_j\cap$ (Ker S) is a subset of Ker $(S\backslash A)$. Moreover, Ker $(S\backslash A)$ is exactly the union of those sets $R_j\cap$ (Ker S) that lie in Ker $(S\backslash A)$.

Proof. By Lemma 1, the only candidates for points in Ker $(S\backslash A)$ are points of Ker S. Clearly no points of D can belong to Ker $(S\backslash A)$, so every point of Ker $(S\backslash A)$ must lie in one of the sets R_j as well as in Ker S. To establish the result, it remains to show that, for each R_j, either $R_j\cap$ (Ker S) is disjoint from Ker $(S\backslash A)$ or $R_j\cap$ (Ker S) is a subset of Ker $(S\backslash A)$.

Without loss of generality, consider the R_j set northwest of A, call it R_1. We will show that one of the two conditions above must hold. If $R_1\cap$ (Ker S) is disjoint from Ker $(S\backslash A)$, then we have satisfied the first condition. Otherwise, for some point x in $R_1\cap$ (Ker S), x also belongs to Ker $(S\backslash A)$. Let w belong to $R_1\cap$ (Ker S) to show that w is in Ker $(S\backslash A)$ as well.

For $y\varepsilon S\backslash A$, we will prove that w sees y via a staircase path in $S\backslash A$. If y is not south, east, or southeast of points in A, then any $w - y$ staircase in S will provide a $w - y$ staircase in $S\backslash A$. Without loss of generality, assume that y is south or southeast of points of A.

Let $\lambda(x,y)$ denote an $x - y$ staircase in $S\backslash A$. Now consider the relative positions of x and w. If w is northwest of x, then let $\mu(w,x)$ be any $w - x$ staircase in S. Certainly $\mu(w,x) \subseteq S\backslash A$, and $\mu(w,x) \cup \lambda(x,y)$ yields a $w - y$ staircase in $S\backslash A$, the desired result. If w is northeast, southeast, or southwest of x, then either there is a point of $\lambda(x,y)$ directly south of w or there is a point of $\lambda(x,y)$ directly east of w. The cases are symmetric, so without loss of generality assume that there is a point t of $\lambda(x,y)$ directly south of w. (Point t need not be unique, since a whole segment of $\lambda(x,y)$ may be south of w.) Since $w\varepsilon$ Ker S, $[w,t] \subseteq S$. Then clearly $[w,t] \subseteq S\backslash A$, and $[w,t] \cup \lambda(t,y)$ provides a $w-y$ staircase in $S\backslash A$, again the desired result. It follows that w sees y via a staircase in $S\backslash A$, and $w\varepsilon$ Ker $(S\backslash A)$. That is, for $1 \leq j \leq 4$, if one point of $R_j\cap$ (Ker S) lies in Ker $(S\backslash A)$, then all points of $R_j\cap$ (Ker S) lie in Ker $(S\backslash A)$, and Ker $(S\backslash A)$ (if nonempty) is exactly the union of the appropriate sets $R_j\cap$ (Ker S). This finishes the proof of Theorem 1. □

Theorem 2. *Let S be an orthogonal polygon, with pairwise disjoint sets $A_1,\ldots,A_n$ each the connected interior of an orthogonal polygon, $A_i \subseteq S, 1 \leq i \leq n$. Then Ker $(S\backslash(A_1 \cup \ldots \cup A_n)) = \cap\{$Ker $(S\backslash A_i) : 1 \leq i \leq n\}$.*

Proof. We use an inductive argument. If $n = 1$, the result is immediate. Assume that the result is true when $n = k \geq 1$ to prove for $n = k+1$. To show that Ker $(S\backslash(A_1 \cup \ldots \cup A_{k+1})) \subseteq \cap\{$Ker $(S\backslash A_i) : 1 \leq i \leq k+1\}$, let x belong to the left set to show that x belongs to Ker $(S\backslash A_i), 1 \leq i \leq k+1$. For convenience, it suffices to show that x belongs to Ker $(S\backslash A_{k+1})$. Let $T = S\backslash(A_2 \cup \ldots \cup A_{k+1})$. Then $S\backslash(A_1 \cup \ldots \cup A_{k+1}) = T\backslash A_1$, so $x\varepsilon$ Ker $(T\backslash A_1)$, and by Lemma 1, $x\varepsilon$ Ker T. However, using our induction hypothesis, Ker $T = \cap\{$Ker $(S\backslash A_i) : 2 \leq i \leq k+1\}$, so $x\varepsilon$ Ker $(S\backslash A_{k+1})$, the desired result. We conclude that x lies in every set Ker $(S\backslash A_i), 1 \leq i \leq k+1$, and Ker $(S\backslash(A_1 \cup \ldots \cup A_{k+1})) \subseteq \cap\{Ker(S\backslash A_i) : 1 \leq i \leq k+1\}$.

To establish the reverse inclusion, let x belong to $\cap\{$Ker $(S\backslash A_i) : 1 \leq i \leq k+1\}$. By our induction hypothesis, $\cap\{$Ker $(S\backslash A_i) : 1 \leq i \leq k\}$ $=$ Ker $(S\backslash(A_1 \cup \ldots \cup A_k))$ so $\cap\{$Ker $(S\backslash A_i) : 1 \leq i \leq k+1\} =$ Ker $(S\backslash(A_1 \cup \ldots A_k))\cap$ Ker $(S\backslash A_{k+1})$. Using Theorem 1, point x necessarily belongs to one of the four regions R_i, one at each corner of the box of A_{k+1}. Without loss of generality we assume that x lies in the region R_1 northwest of A_{k+1}. Let y belong to $S\backslash(A_1 \cup \ldots \cup A_{k+1})$ to show that x sees y via a staircase path in $S\backslash(A_1 \cup \ldots \cup A_{k+1})$. As in the proof of Theorem 1, if y is not south, east, or southeast of points in A_{k+1}, then any $x-y$ staircase in $S\backslash(A_1 \cup \ldots \cup A_k)$ will provide an $x-y$ staircase in $S\backslash(A_1 \cup \ldots \cup A_{k+1})$.

Now consider all the $x-y$ staircases $\lambda(x,y)$ in $S\backslash(A_1 \cup \ldots \cup A_k)$. Each $\lambda(x,y)$ has the same length, which is the rectilinear distance from x to y. We will assume that every such λ meets A_{k+1}, for otherwise the argument is finished. Since $x, y\varepsilon S\backslash A_{k+1}$, for every such λ, $\lambda \cap A_{k+1}$ will be a finite union of relatively open segments. Using the order on λ from x to y, select the last endpoint z_λ of the last of these segments. That is, select z_λ so that there is a segment in $\lambda(x, y) \cap A_{k+1}$ immediately preceding z_λ but $\lambda(z_\lambda, y) \subseteq S \backslash A_{k+1}$. Of course, then $\lambda(z_\lambda, y) \subseteq S \backslash (A_1 \cup \ldots \cup A_{k+1})$.

From all such paths λ, select one path λ_0 and associated $z_{\lambda_0} \equiv z_0$ for which the length of $\lambda_0(z_0, y)$ is as large as possible. That is, the rectilinear distance from z_0 to y is as large as possible, and the rectilinear distance from x to z_0 is as small as possible, for all such paths λ.

Let (z_0', z_0) represent the last segment of $\lambda_0(x, y)$ in A_{k+1}. Since y is southeast of x, the direction of $[z_0', z_0]$ in $\lambda_0(x, y)$ is either south or east. The arguments for these cases are essentially the same, one obtained from the other by appropriately modifying the directions of the vectors involved. Therefore, in the write-up, we consider just the case in which $[z_0', z_0]$ is a south vector in $\lambda_0(x, y)$. Certainly z_0 must lie on a horizontal edge e of $bdry A_{k+1}$.

We assert that z_0 cannot be the west endpoint of e. Otherwise, the vertical edge adjacent to z_0 in $bdry A_{k+1}$ would lie either north of z_0 or south of z_0. Consider each case:

Case 1. If the vertical edge of $bdry A_{k+1}$ adjacent to z_0 were north of z_0, then points of (z_0', z_0) near z_0 would lie in $bdry A_{k+1}$, impossible since (z_0', z_0) lies in the open set A_{k+1}. (See Figure 1.)

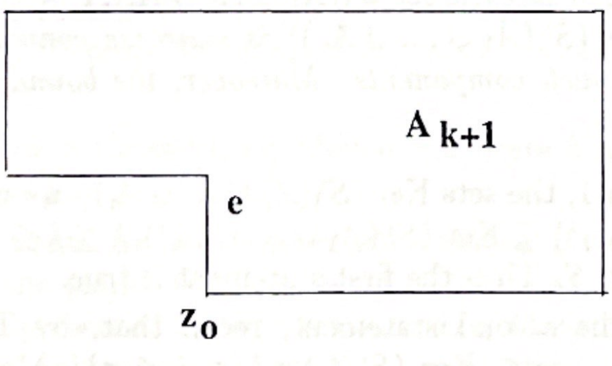

FIGURE 1. Edge e north of z_0.

Case 2. If the vertical edge of $bdryA_{k+1}$ adjacent to z_0 were south of z_0, then x could not see z_0 via a staircase path in $S\backslash A_{k+1}$, impossible since $x\varepsilon$ Ker $(S\backslash A_{k+1})$. (See Figure 2.)

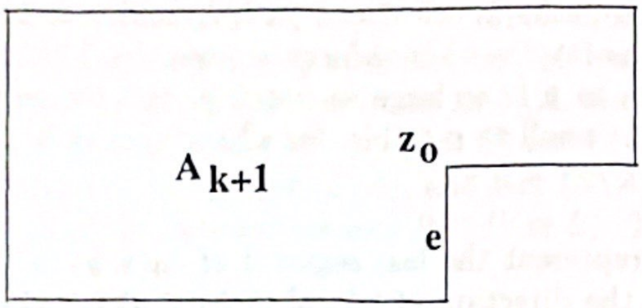

FIGURE 2. Edge e south of z_0.

We conclude that z_0 is not the west endpoint of edge e and hence is either relatively interior to e or the east endpoint of e. Select the west endpoint w_0 of e. Observe that w_0 is strictly closer to x than z_0 is to x. Since $w_0\varepsilon bdryA_{k+1} \subseteq S\backslash(A_1 \cup \ldots \cup A_k)$, there is a staircase path $\mu(x, w_0)$ in $S\backslash(A_1 \cup \ldots \cup A_k)$ from x to w_0. Moreover, $\mu(x, w_0) \cup [w_0, z_0] \cup \lambda_0(z_0, y)$ is again an $x-y$ staircase in $S\backslash(A_1 \cup \ldots \cup A_k)$. Certainly $[w_0, z_0] \cup \lambda_0(z_0, y) \subseteq S\backslash(A_1 \cup \ldots \cup A_{k+1})$, and the length of this path is strictly greater than the length of $\lambda_0(z_0, y)$, contradicting our choice of λ_0. Our assumption that each $x - y$ staircase λ in $S\backslash(A_1 \cup \ldots \cup A_k)$ meets A_{k+1} must be false, and there is some $x - y$ staircase in $S\backslash(A_1 \cup \ldots \cup A_{k+1})$. That is, $x\varepsilon$ Ker $(S\backslash(A_1 \cup \ldots \cup A_{k+1}))$, and $\cap\{$Ker $(S\backslash A_i) : 1 \leq i \leq k+1\} \subseteq$Ker $(S\backslash(A_1 \cup \ldots \cup A_{k+1}))$. Since the reverse inclusion holds as well, the sets are equal. By induction, the set equality holds for all n, finishing the proof of Theorem 2. □

Theorem 3. *Let S be a simply connected orthogonal polygon, with pairwise disjoint sets $A_1, \ldots, A_n$ each the connected interior of an orthogonal polygon, $A_i \subseteq S, 1 \leq i \leq n$. If $S\backslash(A_1 \cup \ldots \cup A_n)$ is staircase starshaped, then so is S and so is each set $S\backslash(A_1 \cup \ldots \cup A_i), 1 \leq i \leq n-1$. Each component of Ker $(S\backslash(A_1 \cup \ldots \cup A_n))$ is staircase convex, and there are at most $(n+1)^2$ such components. Moreover, the bound $(n+1)^2$ is best possible.*

Proof. By Lemma 1, the sets Ker $(S\backslash(A_1 \cup \ldots \cup A_i))$ are nested, with Ker $(S\backslash(A_1 \cup \ldots \cup A_{i+1})) \subseteq$ Ker $(S\backslash(A_1 \cup \ldots \cup A_i)), 1 \leq i \leq n-1$, and with Ker $(S\backslash A_1) \subseteq$ Ker S. Thus the first statement is true.

To establish the second statement, recall that, by Theorem 2, Ker $(S\backslash(A_1 \cup \ldots \cup A_n)) = \cap\{$Ker $(S\backslash A_i) : 1 \leq i \leq n\}$. Moreover, by Theorem 1, each set Ker $(S\backslash A_i)$ is the union of certain sets $R_{ij} \cap$ Ker S, where

R_{ij} is one of the four disjoint regions described in Theorem 1, one at each vertex of the smallest box containing $A_i, 1 \leq j \leq 4$. For convenience, let B denote the smallest box containing S. Then each component of Ker $(S \backslash A_i)$ is the intersection of Ker S with some box $R_{ij} \cap B$. Of course, any intersection of boxes is again a box. It follows that each component of $\cap\{\text{Ker } (S\backslash A_i) : 1 \leq i \leq n\}$ and hence each component of Ker $(S\backslash(A_1 \cup \ldots \cup A_n))$ will be the intersection of Ker S with a box, where the box is an appropriate intersection of boxes of the form $R_{ij} \cap B$ for some i and $j, 1 \leq i \leq n, 1 \leq j \leq 4$. Since Ker S is staircase convex (by [4, Theorem 1]), each component of Ker $(S\backslash(A_1 \cup \ldots \cup A_n))$ will be staircase convex, too.

It remains to count the possible components of Ker $(S\backslash(A_1 \cup \ldots \cup A_n))$. Using the discussion in Theorem 1, each set A_i introduces one associated horizontal strip and one associated vertical strip, $1 \leq i \leq n$. Since some of the strips may share points, in all we have at most n disjoint horizontal strips and at most n disjoint vertical strips. Let $\mathcal{V}$ represent the corresponding collection of strips. The set $\mathbb{R}^2\backslash(\cup\{V : V \text{ in } \mathcal{V}\})$ creates a grid consisting of at most $(n+1)^2$ regions, $n+1$ in each horizontal row and $n+1$ in each vertical column. The intersection of one of these regions with Ker S is a candidate for a component of Ker $(S\backslash(A_1 \cup \ldots \cup A_n))$. Thus there are at most $(n+1)^2$ such components, finishing the proof. □

The following easy example shows that the bound $(n+1)^2$ in Theorem 3 is best possible.

Example 1. Let S be a rectangular polygon in the plane, and let G denote the diagonal of S that extends from the northwest vertex to the southeast vertex. Along G, arrange n pairwise disjoint sets $A_1, \ldots, A_n$, each the interior of a rectangular region, $A_i \subseteq S, 1 \leq i \leq n$. For each i, place the northwest and southeast vertices of clA_i on G. Using the notation in Theorem 3, let $\mathcal{V}$ denote the collection of horizontal and vertical strips determined by the A_i sets, $1 \leq i \leq n$. Then $\mathbb{R}^2\backslash(\cup\{V : V \text{ in } \mathcal{V}\})$ has exactly $(n+1)^2$ regions, and each region contributes to Ker $(S\backslash(A_1 \cup \ldots \cup A_n))$. In fact, this kernel is exactly $S\backslash(A_1 \cup \ldots \cup A_n)$. Thus Ker $(S\backslash(A_1 \cup \ldots \cup A_n))$ has exactly $(n+1)^2$ components, and the bound $(n+1)^2$ in Theorem 3 is best possible.

Figure 3 illustrates the situation when $n = 2$. Here $K_1, \ldots, K_9$ represent the components of Ker $(S\backslash(A_1 \cup A_2))$.

Concluding remarks. Let T represent a non simply connected orthogonal polygon in the plane. If T is a union of fully two-dimensional boxes, then T may be represented as $S\backslash(A_1 \cup \ldots \cup A_n)$, where S is a simply connected orthogonal polygon and where $A_1, \ldots, A_n$ are pairwise disjoint sets, each the connected interior of an orthogonal polygon, $A_i \subseteq S, 1 \leq i \leq n$.

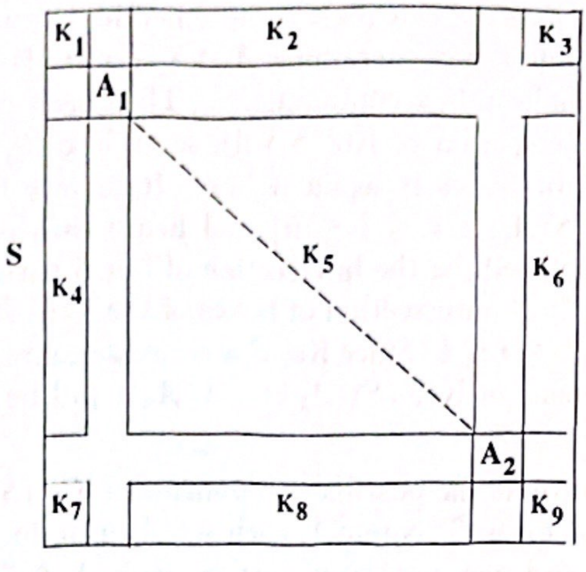

FIGURE 3. The kernel $\cup\{K_i : 1 \leq i \leq 9\}$ of $S\backslash(A_1 \cup A_2)$.

Of course, in this case, we may apply the results above to T. Otherwise, we may write T as the union of such a set (or sets) with line segments contained in $\cup\{A_i : 1 \leq i \leq n\}$. Replacing the segments with sufficiently thin boxes will create a new set to which our results apply and whose staircase kernel is Ker T.

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