New α-Trees and Graceful Unions of α-Graphs and Linear Forests

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Abstract

In this paper we study five methods to construct α-trees by using vertex amalgamations of smaller α-trees. We also study graceful and α-labelings for graphs that are the union of t copies of an α-graph G of order m and size n with a graph H of size t. If $n > m$, we prove that the disjoint union of H and t copies of G is graceful when H is graceful, and that this union is an α-graph when H is any linear forest of size $t-1$. If $n = m$, we prove that this union is an α-graph when H is the path P_{t-1}.

1 Introduction

All graphs used in this work are simple, that is, with no loops nor multiple edges. A *graceful labeling* of a graph G of size n is an injective function $f : V(G) \to \{0, 1, ..., n\}$ such that when each edge uv of G has assigned the *weight* $|f(u) - f(v)|$, all induced weights are distinct. If G admits a graceful labeling is said to be a *graceful graph.* Let f be a graceful labeling of a graph G; suppose that there exists an integer λ such that for each edge uv of G, either $f(u) \leq \lambda < f(v)$ or $f(v) \leq \lambda < f(u)$, then f is said to be an *α-labeling* of G with *boundary value* λ and G is called an *α-graph*. An α-graph is necessarily bipartite and λ is the smaller of the two vertex labels of the edge of weight 1.

There are several methods for combining graceful trees to produce larger graceful trees. The interested reader is referred to Gallian' survey [9] for a detailed account of these methods. We mention here some of them. Burzio and Ferrarese [6] proved that when every edge of a graceful tree is replaced by a path of fixed length, the resulting tree is graceful. Stanton and Zarnke [14] showed a graceful tree obtained by attaching, to every vertex of a graceful tree, a copy of another graceful tree. Thus, given a graceful tree T, we can attach a fixed number of pendant vertices to every vertex of T, that is, the corona $T \odot mK_1$ is graceful if T is graceful. This result was extended later in [2], proving that $G \odot mK_1$ is graceful if G is any graceful graph of order $n+1$ and size n. In [10], Koh, Rogers, and Tan, attached, to every vertex of an α-tree, new α-trees, not necessarily isomorphic. In Section 3 we investigate five new methods to construct α-trees.

Section 4 is devoted to the study of α-labelings of disconnected graphs. Since the first days of this research area, disconnected graphs have been the focus of many works; maybe, the reason of this interest can be found in the fact that the smallest graph, in order and size, that is not graceful, is disconnected, this graph is $C_3 \cup P_2$. Frucht and Salinas [8] explored the gracefulness of the family $C_s \cup P_n$; they conjectured that this graph is graceful if and only if $s+n \geq 6$. Over the years, several authors obtained partial results that confirm this conjecture; in 2012 Traetta [15] solved this conjecture. Recently [5], we considered the gracefulness of graphs of the form $C_s \cup G_n$, where G_n is any caterpillar of size n. A comprehensive list of disconnected graceful and α-graphs can be found in Gallian' survey [9]. If G_1 and G_2 are two graphs, $G_1 \cup G_2$ denotes the disjoint union of them, and tG_1 denotes the disjoint union of t copies of G_1. In Section 4 we prove that $tG \cup H$ is graceful if H is any graceful graph of size $t-1$ and G is an α-graph of order m and size n, where $m \leq n$. We also prove that $tK_{m,n} \cup L_{t-1}$ is an α-graph when $m \geq n$, $m \geq 3$, $n \geq 2$, $t \geq 2$, and L_{t-1} is any linear forest of size $t-1$. We extend this result to show that $tG \cup L_{t-1}$ is an α-graph when G is an α-graph of order m and size n with $m < n$. In addition, we conclude that if $m = n$ and L_{t-1} is replaced by the path P_t, of order t, the graph $tG \cup P_t$ is an α-graph.

In this work we follow the notation and terminology used in [7] and [9].

2 Preliminary Results

Suppose G is an α-graph of size n and f is an α-labeling of G with boundary value λ. Let A and B be the stable sets of the bipartition of $V(G)$; without loss of generality we assume that $A = \{v \in V(G) : f(v) \leq \lambda\}$ and $B = \{v \in V(G) : f(v) > \lambda\}$. When a positive constant $k-1$ is added to every vertex in B, the resulting labeling is a *k-graceful labeling*, the set of induced weights is $\{k, k+1, ..., n+k-1\}$. This operation was introduced independently by Maheo and Thuillier [11] and Slater [13]. Note that when an α-labeling of a tree of size n is transformed into a k-graceful labeling, the set formed by the labels assigned, by the new labeling, is $\{0, 1, ..., \lambda\} \cup \{k+\lambda, k+\lambda+1, ..., k+n-1\}$.

Let $f : V(G) \to \{0, 1, ..., n\}$ be a graceful labeling of a graph G of size n.

- $\overline{f} : V(G) \to \{0, 1, ..., n\}$, defined for every $v \in V(G)$ as $\overline{f}(v) = n - f(v)$, is the *complementary* labeling of f. This is also a graceful labeling of G. Moreover, if f is an α-labeling with boundary value λ, then $\overline{f}$ is an α-labeling with boundary value $n - \lambda$.
- $g : V(G) \to \{c, c+1, ..., c+n\}$, defined for every $v \in V(G)$ and $c \in \mathbb{Z}$ as $g(v) = c + f(v)$, is the *shifting* of f in c units. Note that this labeling preserves the weights induced by f.
- $g : V(G) \to \{0, \kappa, ..., n\kappa\}$, defined for every $v \in V(G)$ and $\kappa \in \mathbb{Z}^+$ as $g(v) = \kappa f(v)$, is the *amplification* of f in κ units. The induced weights are $\kappa, 2\kappa, ..., n\kappa$.

Suppose now that f is an α-labeling with boundary value λ.

- $f_r : V(G) \to \{0, 1, ..., n\}$, defined for every $v \in V(G)$ as $f_r(v) = \lambda - f(v)$ if $f(v) \leq \lambda$, and $f_r(v) = n + \lambda + 1 - f(v)$ if $f(v) > \lambda$, is the *reverse* labeling of f. Note that f_r is also an α-labeling with boundary value λ.

In the following sections, we use these labelings to prove our main results.

3 New α-Trees

In this section we present five new families of α-trees that are constructed using vertex amalgamation. The following result was proved by Barrientos [4], for the sake of completeness and its relevance in the present work, we prove it here again.

Lemma 1. *Suppose that for each $i \in \{1, 2\}$, G_i is an α-graph of size $n_i \geq 1$ and f_i is an α-labeling of G_i with boundary value λ_i. If $u_i, v_i \in V(G_i)$ satisfy $f_i(u_i) = 0$ and $f_i(v_i) = \lambda_i$, then the graph G obtained amalgamating v_1 and u_2 is an α-graph.*

Proof. The labeling f_1 of G_1 is transformed into a (n_2+1)-graceful labeling g_1. Thus, $g_1(v_1) = f_1(v_1) = \lambda_1$. The labels used by g_1 are in $\{0, 1, ..., \lambda_1\} \cup \{n_2 + \lambda_1 + 1, n_2 + \lambda_1 + 2, ..., n_1 + n_2\}$ and the induced weights are $1 + n_2, 2 + n_2, ..., n_1 + n_2$. The labeling f_2 of G_2 is shifted λ_1 units; the labels used by the resulting labeling g_2 of G_2 are in $\{\lambda_1, 1 + \lambda_1, ..., n_2 + \lambda_1\}$, the weights induced by g_2 are $1, 2, ..., n_2$. Note that $g_2(u_2) = \lambda_1$. Amalgamating v_1 and u_2 we obtain the graph G with an α-labeling which boundary value is $\lambda = g(v_2) = \lambda_1 + \lambda_2$. □

Since the α-labeling of G_2 is just shifted, it can be replaced by any graceful labeling of G_2, in this case the resulting labeling is only graceful.

From a general perspective, every α-graph G has, at least, four vertices that can be labeled zero by an α-labeling f; indeed, there exists an α-labeling g such that $g(v) = 0$ if $f(v) \in \{0, \lambda, \lambda + 1, n\}$, where n is the size of G and λ is the boundary value of f. Thus, given two α-labeled graphs G_1 and G_2, the construction in Lemma 1, may produce up to 16 different α-labeled trees.

In the next theorem, we use Lemma 1 in an inductive manner, to concatenate any number of α-trees. Suppose that $T_1, T_2, ..., T_s$ are trees. Let $u_i, v_i \in V(T_i)$; the vertex amalgamation of v_i with u_{i+1}, $1 \leq i \leq s-1$, produces a new tree T, called *chain tree*. In the next theorem we prove that when all the T_i's are α-trees, there exists a chain tree T constructed using them, that also admits an α-labeling.

Theorem 3.1. *If $T_1, T_2, ..., T_s$ are α-trees, then there exists a chain α-tree T constructed using $T_1, T_2, ..., T_s$.*

Proof. Denote by n_i the size of T_i, by f_i an α-labeling with boundary value λ_i of T_i, and by $\{A_i, B_i\}$ the bipartition of $V(T_i)$. Without loss of generality, we assume that f_i assigns the integers $0, 1, ..., \lambda_i = |A_i| - 1$ to the vertices of A_i. Let $u_i, v_i \in V(T_i)$ such that $f_i(u_i) = 0$ and $f_i(v_i) = \lambda_i$. For every $1 \leq i \leq s-1$, identify the vertices v_i and u_{i+1}. The tree so obtained is a chain tree T.

For every $1 \leq i \leq s-1$, let $\overline{n}_i = 1 + \sum_{j=i+1}^{s} n_j$ and $\overline{n}_s = 1$. We transform f_i into the $\overline{n}_i$-graceful labeling g_i. Thus, the weights induced by g_i on the edges of T_i are $\overline{n}_i, \overline{n}_i + 1, ..., \overline{n}_i + n_i - 1$. Therefore the weights on T are:

$$
\begin{aligned}
&1, 2, ..., n_s, \\
&n_s + 1, n_s + 2, ..., n_s + n_{s-1}, \\
&n_s + n_{s-1} + 1, n_s + n_{s-1} + 2, ..., n_s + n_{s-1} + n_{s-2}, \\
&\vdots \\
&n_s + ... + n_2 + 1, n_s + ... + n_2 + 2, ..., n_s + ... + n_1,
\end{aligned}
$$

that is, $1, 2, \ldots, n$, where n is the size of T.

Now, for every $i \geq 2$, we shift the labeling g_i of T_i by adding the constant $\sum_{j=1}^{i-1} \lambda_j$. Thus the vertices u_{i-1} and v_i have the same label, $\sum_{j=1}^{i-1} \lambda_j$. Since the labels on T are in the set $\{0, 1, \ldots, n\}$, the final labeling of T is an α-labeling with boundary value $\lambda = \sum_{i=1}^{s} \lambda_i$. □

In Figure 1 we show an example of this construction. On the left side of the figure, we give the α-labeling of the trees T_1, T_2, and T_3 used to construct T. In order to obtain the α-labeling of T, the α-labeling of T_1 is transformed into a 19-graceful labeling, the α-labeling of T_2 is transformed into a 10-graceful labeling shifted 4 units, and the α-labeling of T_3 is shifted 8 units.

3.1 α-Labelings of T_v^{+r}

Let T be a tree of size n and v be a fixed vertex of T. By T_v^{+r} we denote the tree of size $2n+r$, $r \geq 0$, obtained using two copies of T connected with a path of length r, in such a way that the endvertices of P_{r+1} corresponds to the vertices v of each copy of T. Note that when $r = 0$, T_v^{+r} is just the vertex amalgamation of two copies of T through the vertex v.

In this subsection we present two results associated to this type of trees. In Theorem 3 we show some necessary conditions for the existence of an α-labeling for a tree T_v^{+2} where v is any of the vertices labeled $\lambda, \lambda - 1, \ldots$, or $\lambda - \deg(v) - 1$ by an α-labeling f of T with boundary value λ. In the last proposition we show an α-labeling for a tree T_v^{+4} where v is the vertex labeled $\lambda - 1$ by an α-labeling f of T with boundary value λ.

Theorem 3.2. *If f is an α-labeling of a tree T of size n such that the labels $\lambda+1, \lambda+2, \ldots, \lambda+\kappa$ are assigned to leaves adjacent to the vertex labeled λ, then any tree T_v^{+2} is an α-tree when v is any of the vertices labeled $\lambda, \lambda-1, \ldots, \lambda-\kappa$.*

Proof. Suppose that T is an α-tree of size n and $V(T) = \{v_0, v_1, \ldots, v_n\}$. Let f be an α-labeling of T with boundary value λ, where $f(v_i) = i$ for all $0 \leq i \leq n$. Let T_1 and T_2 be the two copies of T used to construct T_v^{+2}, both copies are pre-labeled using f. Assume that $v_{\lambda+1}, v_{\lambda+2}, \ldots, v_{\lambda+\kappa}$ are leaves of T all of them adjacent to v_λ.

The α-labeling of T_1 is transformed into a $(n+3)$-graceful labeling, using the labels in $\{0, 1, \ldots, \lambda\} \cup \{n+\lambda+3, n+\lambda+4, \ldots, 2n+2\}$ and inducing the weights in $\{n+3, n+4, \ldots, 2n+2\}$. Note that the edges $v_\lambda v_{\lambda+1}, v_\lambda v_{\lambda+2}, \ldots, v_\lambda v_{\lambda+\kappa}$ now have weights $n+3, n+4, \ldots, n+2+\kappa$, respectively.

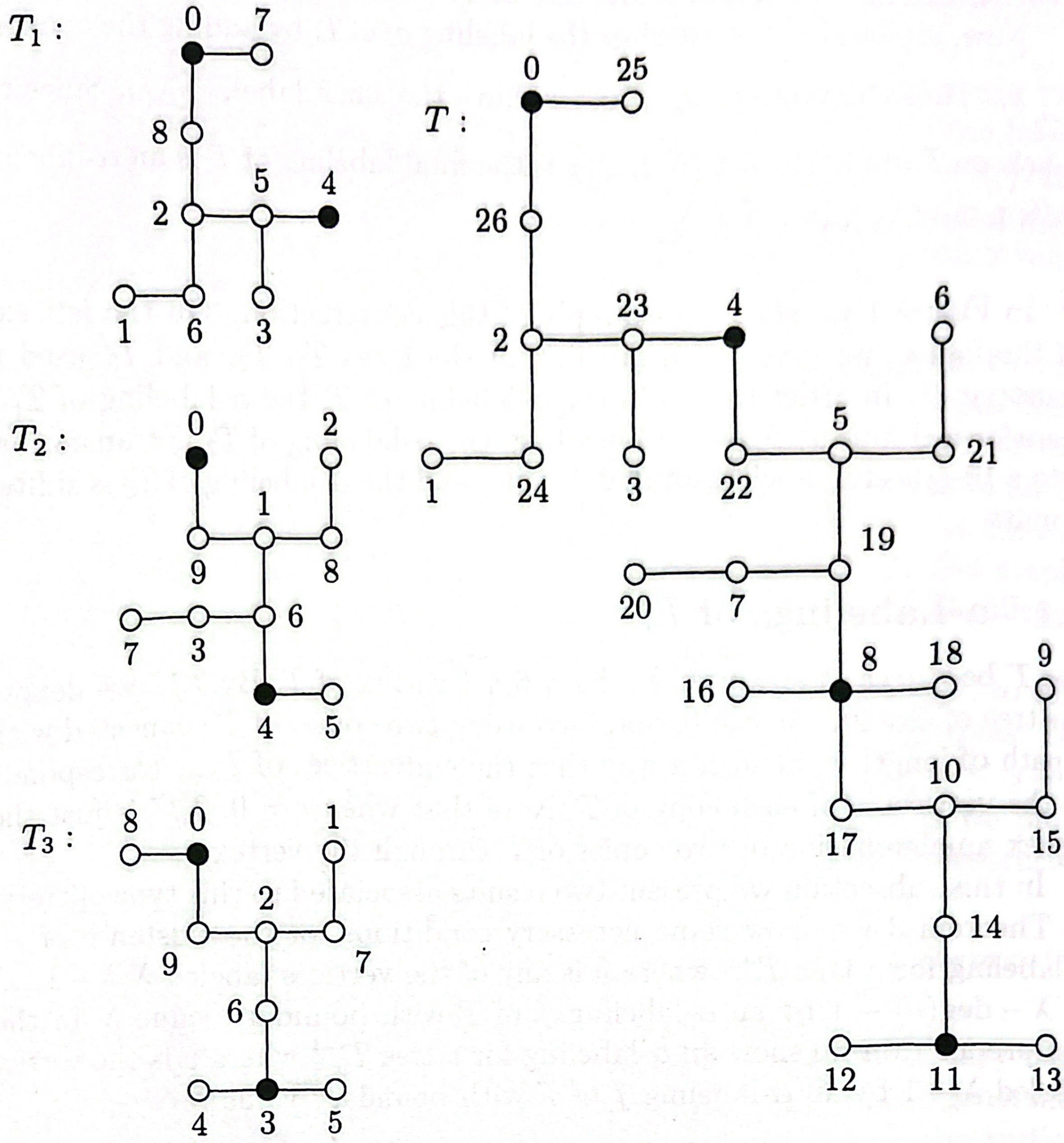

Figure 1: α-labeling of a chain tree

The α-labeling of T_2 is transformed into f_r shifted $\lambda+1$ units. Thus, the labels used here are in the set $\{\lambda+1, \lambda+2, ..., \lambda+1+n\}$ and the induced weights form the set $\{1, 2, ..., n\}$. Note that the edges $v_\lambda v_{\lambda+1}, v_\lambda v_{\lambda+2}, \dots, v_\lambda v_{\lambda+\kappa}$ have now weights $n, n-1, ..., n+1-\kappa$, respectively.

The fixed vertex v of T is chosen among the vertices labeled $\lambda, \lambda - 1, ..., \lambda - \kappa$. Suppose that $v = v_{\lambda-x}$; its image in T_1 has label $\lambda - x$ and in T_2 has label $\lambda + 1 + x$. The central vertex of the path connecting these vertices, has label $n + 2 + \kappa$; thus the edges of this path have weights $n + 2 + \kappa + x - \lambda$ and $n + 1 + \kappa - x - \lambda$, respectively.

Note that both weights have been obtained before. On the other side, there are no edges with weights $n + 1$ or $n + 2$. In order to fix this issue, we relabel some vertices. For each $1 \le i \le x$, $v_{\lambda+i}$ of T is labeled now $n+\lambda+2-i$ in T_1 and $n+\lambda+2+i$ in T_2. Thus, the edges of T_1 with weights $n+3, n+4, \ldots, n+2+x$ have now weights $n+3-(x+1), n+4-(x+1), \ldots, n+2+x-(x+1)$, that is, $n+2-x, n+3-x, \ldots, n+1$. The edges of T_2 with weights $n, n-1, \ldots, n+1-x$ have now weights $n+(x+1), n-1+(x+1), \ldots, n+1-x+(x+1)$, that is, $n+1+x, n+x, \ldots, n+2$.

The weights induced on the edges of T_1 form the set $\{n+2-x, n+3-x, \ldots, n+1\} \cup \{n+3+x, n+4+x, \ldots, 2n+2\}$. The weights induced on the edges of T_2 form the set $\{1, 2, \ldots, n-x\} \cup \{n+2, n+3, \ldots, n+1+x\}$. Since the weights on the connecting path are $n+2+x$ and $n+1-x$, we have a labeling of T_v^{+2} that induces the weights $1, 2, \ldots, 2n+2$. Since the labelings on T_1 and T_2 are modifications of their original α-labelings and the vertices of T_1 and T_2 connected with the path of length 2 are in the same stable set, the final labeling of T is an α-labeling which boundary value is $2\lambda+1$. □

In Figure 2 we show the four α-trees obtained using the previous construction on a tree whose vertex labeled λ has three pendant vertices attached. The vertices of the path P_3 are highlighted and the vertices that needed to be relabeled are represented with grey circles.

Proposition 1. *Let T be an α-tree of size $n \ge 3$. If there exists an α-labeling f of T such that $f(v) = \lambda - 1$, then T_v^{+4} is an α-tree.*

Proof. Let f be an α-labeling of T with boundary value λ. Suppose that there exists $v \in V(T)$ such that $f(v) = \lambda - 1$. Let T_1 and T_2 be the two copies of T used to construct T_v^{+4}; we assume that both copies have been pre-labeled using the function f.

The α-labeling of T_1 is transformed into a $(n+5)$-graceful labeling that uses the labels in $\{0, 1, \ldots, \lambda\} \cup \{n+5+\lambda, n+6+\lambda, \ldots, 2n+4\}$ and induces the weights $n+5, n+6, \ldots, 2n+4$. The α-labeling of T_2 is tranformed into f_r shifted $\lambda+2$ units. The new labeling uses the labels in $\{\lambda+2, \lambda+3, \ldots, n+2+\lambda\}$ and induces the weights $1, 2, \ldots, n$. The vertices of the path connecting T_1 and T_2 are labeled $\lambda-1, n+3+\lambda, \lambda+1, n+4+\lambda$, and $\lambda+3$, respectively. Thus the weights induced are $n+4, n+2, n+3$, and $n+1$, respectively.

The tree T_v^{+4} is obtained identifying the extreme vertices of this path with the vertices of T_1 and T_2 with the labels $\lambda-1$ and $\lambda+3$. Thus, we have a labeling of T_v^{+4} that uses the labels $0, 1, \ldots, 2n+4$ and induces the weights $1, 2, \ldots, 2n+4$. Since the labelings used on T_1 and T_2 are α-labelings and the vertices connected with the path of length 4 are "the same," this is an α-labeling of T_v^{+4} which boundary value is $2\lambda+2$. □

In Figure 3 we show an example of this construction for a tree of size 11 and boundary value $\lambda = 5$.

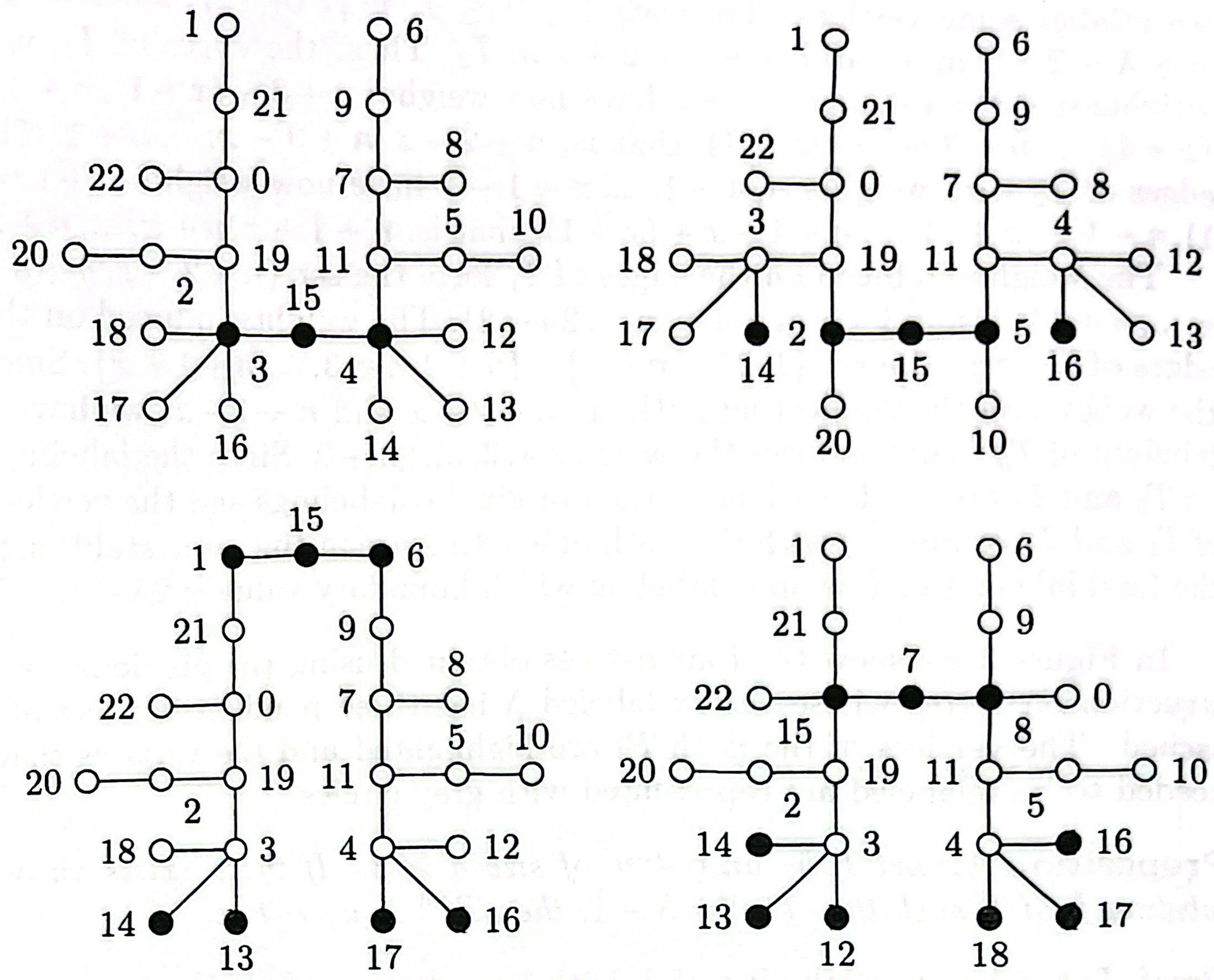

Figure 2: α-labelings of trees of the form T_v^{+2}

3.2 α-Labelings of $\oplus_w(T_1, T_2, T_3, T_4)$

Wu [16] introduced the following notation. Suppose that for each $1 \leq i \leq r$, G_i is a connected graph. Let v_i be an arbitrary vertex of G_i and w a new vertex, the graph obtained by connecting w to each v_i is denoted by $\oplus_w(G_1, G_2, ..., G_r)$. Wu proved that $\oplus_w(G_1, G_2, ..., G_r)$ is graceful provided that $G_1, G_2, ..., G_r$ are copies of a gracefully labeled graph G and, for each $1 \leq i \leq r$, v_i is the vertex labeled 0.

In Theorem 3 we proved that the tree T_v^{+2} was an α-tree under certain conditions; this tree can be seen as $\oplus_w(T_1, T_2)$, where the vertices v_1 and v_2 correspond to the vertex v of Theorem 3. In this subsection we present an α-labeling of $\oplus_w(T_1, T_2, T_3, T_4)$ and use this labeling to create an α-labeling of $\oplus_w(T_1, T_2, T_3)$.

Let $\mathscr{M}$ be the family of all α-trees of even size n such that when $T \in \mathscr{M}$, the stable sets of T contain $n/2$ and $n/2+1$ elements. For each $i \in \{1, 2, 3, 4\}$, let f_i be an α-labeling of $T_i \in \mathscr{M}$ with boundary value $\lambda = n/2 - 1$. By $\oplus_w(T_1, T_2, T_3, T_4)$ we denote the tree of size $4(n+1)$ obtained

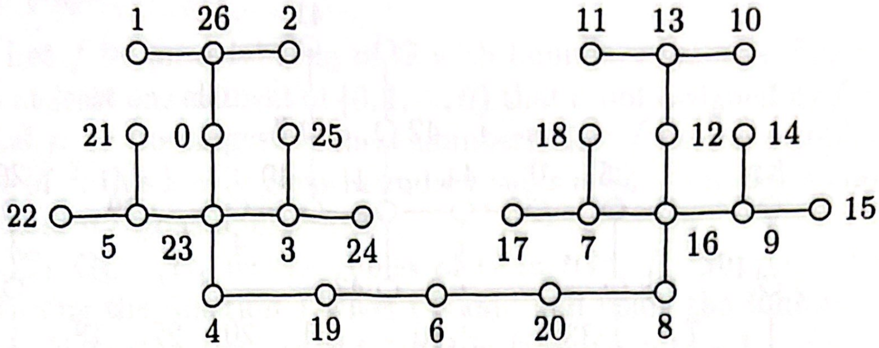

Figure 3: α-labeling of a tree T_v^{+4}

connecting, to a new vertex w, the vertices labeled n in T_1 and T_3 and the vertices labeled $n/2$ in T_2 and T_4. We claim that $\oplus_w(T_1, T_2, T_3, T_4)$ is an α-tree.

Theorem 3.3. *If T_1, T_2, T_3, and T_4 are members of $\mathscr{M}$, not necessarily isomorphic, then the tree $\oplus_w(T_1, T_2, T_3, T_4)$ is an α-tree.*

Proof. Suppose that for each $i \in \{1, 2, 3, 4\}$, f_i is an α-labeling of T_i with boundary value $\lambda = n/2 - 1$. The labeling f_1 is transformed into a $(3n+5)$-graceful labeling of T_1; thus the labels used are in $\{0, 1, ..., n/2-1\} \cup \{7n/2+4, 7n/2+5, ..., 4n+4\}$ and the induced weights form the set $\{3n+5, 3n+6, ..., 4n+4\}$. The labeling f_2 is transformed into a $(2n+4)$-graceful labeling of T_2 shifted $n/2$ units; thus the labels used are in $\{n/2, n/2+1, ..., n-1\} \cup \{3n+3, 3n+4, ..., 7n/2+3\}$ and the induced weights form the set $\{2n+4, 2n+5, ..., 3n+3\}$. The labeling f_3 is transformed into a $(n+2)$-graceful labeling of T_3 shifted $n+1$ units; thus the labels used are in $\{n+1, n+2, ..., 3n/2\} \cup \{5n/2+2, 5n/2+3, ..., 3n+2\}$ and the induced weights form the set $\{n+2, n+3, ..., 2n+1\}$. The labeling f_4 is shifted $3n/2+1$ units; thus the labels used are in $\{3n/2+1, 3n/2+2, ..., 5n/2+1\}$ and the induced weights form the set $\{1, 2, ..., n\}$. Finally, the vertex w is labeled n. In this way, the new edges have weights $3n+4, 2n+3, 2n+2$, and $n+1$. Hence, the weights of the edges of $\oplus_w(T_1, T_2, T_3, T_4)$ are $1, 2, ..., 4n+4$; the labels used on its vertices are $0, 1, ..., 4n+4$. Therefore, $\oplus_w(T_1, T_2, T_3, T_4)$ admits an α-labeling with boundary value $\lambda = 2n$. □

In Figure 4 we show the α-labeling of a tree $\oplus_w(T_1, T_2, T_3, T_4)$ where each T_i is an α-tree of size 10. The trees T_i are presented counterclockwise, starting above the highlighted vertex w.

Remark 1. *Since the labeling of T_4 is just shifted, the tree T_4 can be replaced by any graceful tree of size n. In this case, the resulting tree is graceful.*

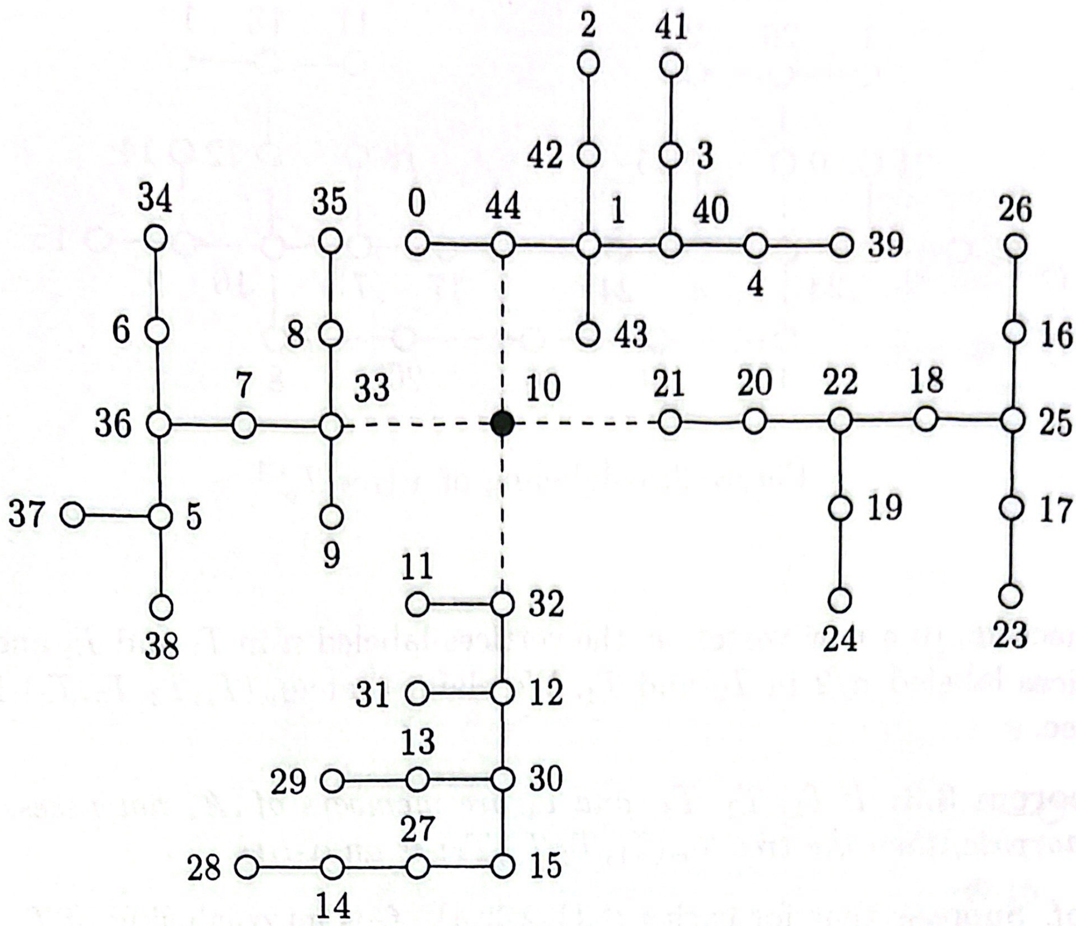

Figure 4: α-labeling of a tree $\oplus_w(T_1, T_2, T_3, T_4)$

If the tree T_4 in $\oplus_w(T_1, T_2, T_3, T_4)$ is deleted and the labels of the vertices in the largest stable set of $\oplus_w(T_1, T_2, T_3)$ are adjusted conveniently, we obtain an α-labeling of $\oplus_w(T_1, T_2, T_3)$. Thus we can prove the following corollary.

Corollary 1. *The tree $\oplus_w(T_1, T_2, T_3)$ is an α-tree.*

4 Disconnected α-Graphs

In this section we study graceful and α-labelings of graphs obtained from the disjoint union of some α-graphs. Let G be a graph, by tG we understand the disjoint union of t copies of G. First we prove that when G is an α-graph of order m and size n, $m \leq n$, the graph $tG \cup H$ is graceful if H is a graceful graph of size $t-1$. In the last result, we prove that $tK_{m,n} \cup L_{t-1}$ is an α-graph when $m \geq 3$, $n \geq 2$, $t \geq 2$, and L_{t-1} is any linear forest of size $t-1$.

Theorem 4.1. *If G is an α-graph of order m and size n, with $m \leq n$, and H is a graceful graph of size $t-1$, then the disconnected graph $tG \cup H$ is a* *graceful graph.*

Proof. Let f be an α-labeling of G with boundary value λ. Since $m \leq n$, there is at least one element of $\{0, 1, ..., n\}$ that is not assigned by f as a label of G. Let μ be the largest of these numbers. Let $\overline{f}$ be the complementary labeling of f; this labeling has boundary value $n-1-\lambda$ and does not assign the number $n - \mu$ as a label of G.

Let $G_1, G_2, ..., G_t$ be the copies of G in $tG \cup H$. Suppose that G_i is labeled using the function f when i is odd and using the function $\overline{f}$ when i is even. Note that the size of $tG \cup H$ is $N = t(n+1) - 1$.

The labeling of G_i is transformed into a $((t-i)(n+1)+1)$-graceful labeling, shifted $(n+1)(i-1)/2$ units when i is odd, and $(n+1)(i-2)/2+\lambda+1$ units when i is even. In this way, the elements of $\{0, 1, ..., N\}$ that have not been used as labels of the subgraph tG form the set $\{\mu + (n+1)(i-1) : 1 \leq i \leq t\}$. The weights induced on the edges of tG form the set $\{1, 2, ..., N\} - \{(n+1)i : 1 \leq i \leq t\}$.

Let h be a graceful labeling of H. If we multiply by $n+1$ the labels assigned by h, and shift them μ units, we obtain a labeling of H with labels in the set $\{\mu, \mu + (n+1), ..., \mu + (t-1)(n+1)\}$ and weights $n+1, 2(n+1), ..., (t-1)(n+1)$.

Thus, we have obtained a labeling of $tG \cup H$ with weights $1, 2, ..., N$ and labels in $\{0, 1, ..., N\}$. Therefore $tG \cup H$ is graceful. □

In Figure 5 we show an example for a graph of the form $7C_4 \cup H$, where H is the smallest tree that is not an α-tree.

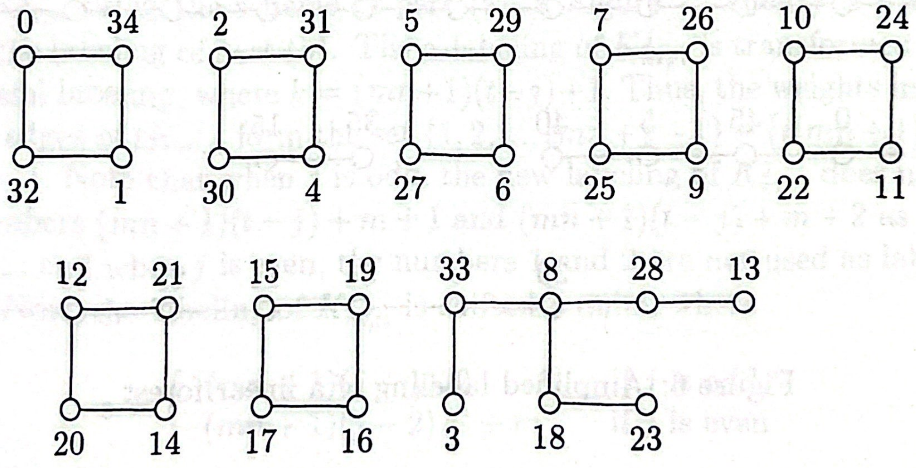

Figure 5: Graceful labeling of $7C_4 \cup H$

Lemma 2. *For every linear forest L of size n and a positive integer $\kappa \geq 2$, there exists a labeling $f : V(L) \to [0, n\kappa]$ such that the induced weights are $\kappa, 2\kappa, ..., n\kappa$.*

Proof. Let P_{n+1} be the path of size n and f be the α-labeling of P_{n+1} that places the label n on a leaf. Multiplying every vertex label by a positive constant κ, we obtain a labeling g of P_{n+1} that induces the weights $\kappa, 2\kappa, ..., n\kappa$ using the labels $0, \kappa, ..., n\kappa$.

Let L be a linear forest of size n with components $\Pi_1, \Pi_2, ..., \Pi_t$. For every $1 \leq i \leq t$, let u_i and v_i be the leaves of Π_i. Identifying v_i with u_{i+1}, for each $i \leq t-1$, we obtain the path P_{n+1}. We assume that P_{n+1} is labeled using the function g described before. Now we disengage each Π_i from P_{n+1} keeping the vertex labels. In this way the weights of the edges of L are $\kappa, 2\kappa, ..., n\kappa$; however, the vertices v_i and u_{i+1} have the same label. In order to eliminate the overlapping of labels, we subtract one unit from each vertex label of Π_i when i is even. Thus the weights remain the same and the labels assigned to v_i and u_{i+1} differ by one unit. Therefore, the labeling of L induces the weights $\kappa, 2\kappa, ..., n\kappa$ using labels in the interval $[0, n\kappa]$. □

In Figure 6 we show an example of this labeling for a linear forest of size 10 and four components; we show the α-labeling of P_{11}, its amplification by a factor $\kappa = 5$, and the final labeling of the linear forest $P_5 \cup P_3 \cup P_2 \cup P_4$.

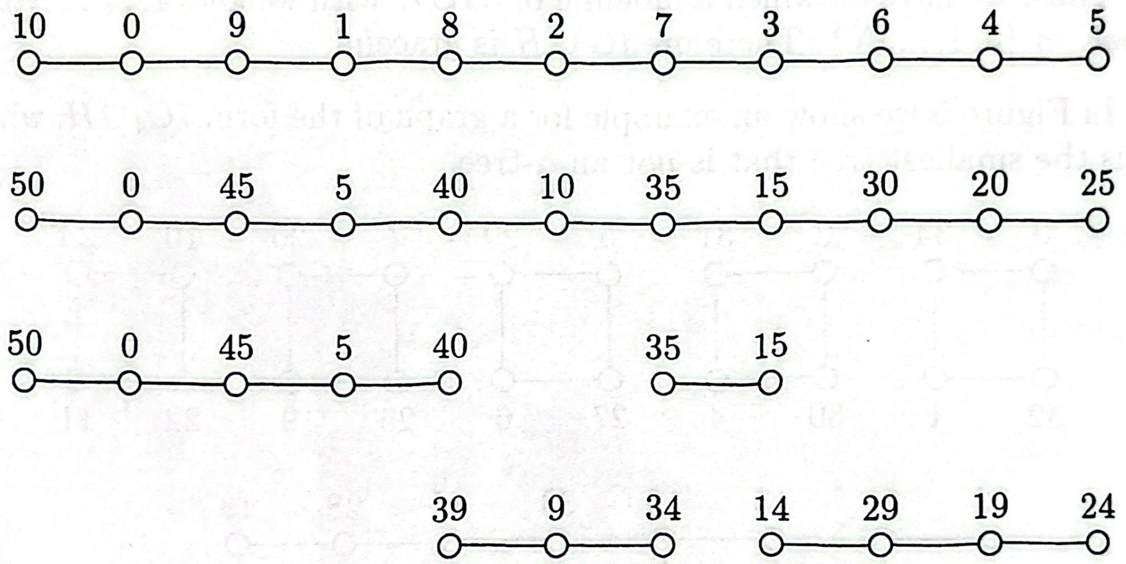

Figure 6: Amplified labeling of a linear forest

Consider the two α-labelings of $K_{m,n}$ shown in Figure 7; note that one is the reverse complementary labeling of the other. Suppose that $m \geq n$, $m \geq 3$, and $n \geq 2$. Since $m \geq 3$, the labeling in part (a) does not use the numbers $m+1$ and $m+2$ as labels, the labeling in part (b) does not use the numbers 1 and 2 as labels.

In the following theorem, we use these labelings to prove that the union of t copies of $K_{m,n}$ and any linear forest of size $t-1$ results in an α-graph.

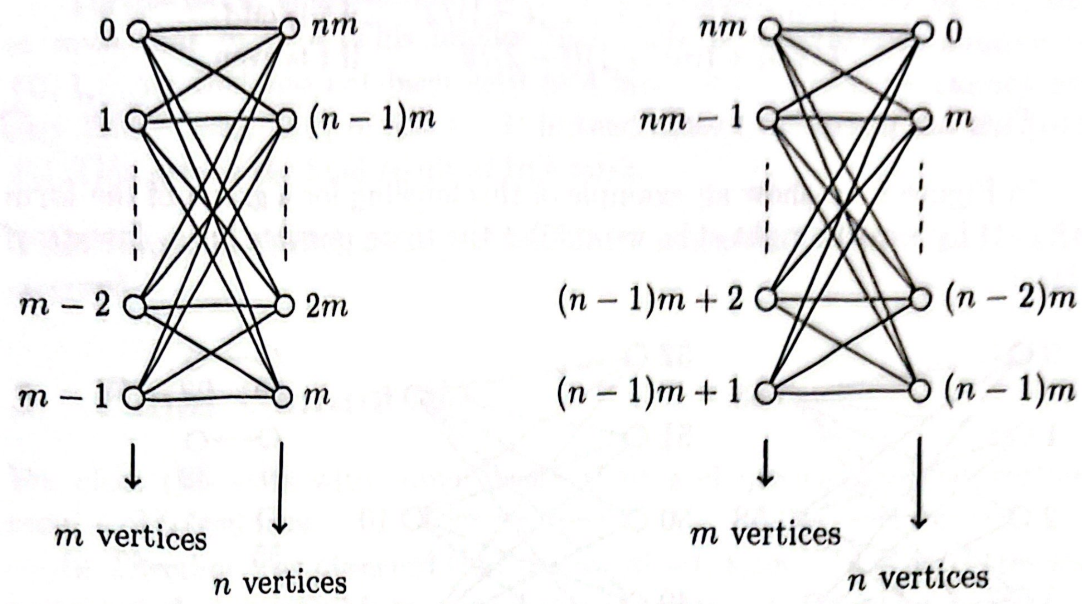

Figure 7: α-labelings of $K_{m,n}$

Theorem 4.2. *For every $m \geq n$, $m \geq 3$, $n \geq 2$, and $t \geq 2$, $G = tK_{m,n} \cup L_{t-1}$ is an α-graph where L_{t-1} is any linear forest of size $t-1$.*

Proof. First we label the t components of G isomorphic to $K_{m,n}$. Let $K^j_{m,n}$ denote the jth of these components, where $1 \leq j \leq t$. When j is odd, we label $K^j_{m,n}$ using the scheme of part (a) in Figure 7; when j is even, we apply the labeling of part (b). The α-labeling of $K^j_{m,n}$ is transformed into a k-graceful labeling, where $k = (mn+1)(t-j)+1$. Thus, the weights induced on the edges of $tK_{m,n}$ form the set $\{1, 2, ..., tmn+t-1\} - \{i(mn+1) : 1 \leq i \leq t-1\}$. Note that when j is odd, the new labeling of $K^j_{m,n}$ does not use the numbers $(mn+1)(t-j)+m+1$ and $(mn+1)(t-j)+m+2$ as labels of $K^j_{m,n}$, and when j is even, the numbers 1 and 2 are not used as labels of $K^j_{m,n}$. Now, the labeling of $K^j_{m,n}$ is shifted ε units, where

$$\varepsilon = \begin{cases} (mn+1)(j-1)/2 & \text{if } j \text{ is odd,} \\ (mn+1)(j-2)/2 + m & \text{if } j \text{ is even.} \end{cases}$$

In this way, the labeling of $tK_{m,n}$ is an injective function whose range is a subset of $\{0, 1, ..., tmn+t-1\}$. In addition, the numbers in $\{(j-1)(mn+1)+m+i : 1 \leq j \leq t \text{ and } i = 1, 2\}$ have not been used as labels.

The elements of this set can be separated to form two independent arithmetic progressions of difference $mn+1$. So, we can use them to label the vertices of L_{t-1} using the labeling described in Lemma 9. In this way, the edges of L_{t-1} have weights $mn+1, 2(mn+1), ..., (t-1)(mn+1)$ and the final labeling of G is an α-labeling with boundary value

$$\lambda = \begin{cases} m - 1 + (mn+1)(t-1)/2 & \text{if } t \text{ is odd,} \\ mn + (mn+1)(t-2)/2 & \text{if } t \text{ is even.} \end{cases}$$

This completes the proof. □

In Figure 8 we show an example of this labeling for a graph of the form $4K_{5,3} \cup L_3$. On the right side, we exhibit the three possible linear forests of size 3.

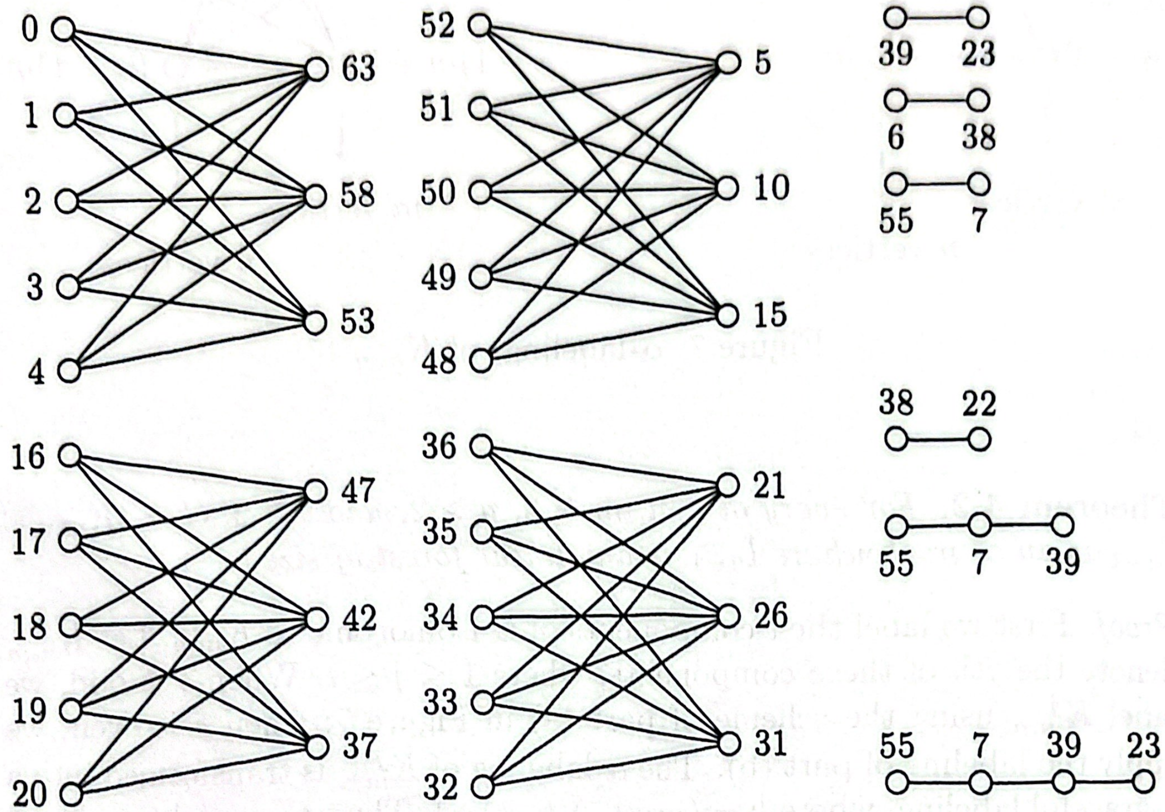

Figure 8: α-labelings of graphs of the form $4K_{5,3} \cup L_3$

Note that the graph $K_{m,n}$ can be replaced by any α-graph G of size q that admits an α-labeling that does not use any two numbers from $\{0, 1, ..., q\}$ as labels. Thus, the following theorem can be proved using the same argument, the difference is in the labeling of L_{t-1}. Let μ and ξ be the two largest numbers in $\{0, 1, ..., q\}$ not used as labels of G. In the last step of the construction of the labeling of L_{t-1}, we subtracted one unit from the labels of the components Π_i with i even; in the new case, we subtract $|\mu - \xi|$ from the labels of these components. We left the rest of the proof to the interested reader.

Theorem 4.3. *Let G be any α-graph of order m and size n. If $m < n$, then $tG \cup L_{t-1}$ is an α-graph.*

Furthermore, suppose now that G is an α-graph of order m and size n such that $m = n$. This implies that there is exactly one number in $\{0, 1, ..., n\}$ that has not been used as a label of G. Thus, we cannot use any linear forest L_{t-1} of size $t-1$; instead of L_{t-1} we must use the path P_t. This proves the final result of this work.

Theorem 4.4. *If G is an α-graph of order and size n, then $tG \cup P_t$ is an α-graph.*

5 Final Comments

We close this work with some observations and questions about certain results obtained here.

In Theorem 3 we observed that if $\kappa = |A| - 1$, then T_v^{+2} is an α-tree for every $v \in A$. Is it possible to extend this result to $v \in B$ and/or $r \neq 2$?

In Theorem 5 we analyzed the one-point union (or vertex amalgamation) of four α-trees of size n. Can we go beyond the four copies?

The result in Theorem 8 uses a graceful labeling of a graph H of size $t-1$. In Lemma 9 we learned how to use the α-labeling of P_t to produce a suitable labeling of a linear forest. If the graph H is a graceful tree of size $t-1$, it is possible to apply the technique in Lemma 9 to decompose H into a forest with a suitable labeling. Thus, a more general version of Theorem 8 is: If G is an α-graph of order m and size n, with $m \leq n$, and H is a graceful graph of size $t-1$ or any forest obtained by decomposing H into subtrees, then the disconnected graph $tG \cup H$ is a graceful graph.

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