On a Conjecture on Spanning Trees with few Branch Vertices

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Abstract A branch vertex of a tree is a vertex of degree at least three. Matsuda, et. al. [7] conjectured that, if n and k are non-negative integers and G is a connected claw-free graph of order n, there is either an independent set on $2k+3$ vertices whose degrees add up to at most $n-3$, or a spanning tree with at most k branch vertices. The authors of the conjecture proved it for $k=1$; we prove it for $k=2$.

Keywords: Spanning trees, Branch vertices, Claw-free graphs

Introduction

In a tree, vertices of degree one and vertices of degree at least three are called *leaves* and *branch vertices*, respectively. A hamiltonian path can be regarded as a spanning tree with maximum degree at most two, a spanning tree with at most two leaves, or a spanning tree with no branch vertex. A natural extension of the hamiltonian path problem is, therefore, to look for conditions that guarantee the existence of a spanning tree with low maximum degree, few leaves, or few branch vertices. A survey of spanning trees by Ozeki and Yamashita [12] examines many of these efforts, including independence number and degree sum conditions for the existence of such spanning trees; low maximum degree [3, 8, 11, 14], few leaves [1, 13, 15], and few branch vertices [2, 4, 5, 6, 9].

We denote by $\sigma_m(G)$ the smallest possible sum of degrees of an independent set of m vertices in G. If there is no such independent set, we say $\sigma_m(G)=\infty$. We also denote by $G[V]=G[v_1,v_2,\dots,v_t]$ the subgraph induced by $V=\{v_1,v_2,\dots,v_t\}$. A paper of Matsuda, Ozeki, and Yamashita

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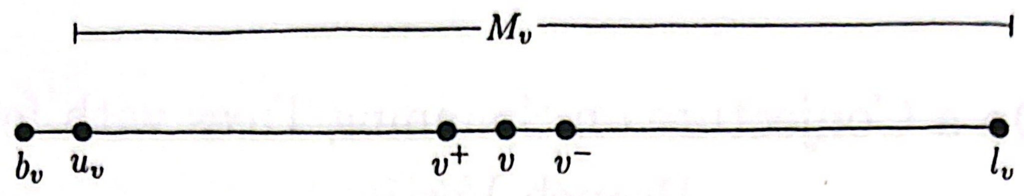

Figure 1: A vertex v outside the internal subtree S_T and some nearby vertices. In this diagram, only $\deg_T(b_v) \geq 3$ while $\deg_T(l_v) = 1$, and all other vertices in the diagram have degree 2. The only vertex of S_T shown in this diagram is b_v.

[9] conjectures a particular condition on connected claw-free graphs which ensures the existence of a spanning tree with at most k branch vertices.

Conjecture 1. (*Matsuda, et. al.* [9]) Let G be a connected claw-free graph on n vertices. If $\sigma_{2k+3}(G) \geq n-2$, then G has a spanning tree with at most k branch vertices.

The $k = 0$ case, as they point out, follows from a theorem of Matthews and Sumner [10]; they prove the $k = 1$ case. In this paper, we prove the $k = 2$ case.

Theorem 1. *Let G be a connected claw-free graph on n vertices. If $\sigma_7(G) \geq n-2$, then G has a spanning tree with at most 2 branch vertices.*

The following definitions and notation will be useful in our proofs. For any tree T, we denote by $B = B(T)$ the set of its branch vertices, and by $L = L(T)$ the set of its leaves. Any two of its vertices u and v are joined by a unique path, which we will denote uTv, and we call the set $S_T = \bigcup_{u,v \in B} uTv$ the **internal subtree** of T. Also, in this paper, $[t]$ refers to the set of all positive integers less than or equal to t. Some additional notation will be helpful:

Definition 1. *Let $v \in V(T) \setminus V(S_T)$. The induced subgraph of T given by those vertices in the same component of $T[V(T) \setminus V(S_T)]$ as v must form a path, which we call M_v. We denote the end of this path which is a leaf in T as l_v, and the other end as u_v. We define $b_v = N_T(u_v) \cap V(S_T)$. Furthermore, we define $v^+ = N_T(v) \cap vTb_v$, and if v is not a leaf we define $v^- = N_T(v) \cap vTl_v$.*

Proof of the Main Result

To prove Theorem 1, let G be a connected claw-free graph. Assume $\sigma_7(G) \geq n-2$. By way of contradiction, assume every spanning tree of G has at least 3 branch vertices. The following result, which guarantees the existence of a spanning tree with few leaves, will be useful in our proof:

Theorem 2. (*Kano et. al.* [7]) Let k be a non-negative integer and let G be a connected claw-free graph. If $\sigma_{k+3} \geq n-k-2$, then G has a spanning tree with at most $k+2$ leaves.

By Theorem 2 with $k = 4$, G has a spanning tree with at most 6 leaves. Among all spanning trees of G with at most 6 leaves, choose a spanning tree T also satisfying:

(T1) T has as few branch vertices as possible.

(T2) T has as few leaves as possible, subject to (T1).

Given that T has at most six leaves, it must have at most four branch vertices. Define the *derived tree* $\tau = \tau(T)$ by homeomorphically reducing T (so there are no more degree two vertices) and deleting all leaves. It is not hard to show that τ is also a tree, as any cycle in τ would correspond to a cycle in T, of which there are none. Now since T has at most six leaves, it can have either three or four branch vertices. If T has only three branch vertices, then necessarily $\tau \cong P_3$, and at most one of the branch vertices of T has degree four in T. If one vertex of T has degree 4, it can correspond to either the middle vertex of $\tau(T)$ or an end vertex. We can thus impose two more conditions (the second of which applies regardless of the structure of T):

(T3) Suppose two trees A and B exist satisfying (T2), each with exactly one vertex of degree 4, and suppose the middle vertex of $\tau(A)$ corresponds to the degree 4 vertex of A, while an end vertex of $\tau(B)$ corresponds to the degree 4 vertex of B. We select A over B.

(T4) S_T is as small as possible, subject to (T3) if applicable or (T2) otherwise.

Once this T is chosen, several lemmas follow.

Lemma 1. *If $N_T(v) = \{a, b, c\}$ and $a, b \notin S_T$, then $ab \in E(G)$.*

Proof. Let $v \in V(G)$ such that $N_T(v) = \{a, b, c\}$, and assume $a, b \notin S_T$. Since T has more than one branch vertex, $c \in S_T$. Now if $ac \in E(G)$, then $T' := T - \{va\} + \{ac\}$ either has fewer branch vertices than T (if $c \in B(T)$) or else it has the same number of branch vertices and leaves as T, with the same structure, but a smaller internal subtree. Thus either (T1) or (T4) is violated. □

Lemma 2. *If $v \in V(T) \setminus V(S_T)$ and $v^+l_v \in E(G)$, then $vl \notin E(G)$ if l is any leaf of T other than l_v. In particular, $L(T)$ is an independent set.*

Proof. Let $v \in V(T) \setminus V(S_T)$, and assume $v^+l_v \in E(G)$. Let l be a leaf of T other than l_v. Then $T' := T - \{vv^+, b_vu_v\} + \{vl, l_vv^+\}$ has no more branch vertices than T and fewer leaves, violating either (T1) or (T2). □

Lemma 3. *If $v \in V(T) \setminus V(S_T)$, $v^+l_v \in E(G)$, and $\deg_T(b_v) = 3$, then $vb \notin E(G)$ if b is any branch vertex of T other than b_v. In particular, if $b \in B(T)$ and $l \in L(T)$ such that $\deg_T(b_l) = 3$ and $b \neq b_l$, then $lb \notin E(G)$.*

Proof. Let $v \in V(T) \setminus V(S_T)$, and assume $v^+l_v \in E(G)$ and $\deg_T(b_v) = 3$. Let b be a branch vertex of T other than b_v. Then $T' := T - \{vv^+, b_vu_v\} + \{vb, l_vv^+\}$ has fewer branch vertices than T, violating (T1). □

Lemma 4. *Let $v \in V(T) \setminus V(S_T)$ such that $\deg_T(b_v) = 3$, $vb_v \in E(G)$, and $|N_T(b_v) \cap S_T| = 1$. Then $v^+l_v \notin E(G)$. In particular, if $l \in L(T)$ such that $\deg_T(b_l) = 3$ and $|N_T(b_l) \cap S_T| = 1$, then $lb_l \notin E(G)$.*

Proof. Suppose $v^+l_v \in E(G)$. Define $u' = N_T(b_v) \setminus (S_T \cup \{u_v\})$, so Lemma 1 gives that $u_vu' \in E(G)$. It follows that $T' := T - \{vv^+, b_vu_v, b_vu'\} + \{vb_v, v^+l_v, u_vu'\}$ violates (T1). □

Lemma 5. *If $a, c \in L(T)$ and $v \in V(T) \setminus V(S_T)$ and $c \neq l_v \neq a$, then $v \notin N_G(a) \cap N_G(c)$.*

Proof. Suppose $av, cv \in E(G)$ for some a, c, v as above. Since v is not a leaf (by Lemma 2), there exists v^-. Since $G[v, v^-, a, c]$ is not a claw and Lemma 2 ensures that $ac \notin E(G)$, it follows that either $av^- \in E(G)$ or $cv^- \in E(G)$. Without loss of generality, assume $av^- \in E(G)$. Then $T' := T - \{vv^-, u_vb_v\} + \{av^-, cv\}$ has no more branch vertices than T and fewer leaves, violating either (T1) or (T2). □

Lemma 6. *Let $l \in L(T)$, $b \in B(T)$, and $v \in V(T) \setminus V(S_T)$ such that $l \neq l_v$, $b_l \neq b \neq b_v$, $lb \notin E(G)$, and $\deg_T(b_v) = 3$. Then $v \notin N_G(l) \cap N_G(b)$.*

Proof. Assume $lv, bv \in E(G)$ for some l, b, v as above. Lemma 2 ensures that v is not a leaf, so there exists v^-. Since $G[v, v^-, l, b]$ is not a claw and $lb \notin E(G)$, either $lv^- \in E(G)$ or $bv^- \in E(G)$. If $lv^- \in E(G)$, then $T' := T - \{vv^-, b_vu_v\} + \{lv^-, bv\}$ has fewer branch vertices than T, violating (T1). Otherwise $bv^- \in E(G)$, so $T' := T - \{vv^-, b_vu_v\} + \{lv, bv^-\}$ has fewer branch vertices than T, still violating (T1). □

Lemma 7. *Let $u \in V(T) \setminus V(S_T)$ such that $ub_u \in E(T)$, and let $l_u \neq l \in L(T)$. Then $ul \notin E(G)$.*

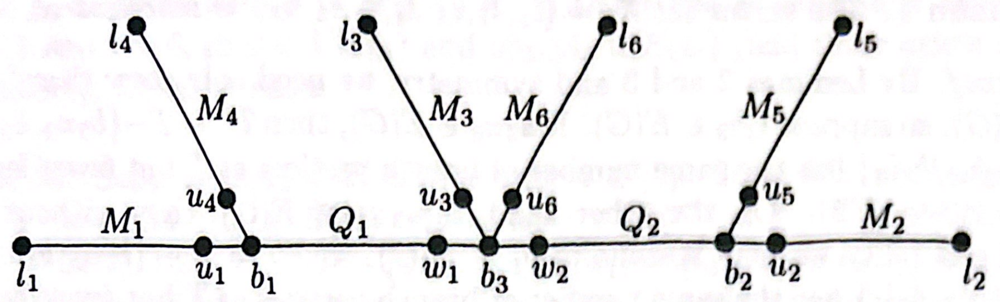

Figure 2: If $\tau \cong P_3$, T may have a degree 4 vertex corresponding to the middle vertex of τ. Each vertex labeled b_i is also called b_{i+3}.

Proof. Suppose $ul \in E(G)$ for some u, l as above. Then $T' := T - \{ub_u\} + \{ul\}$ has no more branch vertices than T and fewer leaves, violating either (T1) or (T2). □

Lemma 8. *Let $u \in V(T) \setminus V(S_T)$ such that $ub_u \in E(T)$ and $\deg_T(b_u) = 3$, and let $b_u \neq b \in B(T)$. Then $ub \notin E(G)$.*

Proof. Suppose $ub \in E(G)$. Then $T' := T - \{ub_u\} + \{ub\}$ has fewer branch vertices than T, violating (T1). □

We now prove several results about T.

Theorem 3. *Figure 2 does not accurately represent T. That is, under our assumptions, it is not the case that $\tau(T) \cong P_3$ with its middle vertex corresponding to a degree 4 vertex of T.*

Proof. By contradiction, suppose Figure 2 accurately represents T. As shown in Figure 2, we select two leaves with the same nearest branch vertex, which has degree three, and call them l_1 and l_4. We then call the other two such l_2 and l_5. We also call the two leaves whose nearest branch vertex has degree four l_3 and l_6, and we then abbreviate u_{l_i} as u_i, and b_{l_i} as b_i, and M_{l_i} as M_i, for each $i \in [6]$. We also define $w_j = N_T(b_3) \cap V(b_3Tb_j)$ and $Q_j = w_jTb_j$ for each $j \in [2]$. Note that $b_3 = b_6$ is in none of the labeled paths.

Since G is claw-free, there can be no induced claw centered at b_3. Among the four vertices of $N_T(b_3)$, therefore, there must be two disjoint cliques whose union is all of $N_T(b_3)$. If these are a singleton and a triplet, the singleton cannot be u_{3i} for any $i \in [2]$, since otherwise $T' := T - \{u_{9-3i}b_3, b_3w_2\} + \{u_{9-3i}w_1, w_1w_2\}$ violates either (T1) or (T4). Therefore either $u_3u_6 \in E(G)$ or $u_3w_1, u_6w_2 \in E(G)$ or $u_3w_2, u_6w_1 \in E(G)$. Also, $u_1, u_2, u_4, u_5 \notin S_T$ are neighbors of b_1 and b_2, so Lemma 1 gives that $u_1u_4, u_2u_5 \in E(G)$.

Claim 1. *The vertex set $X := \{l_1, l_2, l_3, l_4, l_5, l_6, b_3\}$ is independent.*

Proof. By Lemmas 2 and 3 and symmetry, we need only show that $l_3b_3 \notin E(G)$, so suppose $l_3b_3 \in E(G)$. If $u_3u_6 \in E(G)$, then $T' := T - \{b_3u_3, b_3u_6\} + \{u_3u_6, b_3l_3\}$ has the same number of branch vertices as T but fewer leaves, violating (T2). On the other hand, if $u_3u_6 \notin E(G)$, then without loss of generality we may assume $u_3w_1 \in E(G)$, so $T' := T - \{b_3u_3, b_3w_1\} + \{u_3w_1, b_3l_3\}$ has the same number of branch vertices as T but fewer leaves, still violating (T2). □

Claim 2. *For every $h \in [6]$, $(N_G(l_h) \cap V(M_h))^- \cap N_G(b_3) = \emptyset$.*

Proof. Suppose some $v \in (N_G(l_h) \cap V(M_h))^- \cap N_G(b_3)$. By Lemma 3, we may assume $3 \mid h$. Now if $u_3u_6 \in E(G)$, then we may assume $h = 3$ without loss of generality, so $T' := T - \{vv^+, u_3b_3, u_6b_3\} + \{vb_3, v^+l_3, u_3u_6\}$ has the same number of branch vertices as T and one less leaf, violating (T2). Otherwise, either $u_3w_1, u_6w_2 \in E(G)$ or $u_3w_2, u_6w_1 \in E(G)$. Without loss of generality, we may assume $h = 3$ and $u_3w_1 \in E(G)$. Then $T' := T - \{vv^+, b_3u_3, b_3w_1\} + \{b_3v, l_3v^+, u_3w_1\}$ has the same number of branch vertices as T and one less leaf, violating (T2). □

Claim 3. *If $i \neq h$, then $N_G(l_i) \cap V(M_h) \cap N_G(b_3) = \emptyset$.*

Proof. Suppose $v \in N_G(l_i) \cap V(M_h) \cap N_G(b_3)$. Lemma 6 ensures that either $3 \mid h$ or $3 \mid i$. Consider cases:

Case 1: Suppose $3 \nmid h$. Then $3 \mid i$, and since $v \neq l_h$ by Lemma 2, there exists v^-. Since $G[v, v^-, b_3, l_i]$ is not a claw and $b_3l_i \notin E(G)$ by Claim 1, either $b_3v^- \in E(G)$ or $l_iv^- \in E(G)$. If $b_3v^- \in E(G)$, then $T' := T - \{vv^-, b_hu_h\} + \{vl_i, b_3v^-\}$ has fewer branch vertices than T, violating (T1). Otherwise $l_iv^- \in E(G)$, so $T' := T - \{vv^-, b_hu_h\} + \{vb_3, l_iv^-\}$ has fewer branch vertices than T, still violating (T1).

Case 2: Suppose $3 \nmid i$. Then $3 \mid h$, and since $v \neq l_h$ by Lemma 2, there exists v^-. Since $G[v, v^-, l_i, b_3]$ is not a claw and $l_ib_3 \notin E(G)$, it follows that either $l_iv^- \in E(G)$ or $b_3v^- \in E(G)$. If $b_3v^- \in E(G)$, then $T' := T - \{vv^-, u_ib_i\} + \{b_3v^-, l_iv\}$ has fewer branch vertices than T, contradicting (T1). On the other hand, if $l_iv^- \in E(G)$, we consider whether or not $u_3u_6 \in E(G)$. If $u_3u_6 \in E(G)$, then $T' := T - \{vv^-, b_3u_3, b_3u_6\} + \{l_iv^-, b_3v, u_3u_6\}$ has the same number of branch vertices as T but fewer leaves, contradicting (T2). If $u_3u_6 \notin E(G)$, then $u_hw_j \in E(G)$ for some $j \in [2]$, and $T' := T - \{b_3u_h, b_3w_j, vv^-\} + \{u_hw_j, b_3v, l_iv^-\}$ has the same number of branch vertices as T but fewer leaves, contradicting (T2).

Case 3: Suppose both $3 \mid i$ and $3 \mid h$. Without loss of generality, assume $h = 3$ and $i = 6$, so $v \in V(M_3)$ and $vb_3, vl_6 \in E(G)$ (and there exists v^-, as before). Consider cases:

Case 3a: Suppose $w_i u_3 \in E(G)$ for some $i \in [2]$. Since $G[v, v^-, b_3, l_6]$ is not a claw and $b_3 l_6 \notin E(G)$, either $l_6 v^- \in E(G)$ or $b_3 v^- \in E(G)$. If $l_6 v^- \in E(G)$, then $T' := T - \{vv^-, b_3 u_3, b_3 w_i\} + \{l_6 v^-, b_3 v, u_3 w_i\}$ has the same number of branch vertices as T and fewer leaves, contradicting (T2). On the other hand, if $b_3 v^- \in E(G)$, then $T' := T - \{vv^-, b_3 u_3, b_3 w_i\} + \{b_3 v^-, l_6 v, u_3 w_i\}$ has the same number of branch vertices as T but fewer leaves, still contradicting (T2).

Case 3b: Suppose $w_i u_6 \in E(G)$ for some $i \in [2]$. Since $G[v, v^-, b_3, l_6]$ is not a claw and $b_3 l_6 \notin E(G)$, either $l_6 v^- \in E(G)$ or $b_3 v^- \in E(G)$. If $l_6 v^- \in E(G)$, then $T' := T - \{vv^-, b_3 u_6, b_3 w_i\} + \{l_6 v^-, l_6 v, u_6 w_i\}$ has the same number of branch vertices as T and fewer leaves, contradicting (T2). On the other hand, if $b_3 v^- \in E(G)$, then $T' := T - \{vv^-, b_3 u_6, b_3 w_i\} + \{b_3 v^-, l_6 v, u_6 w_i\}$ has the same number of branch vertices as T but fewer leaves, still contradicting (T2).

Case 3c: Suppose $w_1 u_3, w_1 u_6, w_2 u_3, w_2 u_6 \notin E(G)$. In this case, since $G[b_3, w_1, u_3, u_6]$ is not a claw and $u_3 w_1, u_6 w_1 \notin E(G)$, it follows that $u_3 u_6 \in E(G)$. Also, since $G[b_3, w_1, w_2, u_3]$ is not a claw and $w_1 u_3, w_2 u_3 \notin E(G)$, it follows that $w_1 w_2 \in E(G)$. As before, since $G[v, v^-, b_3, l_6]$ is not a claw and $b_3 l_6 \notin E(G)$, either $l_6 v^- \in E(G)$ or $b_3 v^- \in E(G)$. If $l_6 v^- \in E(G)$, then $T' := T - \{b_3 u_3, b_3 u_6, vv^-\} + \{u_3 u_6, b_3 v, l_6 v^-\}$ has the same number of branch vertices as T but one less leaf, contradicting (T2). We consider separately the case where $b_3 v^- \in E(G)$:

Case 3c′: Suppose $u_3 u_6, w_1 w_2, b_3 v^- \in E(G)$. For each $i \equiv 0 \pmod 3$, $j \in [2]$, since $G[b_3, v^-, u_i, w_j]$ is not a claw and $u_i w_j \notin E(G)$, it follows that either $v^- u_i \in E(G)$ or $v^- w_j \in E(G)$. In other words, there does not exist a pair (i, j) such that $v^- u_i, v^- w_j \notin E(G)$. Therefore either $v^- w_1, v^- w_2 \in E(G)$, or else $v^- u_3, v^- u_6 \in E(G)$. If $v^- w_1, v^- w_2 \in E(G)$, then $T' := T - \{vv^-, b_3 w_2, b_3 u_3, b_3 u_6\} + \{w_1 v^-, w_1 w_2, b_3 v, u_3 u_6\}$ is a tree with the same number of branch vertices (barring $w_1 = b_1$, which would violate (T1)) and leaves, with the same structure, but $|V(S_{T'})| < |V(S_T)|$, contradicting (T4). On the other hand, if $v^- u_3, v^- u_6 \in E(G)$, then $T' := T - \{vv^-, b_3 u_3\} + \{l_6 v, v^- u_3\}$ has the same number of branch vertices at T and fewer leaves, contradicting (T2) and completing the proof of Claim 3. □

Claim 4. *If $i \equiv j \pmod 3$, then $N_G(l_i) \cap V(Q_j) = \emptyset$.*

Proof. Suppose $v \in N_G(l_i) \cap V(Q_j)$. Then $v \neq b_i$ by Lemma 4, so $T' := T - \{b_i u_i\} + \{v l_i\}$ has the same number of branch vertices and leaves as T, still with the same structure, but $|V(S_{T'})| < |V(S_T)|$, violating (T4). □

Claim 5. *If $i + j \equiv h \equiv 0 (\text{mod } 3)$, then $N_G(l_i) \cap V(Q_j) \cap N_G(l_h) = \emptyset$.*

Proof. Suppose some $v \in N_G(l_i) \cap V(Q_j) \cap N_G(l_h)$. Lemma 3 ensures that $v \neq b_j$, so $T' := T - \{b_i u_i, b_3 u_h\} + \{l_i v, l_h v\}$ matches the structure of T but $|V(S_{T'})| < |V(S_T)|$, violating (T4). □

Claim 6. *If $i + j \equiv 0 (\text{mod } 3)$, then $(N_G(b_3) \cap V(Q_j))^- \cap N_G(l_i) = \emptyset$.*

Proof. Suppose some $v \in (N_G(b_3) \cap V(Q_j))^- \cap N_G(l_i)$. Then $v^+ b_3 \in E(G)$, so $T' := T - \{vv^+, b_i u_i\} + \{v^+ b_3, l_i v\}$ violates (T1). □

Claim 7. *If $i + j = 3$, then $N_G(l_i) \cap V(Q_j) \cap N_G(l_{i+3}) = \emptyset$.*

Proof. Suppose some $v \in N_G(l_i) \cap V(Q_j) \cap N_G(l_{i+3})$. Then $T' := T - \{b_i u_i, b_3 w_i\} + \{l_i v, l_{i+3} v\}$ violates (T4) since $|V(S_{T'})| < |V(S_T)|$. □

Claim 8. *For every $j \in [2]$, $N_G(l_3) \cap V(Q_j) \cap N_G(l_6) = \emptyset$.*

Proof. Suppose $v \in N_G(l_3) \cap V(Q_j) \cap N_G(l_6)$. Now if $u_3 u_6 \in E(G)$, then $T' := T - \{b_3 u_3, b_3 u_6\} + \{v l_3, u_3 u_6\}$ has no more branch vertices than T and fewer leaves, violating either (T1) or (T2). Otherwise, either $u_3 w_1, u_6 w_2 \in E(G)$ or $u_3 w_2, u_6 w_1 \in E(G)$. Without loss of generality, assume $u_3 w_1, u_6 w_2 \in E(G)$ and $j = 1$. Then $T' := T - \{b_3 u_6, b_3 w_2\} + \{v l_6, u_6 w_2\}$ has at most as many branch vertices as T and fewer leaves, again violating (T1) or (T2). □

Claim 9. *If $3|i$, then $(N_G(b_3) \cap V(Q_j))^- \cap N_G(l_i) = \emptyset$.*

Proof. Suppose $v \in (N_G(b_3) \cap V(Q_j))^- \cap N_G(l_i)$. Then $v^+ b_3 \in E(G)$, so $T' := T - \{vv^+, b_3 w_j\} + \{l_i v, v^+ b_3\}$ violates (T4) since $|V(S'_T)| < |V(S_T)|$. □

Claim 1 gives an independent 7-vertex set $X := \{l_1, l_2, l_3, l_4, l_5, l_6, b_3\}$. For every $h, i \in [6]$ with $i \neq h$, $(N_G(l_h) \cap V(M_h))^-$ is disjoint from both $N_G(l_i) \cap V(M_h)$ and $N_G(b_3) \cap V(M_h)$, by Lemma 2 and Claim 2, respectively. Lemma 5 gives that the five sets $N_G(l_i) \cap V(M_h)$ are disjoint from each other, and Claim 3 ensures that $N_G(b_3) \cap V(M_h)$ is disjoint from any of them. Therefore, for every $h \in [6]$, the seven sets $(N_G(l_h) \cap V(M_h))^-$, $N_G(b_3) \cap V(M_h)$, and $N_G(l_i) \cap V(M_h)$ for each $i \neq h$ are all disjoint. Furthermore, Lemmas 7 and 8 show that u_h is in none of these sets if $3 \nmid h$. Therefore:

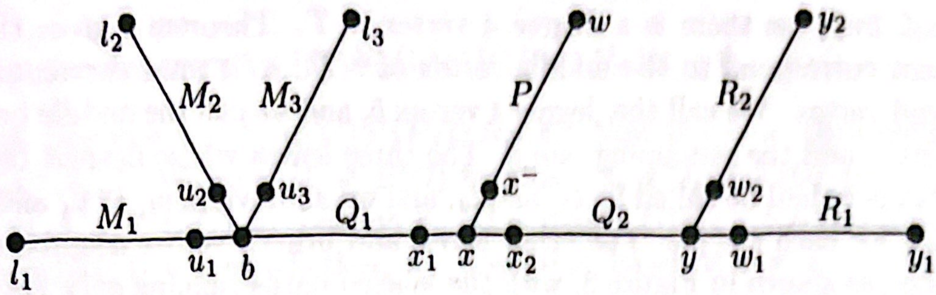

Figure 3: If $\tau \cong P_3$, T may have a degree 4 vertex corresponding to an end vertex of τ.

$$\begin{aligned}
&\sum_{v\in X} |N_G(v)\cap V(M_h)| \\
= \quad & |N_G(b_3)\cap V(M_h)| + |N_G(l_h)\cap V(M_h)| + \sum_{i\neq h} |N_G(l_i)\cap V(M_h)| \\
= \quad & |N_G(b_3)\cap V(M_h)| + |(N_G(l_h)\cap V(M_h))^-| + \sum_{i\neq h} |N_G(l_i)\cap V(M_h)| \\
\leq \quad & \begin{cases} |V(M_h)| & h \equiv 0 (\text{mod } 3) \\ |V(M_h)| - 1 & h \not\equiv 0 (\text{mod } 3). \end{cases}
\end{aligned}$$

Claim 4, meanwhile, shows that for each $j \in [2]$ the only possible neighbors of vertices in $V(Q_j)$ in X are l_{3-j}, l_{6-j}, l_3, l_6, and y_3; Claims 5-9 show that for each $j \in [2]$, the five sets $N_G(l_{3-j}) \cap V(Q_j)$, $N_G(l_{6-j}) \cap V(Q_j)$, $N_G(l_3) \cap V(Q_j)$, $N_G(l_6) \cap V(Q_j)$, and $(N_G(b_3) \cap V(Q_j))^-$ are all disjoint. Therefore, for each $j \in [2]$:

$$\sum_{v\in X} |N_G(v)\cap V(Q_j)| = |N_G(l_{3-j})\cap V(Q_j)| + |N_G(l_{6-j})\cap V(Q_j)| + |N_G(l_3)\cap V(Q_j)| + |N_G(l_6)\cap V(Q_j)| + |(N_G(b_3)\cap V(Q_j))^-| \leq |V(Q_j)|.$$

Since $b_3 \in X$, no vertex of X is adjacent to b_3 in G, so we can sum these inequalities for

$\sum_{v\in X} \deg_G(v) \leq n - 4$, contradicting the assumption that $\sigma_7(G) \geq n - 2$

□

Theorem 4. *There is no degree 4 vertex in T.*

Proof. Suppose there is a degree 4 vertex in T. Theorem 3 gives that it cannot correspond to the middle vertex of $\tau(T)$, so it must correspond to an end vertex. We call the degree 4 vertex b, and we call the middle branch vertex x and the remaining one y. The three leaves whose nearest branch vertex is b shall be called l_1, l_2, and l_3, and we abbreviate u_{l_i} as u_i and M_{l_i} as M_i for each $i \in [3]$. The other leaves and branch vertex neighbors are labeled as shown in Figure 3, with the labeled paths running only between nearest labeled vertices, similar to Figure 2 (for example, $Q_1 = bTx_1$), with one important exception: $P = wTx$.

Recall condition (T3), which prefers trees whose middle branch vertex has degree 4 over trees with an "end" branch vertex of degree 4. This condition, together with our choice of T, rules out the existence of any spanning tree of G whose middle branch vertex (of three) has degree 4.

Once this T is chosen, since G is claw-free, there can be no induced claw centered at b. Define $b^+ := N_T(b) \cap V(bTx)$. If there are two distinct $i, j \in [3]$ such that $u_i b^+, u_j b^+ \in E(G)$, then consider $T' := T - \{bu_i, bu_j\} + \{u_i b^+, u_j b^+\}$. If $b^+ = x$, then T' has fewer branch vertices than T, violating (T1). Otherwise T' has the same number of branch vertices and leaves as T, with the same structure, but $|V(S_{T'})| < |V(S_T)|$, violating (T4). Therefore there is at most one $i \in [3]$ such that $u_i b^+ \in E(G)$. If such an i exists, let $\{j, k\} = [3] \setminus \{i\}$, so it is easily seen that $u_j u_k \in E(G)$. Otherwise, it is easily seen that $\{u_1, u_2, u_3\}$ is a clique. Also, Lemma 1 gives that $w_1 w_2 \in E(G)$.

Claim 1. *The vertex set $X := \{l_1, l_2, l_3, w, y_1, y_2, b\}$ is independent.*

Proof. By Lemmas 2 and 3, we need only show that $l_i b \notin E(G)$ for each $i \in [3]$. Assume $l_i b \in E(G)$. Then either $u_i b^+ \in E(G)$ or $u_i u_j \in E(G)$ for some $j \neq i$. If $u_i b^+ \in E(G)$, then $T' := T - \{bb^+, bu_i\} + \{b^+ u_i, l_i b\}$ has the same number of branch vertices as T but fewer leaves, violating either (T1) or (T2). Otherwise $u_i u_j \in E(G)$ for some $j \neq i$, so $T' := T - \{bu_i, bu_j\} + \{bl_i, u_i u_j\}$ has the same number of branch vertices as T but fewer leaves, violating (T2). □

Claim 2. *For every $h \in [3]$, $(N_G(l_h) \cap V(M_h))^- \cap N_G(b) = \emptyset$.*

Proof. Suppose $v \in (N_G(l_h) \cap V(M_h))^- \cap N_G(b)$. Then $v^+ \in N_G(l_h) \cap V(M_h)$, and either $u_h b^+ \in E(G)$ or $u_h u_i \in E(G)$ for some $i \neq h$. If $u_h b^+ \in E(G)$, then $T' := T - \{bb^+, bu_h, vv^+\} + \{v^+ l_h, vb, b^+ u_h\}$ has the same number of branch vertices as T and fewer leaves, violating (T2). Otherwise $u_h u_i \in E(G)$ for some $i \neq h$, so $T' := T - \{vv^+, bu_h, bu_i\} + \{vb, v^+ l_h, u_h u_i\}$

has the same number of branch vertices as T and fewer leaves, violating (T2). □

Claim 3. *For every $i \in [3]$, $N_G(l_i) \cap V(P) \cap N_G(b) = \emptyset$.*

Proof. Suppose $v \in N_G(l_i) \cap V(P) \cap N_G(b)$. Now if $v = x$, then consider $G[x, x^-, b, l_i]$. We have $bl_i \notin E(G)$ by Claim 1, $x^-l_i \notin E(G)$ by Lemma 7, and $x^-b \notin E(G)$ by Lemma 8. This makes $G[x, x^-, b, l_i]$ an induced claw, which is a contradiction. On the other hand, if $v \neq x$, then since $v \neq w$, there exists v^-. Since $G[v, v^-, b, l_i]$ is not a claw and $bl_i \notin E(G)$, it follows that either $v^-b \in E(G)$ or $v^-l_i \in E(G)$. If $v^-b \in E(G)$, then $T' := T - \{vv^-, xx^-\} + \{v^-b, vl_i\}$ has fewer branch vertices than T; otherwise $v^-l_i \in E(G)$, so $T' := T - \{vv^-, xx^-\} + \{bv, l_iv^-\}$ has fewer branch vertices than T. Either way (T1) is still violated. □

Claim 4. *For every $i \in [3]$ and $h \in [2]$, $N_G(l_i) \cap V(R_h) \cap N_G(b) = \emptyset$.*

Proof. Suppose $v \in N_G(l_i) \cap V(R_h) \cap N_G(b)$. Since $v \neq y_h$, there exists v^-. Since $G[v, v^-, b, l_i]$ is not a claw and $bl_i \notin E(G)$, either $bv^- \in E(G)$ or $l_iv^- \in E(G)$. If $bv^- \in E(G)$, then $T' := T - \{vv^-, yw_h\} + \{bv^-, l_iv\}$ has fewer branch vertices than T; otherwise $l_iv^- \in E(G)$, so $T' := T - \{vv^-, yw_h\} + \{bv, l_iv^-\}$ has fewer branch vertices than T. Either way (T1) is violated. □

Claim 5. *For every $h \in [3]$, $N_G(w) \cap V(M_h) \cap N_G(b) = \emptyset$.*

Proof. Suppose $v \in N_G(w) \cap V(M_h) \cap N_G(b)$. Now either $u_hb^+ \in E(G)$ or there exists some $i \in [3] \setminus \{h\}$ such that $u_hu_i \in E(G)$. Consider two cases:

Case 1: Suppose $u_hb^+ \in E(G)$. Now if $b^+ = x$, then $T' := T - \{bu_h\} + \{xu_h\}$ corresponds to Figure 2, violating (T3). If $b^+ \neq x$, then $T' := T - \{bu_h, bb^+, xx_1\} + \{vw, vb, b^+u_h\}$ corresponds to Figure 2, violating (T3).

Case 2: Suppose $u_hu_i \in E(G)$. Since $v \neq l_h$, there exists v^-, and since $G[v, v^-, b, w]$ is not a claw and $bw \notin E(G)$, it follows that either $bv^- \in E(G)$ or $wv^- \in E(G)$. Now if $bv^- \in E(G)$, then $T' := T - \{vv^-, bu_h, bu_i\} + \{bv^-, wv, u_hu_i\}$ has the same number of branch vertices as T but fewer leaves, violating (T2). Otherwise $wv^- \in E(G)$, so $T' := T - \{vv^-, bu_h, bu_i\} + \{u_hu_i, bv, wv^-\}$ has the same number of branch vertices as T but fewer leaves, violating (T2). □

Claim 6. *For every $h \in [3]$ and $i \in [2]$, $N_G(y_i) \cap V(M_h) \cap N_G(b) = \emptyset$.*

Proof. Suppose $v \in N_G(y_i) \cap V(M_h) \cap N_G(b)$. Now either $u_h b^+ \in E(G)$ or there exists some $j \in [3] \setminus \{h\}$ such that $u_h u_j \in E(G)$. Consider two cases:

Case 1: Suppose $u_h b^+ \in E(G)$. Now if $b^+ = x$, then $T' := T - \{bu_h\} + \{xu_h\}$ corresponds to Figure 2, violating (T3). Otherwise, $T' := T - \{bu_h, bb^+, xx_2\} + \{vy_i, vb, b^+u_h\}$ corresponds to Figure 2, again violating (T3).

Case 2: Suppose $u_h u_j \in E(G)$. Since $v \neq l_h$, there exists v^-, and since $G[v, v^-, b, y_i]$ is not a claw and $by_i \notin E(G)$, it follows that either $bv^- \in E(G)$ or $y_i v^- \in E(G)$. Now if $bv^- \in E(G)$, then $T' := T - \{vv^-, bu_h, bu_j\} + \{bv^-, y_i v, u_h u_j\}$ has the same number of branch vertices as T but fewer leaves, violating (T2). Otherwise, $y_i v^- \in E(G)$, so $T' := T - \{vv^-, bu_h, bu_j\} + \{u_h u_j, bv, y_i v^-\}$ has the same number of branch vertices as T but fewer leaves, again violating (T2). □

Claim 7. *If $h \neq i$, then $N_G(l_i) \cap V(M_h) \cap N_G(b) = \emptyset$.*

Proof. Suppose $v \in N_G(l_i) \cap V(M_h) \cap N_G(b)$. Choose $j \in [3] \setminus \{h, i\}$ and consider two cases:

Case 1: Suppose $u_j b^+ \notin E(G)$. Then either $u_j u_i \in E(G)$ or $u_j u_h \in E(G)$. If $u_j u_i \in E(G)$, then $T' := T - \{bu_i, bu_j\} + \{u_j u_i, vl_i\}$ has the same number of branch vertices as T but fewer leaves, violating (T2). Otherwise $u_j u_h \in E(G)$, so $T' := T - \{bu_h, bu_j\} + \{u_h u_j, vl_i\}$ has the same number of branch vertices as T but fewer leaves, still violating (T2).

Case 2: Suppose $u_j b^+ \in E(G)$. Then $u_h u_i \in E(G)$, and since $v \neq l_h$, there exists v^-. Since $G[v, v^-, b, l_i]$ is not a claw and $bl_i \notin E(G)$, it follows that either $l_i v^- \in E(G)$ or $bv^- \in E(G)$. If $l_i v^- \in E(G)$, then $T' := T - \{bu_h, bu_i, vv^-\} + \{bv, u_h u_i, l_i v^-\}$ has the same number of branch vertices as T but fewer leaves, violating (T2). Otherwise $bv^- \in E(G)$, and since $G[b, u_h, v^-, b^+]$ is not a claw and $u_h b^+ \notin E(G)$, it follows that either $u_h v^- \in E(G)$ or $b^+ v^- \in E(G)$. If $u_h v^- \in E(G)$, then $T' := T - \{vv^-, bu_h\} + \{l_i v, u_h v^-\}$ has the same number of branch vertices as T but fewer leaves, violating (T2). Otherwise $b^+ v^- \in E(G)$, so consider $T' := T - \{vv^-, bu_h, bu_j\} + \{b^+ v^-, b^+ u_j, l_i v\}$. If $b^+ = x$, then T' has fewer branch vertices than T, violating (T1). Otherwise, T' has the same number of branch vertices and leaves as T, with the same structure, but $|V(S_{T'})| < |V(S_T)|$, violating (T4). □

Claim 8. *If $h, i \in [2]$, then $N_G(y_i) \cap V(Q_h) = \emptyset$.*

Proof. Suppose $v \in N_G(y_i) \cap V(Q_h)$. Choose $j \in [2] \setminus \{i\}$ and consider $T' := T - \{yw_i, yw_j\} + \{vy_i, w_iw_j\}$. If $v = b$ or $v = y$, then T' has fewer branch vertices than T, violating (T1). Otherwise, T' has the same number of branch vertices and leaves as T, both matching Figure 3, but $|V(S_{T'})| < |V(S_T)|$, violating (T4). ☐

Claim 9. *If $i \neq j$, then $N_G(l_i) \cap V(Q_1) \cap N_G(l_j) = \emptyset$.*

Proof. Suppose $v \in N_G(l_i) \cap V(Q_1) \cap N_G(l_j)$. Then $v \neq b$, so $T' := T - \{bu_i, bu_j\} + \{vl_i, vl_j\}$ has the same number of branch vertices and leaves as T, with the same structure, but $|V(S_{T'})| < |V(S_T)|$, violating (T4). ☐

Claim 10. *If $i \neq j$, then $N_G(l_i) \cap V(Q_2) \cap N_G(l_j) = \emptyset$.*

Proof. Suppose $v \in N_G(l_i) \cap V(Q_2) \cap N_G(l_j)$. Then consider $T' := T - \{bu_i, bu_j\} + \{vl_i, vl_j\}$. If $v = y$, then T' has fewer branch vertices than T, violating (T1). Otherwise T' corresponds to Figure 2, violating (T3). ☐

Claim 11. *If $i \in [3]$ and $h \in [2]$, then $N_G(l_i) \cap V(Q_h) \cap N_G(w) = \emptyset$.*

Proof. Suppose $v \in N_G(l_i) \cap V(Q_h) \cap N_G(w)$. Then $v \neq b$, and it is easily verified that $v \neq y$, so $T' := T - \{bu_i, xx^-\} + \{vw, vl_i\}$ has corresponds to Figure 2, violating (T3). ☐

Claim 12. *If $i \in [3]$, then $(N_G(b) \cap V(Q_1))^- \cap N_G(l_i) = \emptyset$.*

Proof. Suppose $v \in (N_G(b) \cap V(Q_1))^- \cap N_G(l_i)$. Then $v^+ \in N_G(b) \cap V(Q_1)$, so $T' := T - \{vv^+, bb^+\} + \{l_iv, bv^+\}$ has the same number of branch vertices and leaves as T, with the same structure, but $|V(S_{T'})| < |V(S_T)|$, violating (T4). ☐

Claim 13. *We have $(N_G(b) \cap V(Q_1))^- \cap N_G(w) = \emptyset$.*

Proof. Suppose $v \in (N_G(b) \cap V(Q_1))^- \cap N_G(w)$. Then $v^+ \in N_G(b) \cap V(Q_1)$, so $T' := T - \{vv^+, xx^-\} + \{vw, v^+b\}$ has fewer branch vertices than T, violating (T1). ☐

Claim 14. *If $i \in [3]$, then $(N_G(b) \cap V(Q_2))^- \cap N_G(l_i) = \emptyset$.*

Proof. Suppose $v \in (N_G(b) \cap V(Q_2))^- \cap N_G(l_i)$. Then $v^+ \in N_G(b) \cap V(Q_2)$, so $T' := T - \{vv^+, xx_2\} + \{bv^+, l_iv\}$ has fewer branch vertices than T, violating (T1). ☐

Claim 15. *We have $(N_G(b) \cap V(Q_2))^- \cap N_G(w) = \emptyset$.*

Proof. Suppose $v \in (N_G(b) \cap V(Q_2))^- \cap N_G(w)$. Then $v^+ \in N_G(b) \cap V(Q_2)$, so $T' := T - \{vv^+, xx_2\} + \{wv, bv^+\}$ has fewer branch vertices than T, violating (T1). □

Claim 16. *We have $wx \notin E(G)$.*

Proof. Suppose $wx \in E(G)$. Since $G[x, x^-, x_1, x_2]$ is not a claw, either $x^-x_1 \in E(G)$ or $x^-x_2 \in E(G)$ or $x_1x_2 \in E(G)$. If $x^-x_1 \in E(G)$, then $T' := T - \{xx^-, xx_1\} + \{wx, x^-x_1\}$ violates (T1). If $x^-x_2 \in E(G)$, then $T' := T - \{xx^-, xx_2\} + \{wx, x^-x_2\}$ violates (T1). Otherwise $x_1x_2 \in E(G)$, so $T' := T - \{xx_1\} + \{x_1x_2\}$ violates (T4). □

Lemma 2 ensures that $(N_G(w) \cap V(P))^-$ is disjoint from $N_G(y_i) \cap V(P)$ for each $i \in [2]$ and from $N_G(l_j) \cap V(P)$ for each $j \in [3]$. Lemma 3 ensures that $(N_G(w) \cap V(P))^-$ is disjoint from $N_G(b) \cap V(P)$. Lemma 5 ensures that the five sets $N_G(y_i) \cap V(P)$ for each $i \in [2]$ and $N_G(l_j) \cap V(P)$ for each $j \in [3]$ are all disjoint. Lemma 6 ensures that $N_G(b) \cap V(P)$ is disjoint from $N_G(y_i) \cap V(P)$ for each $i \in [2]$, and Claim 3 ensures that $N_G(l_j) \cap V(P)$ is disjoint from $N_G(b) \cap V(P)$ for each $j \in [3]$. Therefore the seven sets $(N_G(w) \cap V(P))^-$, $N_G(y_i) \cap V(P)$ for $i \in [2]$, $N_G(l_j) \cap V(P)$ for $j \in [3]$, and $N_G(b) \cap V(P)$ are all disjoint. Furthermore, Lemmas 7 and 8 and Claim 16 ensure that none of them contain x^-, so the sum of their cardinalities is at most $|V(P)| - 1$.

Similarly, for each $h \in [2]$, Lemma 2 ensures that $(N_G(y_h) \cap V(R_h))^-$ is disjoint from any of $N_G(y_{3-h}) \cap V(R_h)$, $N_G(w) \cap V(R_h)$, and $N_G(l_j) \cap V(R_h)$ (for $j \in [3]$), and Lemma 3 ensures that $(N_G(y_h) \cap V(R_h))^-$ is disjoint from $N_G(b) \cap V(R_h)$. Lemma 5 ensures that the five sets $N_G(y_{3-h}) \cap V(R_h)$, $N_G(w) \cap V(R_h)$, and $N_G(l_j) \cap V(R_h)$ are all disjoint. Lemma 6 ensures that $N_G(b) \cap V(R_h)$ is disjoint from both $N_G(y_{3-h}) \cap V(R_h)$ and $N_G(w) \cap V(R_h)$, while Claim 4 ensures that $N_G(b) \cap V(R_h)$ is disjoint from $N_G(l_j) \cap V(R_h)$. Therefore the seven sets $(N_G(y_h) \cap V(R_h))^-$, $N_G(y_{3-h}) \cap V(R_h)$, $N_G(w) \cap V(R_h)$, $N_G(l_j) \cap V(R_h)$ for $j \in [3]$, and $N_G(b) \cap V(R_h)$ are all disjoint. Now Lemmas 7 and 8 ensure that none of them contain w_h, so the sum of their cardinalities is at most $|V(R_h)| - 1$.

Similarly, for each $h \in [3]$, Lemma 2 ensures that $(N_G(l_h) \cap V(M_h))^-$ is disjoint from any of $N_G(l_i) \cap V(M_h)$ (for $i \neq h$), $N_G(w) \cap V(M_h)$, and $N_G(y_j) \cap V(M_h)$ (for $j \in [2]$), and Claim 2 ensures that $(N_G(l_h) \cap V(M_h))^-$ is disjoint from $N_G(b) \cap V(M_h)$. Meanwhile, Lemma 5 ensures that the five sets $N_G(l_i) \cap V(M_h)$ for $i \neq h$, $N_G(w) \cap V(M_h)$, and $N_G(y_j) \cap V(M_h)$ are all disjoint. Now $N_G(b) \cap V(M_h)$ is disjoint from $N_G(y_j) \cap V(M_h)$ (by Claim 6), $N_G(w) \cap V(M_h)$ (by Claim 5), and $N_G(l_i) \cap V(M_h)$ (by Claim 7). Therefore the seven sets $(N_G(l_h) \cap V(M_h))^-$, $N_G(l_i) \cap V(M_h)$ for $i \neq h$,

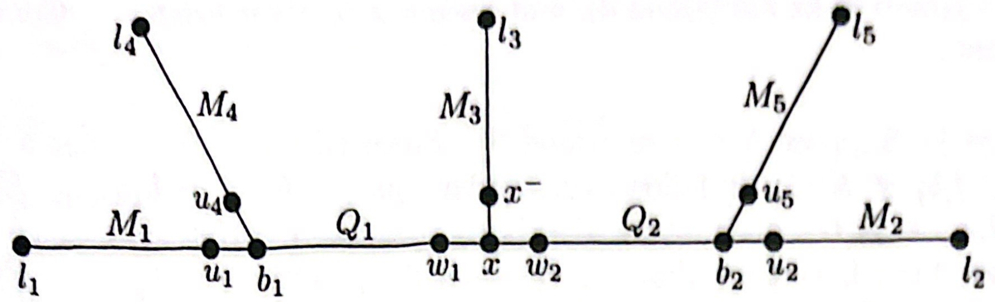

Figure 4: If $\tau \cong P_3$, T may have no degree 4 vertices. Each vertex labeled b_i is also called b_{i+3}.

$N_G(w) \cap V(M_h)$, $N_G(y_j) \cap V(M_h)$ for $j \in [2]$, and $N_G(b) \cap V(M_h)$ are all disjoint, so the sum of their cardinalities is at most $|V(M_h)|$.

Finally, for each $h \in [2]$, Claim 8 gives that the two sets $N_G(y_i) \cap V(Q_h)$ are empty, and Claims 9-15 give that the five sets $N_G(l_i) \cap V(Q_h)$, $N_G(w) \cap V(Q_h)$, and $(N_G(b) \cap V(Q_h))^-$ are all disjoint, so the sum of their cardinalities is at most $|V(Q_h)|$.

Summing these inequalities gives $\sum_{v \in X} \deg_G(v) \le n - 3$, contradicting the assumption of the theorem. □

We now know that T has no degree 4 vertices.

Theorem 5. *Our tree T has at least four branch vertices.*

Proof. By contradiction, suppose T has only three branch vertices. Since Theorem 4 requires that they all have degree 3, we label vertices and paths as shown in Figure 4, with each labeled path connecting only the nearest labeled vertices, as with the other figures, with one important exception: $M_3 = xTl_3$. Lemma 1 gives that $u_1u_4, u_2u_5 \in E(G)$. Furthermore, (T4) gives that $w_1w_2 \notin E(G)$, so either $w_1x^- \in E(G)$ or $w_2x^- \in E(G)$.

Claim 1. *The vertex set $X := \{l_1, l_2, l_3, l_4, l_5, b_1, b_2\}$ is independent.*

Proof. By Lemmas 2, 3, and 4, we need only show that $b_1b_2 \notin E(G)$. If $b_1b_2 \in E(G)$, then $T' := T - \{w_1x\} + \{b_1b_2\}$ has fewer branch vertices than T, violating (T1). □

Claim 2. *If $h \neq i$ and $j \in [2]$, then $N_G(l_i) \cap V(M_h) \cap N_G(b_j) = \emptyset$.*

Proof. Suppose $v \in N_G(l_i) \cap V(M_h) \cap N_G(b_j)$. Lemma 6 requires that either $h \equiv j (\text{mod } 3)$ or $i \equiv j (\text{mod } 3)$, and since $v \neq l_h$, there exists v^-. Consider cases:

Case 1: Suppose $h \equiv i \equiv j (\text{mod } 3)$. Since $G[v, v^-, l_i, b_j]$ is not a claw and $l_i b_j \notin E(G)$, it follows that either $l_i v^- \in E(G)$ or $b_j v^- \in E(G)$. If $l_i v^- \in E(G)$, then $T' := T - \{b_j u_h, b_j u_i, vv^-\} + \{l_i v^-, b_j v, u_h u_i\}$ has fewer branch vertices than T, violating (T1). Otherwise $b_j v^- \in E(G)$, so since $G[b_j, v^-, u_h, b_j^+]$ is not a claw and $u_h b_j^+ \notin E(G)$, it follows that either $b_j^+ v^- \in E(G)$ or $u_h v^- \in E(G)$. If $b_j^+ v^- \in E(G)$, then $T' := T - \{vv^-, b_j u_h\} + \{l_i v, b_j^+ v^-\}$ either has fewer branch vertices than T (if $b_j^+ = x$) or else the same number of branch vertices and leaves, with the same structure, but with a smaller internal subtree. Otherwise $u_h v^- \in E(G)$, so $T' := T - \{vv^-, b_j u_h\} + \{l_i v, u_h v^-\}$ has fewer branch vertices than T. In each case, (T1) or (T4) is violated.

Case 2: Suppose $h \equiv j \not\equiv i (\text{mod } 3)$. If $i = 3$, then $T' := T - \{xx^-, b_j u_j, b_j u_{j+3}\} + \{u_j u_{j+3}, vl_i, vb_j\}$ has fewer branch vertices than T. If $i \neq 3$, then $T' := T - \{b_i u_i, b_j u_j, b_j u_{j+3}\} + \{u_j u_{j+3}, vl_i, vb_j\}$ has fewer branch vertices than T. Either way (T1) is violated.

Case 3: Suppose $h = 3$. Then $i \equiv j (\text{mod } 3)$, so $T' := T - \{xx^-, b_j u_j, b_j u_{j+3}\} + \{b_j v, l_i v, u_j u_{j+3}\}$ has fewer branch vertices than T, violating (T1).

Case 4: Suppose $3 \neq h \not\equiv j \equiv i (\text{mod } 3)$. Then $T' := T - \{b_j u_j, b_j u_{j+3}, xw_j\} + \{vb_j, vl_i, u_j u_{j+3}\}$ has fewer branch vertices than T, violating (T1) and proving the claim. □

Claim 3. *For every $h \in [5]$, $N_G(b_1) \cap V(M_h) \cap N_G(b_2) = \emptyset$.*

Proof. Suppose $v \in N_G(b_1) \cap V(M_h) \cap N_G(b_2)$. Since $v \neq l_h$ by Claim 1, there exists v^-. Consider cases:

Case 1: Suppose $h \neq 3$. Without loss of generality, suppose $h = 1$. Since $G[v, v^-, b_1, b_2]$ is not a claw, either $v^- b_1 \in E(G)$ or $v^- b_2 \in E(G)$. If $v^- b_1 \in E(G)$, then $T' := T - \{vv^-, b_1 u_1, b_1 u_4\} + \{b_2 v, b_1 v^-, u_1 u_4\}$ has fewer branch vertices than T, violating (T1). Otherwise $v^- b_2 \in E(G)$, so $T' := T - \{vv^-, b_1 u_1, b_1 u_4\} + \{b_1 v, b_2 v^-, u_1 u_4\}$ similarly violates (T1).

Case 2: Suppose $h = 3$. If $v = x$, then without loss of generality, assume $x^- w_1 \in E(G)$, so it is easily seen that $b_1 \neq w_1$, so $T' := T - \{xx^-, xw_1\} + \{xb_1, x^- w_1\}$ has fewer branch vertices than T, violating (T1). If $v \neq x$, then since $G[v, v^-, b_1, b_2]$ is not a claw, and $b_1 b_2 \notin E(G)$, either $v^- b_1 \in E(G)$

or $v^-b_2 \in E(G)$. Without loss of generality, assume $v^-b_1 \in E(G)$, so $T' := T - \{vv^-, xx^-\} + \{v^-b_1, vb_2\}$ has fewer vertices than T, violating (T1) and proving the claim. □

Claim 4. *If $i \neq 3$, then $N_G(l_i) \cap V(Q_j) = \emptyset$.*

Proof. Suppose $v \in N_G(l_i) \cap V(Q_j)$ for some $i \neq 3$. For $T' := T - \{b_iu_i\} + \{vl_i\}$, we have $|V(S_{T'})| < |V(S_T)|$, violating (T4). □

Claim 5. *For every $j \in [2]$, $(N_G(b_j) \cap V(Q_j))^- \cap N_G(l_3)$.*

Proof. Suppose $v \in (N_G(b_j) \cap V(Q_j))^- \cap N_G(l_3)$. Then $v^+ \in N_G(b_j) \cap V(Q_j)$, so $T' := T - \{vv^+, xx^-\} + \{v^+b_j, vl_3\}$ has fewer branch vertices than T, violating (T1). □

Claim 6. *If $\{i,j\} = \{1,2\}$, then $(N_G(b_j) \cap V(Q_j))^- \cap N_G(b_i) = \emptyset$.*

Proof. Suppose $v \in (N_G(b_j) \cap V(Q_j))^- \cap N_G(b_i)$. Then $v^+ \in N_G(b_j) \cap V(Q_j)$, so $T' := T - \{vv^+, xw_i\} + \{v^+b_j, vb_i\}$ has fewer branch vertices than T, violating (T1). □

Claim 7. *If $\{i,j\} = \{1,2\}$, then $N_G(b_i) \cap V(Q_j) \cap N_G(l_3) = \emptyset$.*

Proof. Suppose $v \in N_G(b_i) \cap V(Q_j) \cap N_G(l_3)$. Then since $b_jl_3 \notin E(G)$, $v \neq b_j$ so there exists v^-. Since $G[v, v^-, b_i, l_3]$ is not a claw and $b_il_3 \notin E(G)$, either $v^-b_i \in E(G)$ or $v^-l_3 \in E(G)$. If $v^-l_3 \in E(G)$, then $T' := T - \{vv^-, xw_j\} + \{b_iv, l_3v^-\}$ has fewer branch vertices than T, violating (T1). Otherwise $v^-b_i \in E(G)$, so $T' := T - \{vv^-, xw_j\} + \{l_3v, b_iv^-\}$ has fewer branch vertices than T, again violating (T1). □

Claim 8. *We have $xl_3 \notin E(G)$.*

Proof. We already know $x^-w_i \in E(G)$ for some $i \in [2]$, so if $xl_3 \in E(G)$, then $T' := T - \{xx^-, xw_i\} + \{x^-w_i, xl_3\}$ has fewer branch vertices than T, violating (T1). □

Claim 9. *If $\{i,j\} = [2]$, then $w_j \notin N_G(b_i) \cup N_G(l_3)$.*

Proof. Suppose $w_j \in N_G(b_i) \cup N_G(l_3)$. Then either $w_j \in N_G(b_i)$ (in which case $T' := T - \{xw_j\} + \{b_iw_j\}$ violates (T1)) or else $w_j \in N_G(l_3)$ (in which case $T' := T - \{xw_j\} + \{l_3w_j\}$ violates (T1)). □

For every $i \neq h \in [5]$, Lemma 2 ensures that $(N_G(l_h) \cap V(M_h))^-$ is disjoint from $N_G(l_i) \cap V(M_h)$. Lemma 3 ensures that $(N_G(l_h) \cap V(M_h))^-$ is disjoint from $N_G(b_j) \cap V(M_h)$ when $h \not\equiv j (\text{mod } 3)$, and Lemma 4 ensures the same when $h \equiv j (\text{mod } 3)$. Lemma 5 ensures that the four sets $N_G(l_i) \cap V(M_h)$ are all disjoint, and Claim 2 ensures that each $N_G(l_i) \cap V(M_h)$ with $i \neq h$ is disjoint from each $N_G(b_j) \cap V(M_h)$. Finally, Claim 3 ensures that $N_G(b_1) \cap V(M_h)$ does not intersect $N_G(b_2) \cap V(M_h)$, so the seven sets $(N_G(l_h) \cap V(M_h))^-$, $N_G(l_i) \cap V(M_h)$ (for each $i \neq h$), and $N_G(b_j) \cap V(M_h)$ (for $j \in [2]$) are disjoint, so the sum of their cardinalities equals the cardinality of their union, which cannot exceed the cardinality of $V(M_h)$. Furthermore, none of these contain x^- by Lemmas 7 and 8 and Claim 8, so:

$$
\begin{aligned}
&\sum_{v \in X} |N_G(v) \cap V(M_h)| \\
&= \sum_{i=1}^{5} |N_G(l_i) \cap V(M_h)| + \sum_{j=1}^{2} |N_G(b_j) \cap V(M_h)| \\
&= |N_G(l_h) \cap V(M_h)| + \sum_{i \neq h} |N_G(l_i) \cap V(M_h)| + \sum_{j=1}^{2} |N_G(b_j) \cap V(M_h)| \\
&= |(N_G(l_h) \cap V(M_h))^-| + \sum_{i \neq h} |N_G(l_i) \cap V(M_h)| + \sum_{j=1}^{2} |N_G(b_j) \cap V(M_h)| \\
&\leq |V(M_h) \setminus \{x^-\}| = \begin{cases} |V(M_h)| & h \neq 3 \\ |V(M_h)| - 1 & h = 3. \end{cases}
\end{aligned}
$$

Meanwhile, for each $j \in [2]$ (and $\{i\} = [2] \setminus \{j\}$), Claim 4 gives that b_1, b_2, and l_3 are the only vertices in X that can be adjacent to any vertex of $V(Q_j)$, and Claims 5, 6, and 7 give that the three sets $(N_G(b_j) \cap V(Q_j))^-$, $N_G(l_3) \cap V(Q_j)$, and $N_G(b_i) \cap V(Q_j)$ are disjoint, and none of them contain w_j by Claim 9, so the sum of their cardinalities is at most $|V(Q_j) \setminus \{w_j\}| = |V(Q_j)| - 1$, so

$$
\begin{aligned}
&\sum_{v \in X} |N_G(v) \cap V(Q_j)| \\
&= \sum_{h=1}^{5} |N_G(l_h) \cap V(Q_j)| + \sum_{k=1}^{2} |N_G(b_k) \cap V(Q_j)| \\
&= |N_G(l_3) \cap V(Q_j)| + |N_G(b_i) \cap V(Q_j)| + |N_G(b_j) \cap V(Q_j)| \\
&= |N_G(l_3) \cap V(Q_j)| + |N_G(b_i) \cap V(Q_j)| + |(N_G(b_j) \cap V(Q_j))^-| \\
&\leq |V(Q_j)| - 1.
\end{aligned}
$$

Summing these inequalities gives $\sum_{v \in X} \deg_G(v) \leq n - 3$, contradicting the assumption of the theorem. □

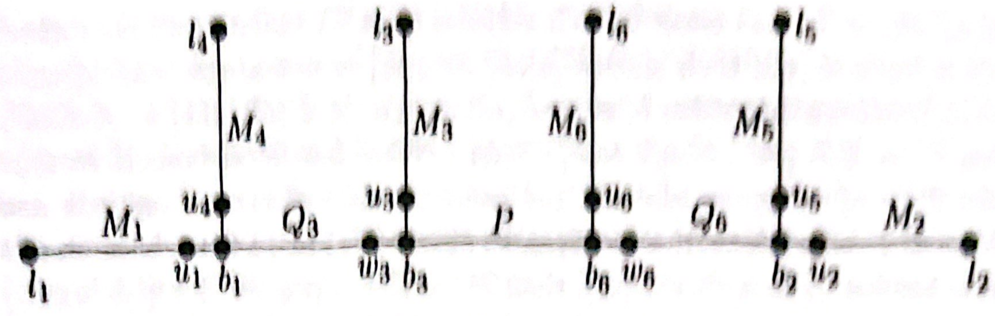

Figure 5: If T has 4 branch vertices, we may have $\tau \cong P_4$. Here, b_1 is also called b_4, while b_2 is also called b_5.

Therefore T must have at least 4 branch vertices (all with degree 3 of course), so either $\tau \cong P_4$ or τ is a claw.

Theorem 6. *The derived tree* $\tau(T) \ncong P_4$.

Proof. By contradiction, suppose $\tau(T) \cong P_4$. We then label vertices and paths as shown in Figure 5. Note that each vertex is in exactly one labeled path. Once this T is chosen, we choose a (potentially different, but still with $\tau \cong P_4$) T such that

(T5) P is as short as possible.

By Lemma 1, $u_1u_4 \in E(G)$ and $u_2u_5 \in E(G)$. Similarly, since no induced claw is centered at b_{3i} for any $i \in [2]$, (T4) and (T5) give that $u_{3i}w_{3i} \in E(G)$. Meanwhile, Lemmas 2, 3, and 4 ensure that $X := \{l_1, l_2, l_3, l_4, l_5, l_6, b_1\}$ is an independent set. Define $b_1^+ = N_T(b_1) \cap V(S_T)$.

Claim 1. *If* $h \neq i$, *then* $N_G(l_i) \cap V(M_h) \cap N_G(b_1) = \emptyset$.

Proof. Suppose $v \in N_G(l_i) \cap V(M_h) \cap N_G(b_1)$. By Lemma 6, we may assume $h \equiv 1 (\text{mod } 3)$ or $i \equiv 1 (\text{mod } 3)$. Consider several cases:

Case 1: Suppose $i \not\equiv 1 \equiv h (\text{mod } 3)$. Then $T' := T - \{b_iu_i, b_1u_1, b_1u_4\} + \{u_1u_4, vb_1, vl_i\}$ has fewer branch vertices than T, violating (T1).

Case 2: Suppose $i \equiv 1 \not\equiv h (\text{mod } 3)$. Then $T' := T - \{b_hu_h, b_1u_1, b_1u_4\} + \{u_1u_4, vb_1, vl_i\}$ has fewer branch vertices than T, violating (T1).

Case 3: Suppose $i \equiv 1 \equiv h (\text{mod } 3)$. Since $v \neq l_h$, there exists v^-. Since $G[v, b_1, l_i, v^-]$ is not a claw and $b_1l_i \notin E(G)$, either $v^-l_i \in E(G)$ or $v^-b_1 \in E(G)$. Now if $v^-l_i \in E(G)$, then $T' := T - \{vv^-, b_1u_1, b_1u_4\} +$

$\{u_1u_4, vb_1, v^-l_i\}$ has fewer branch vertices than T, violating (T1). Otherwise $v^-b_1 \in E(G)$, then since $G[b_1, b_1^+, v^-, u_h]$ is not a claw and $b_1^+u_h \notin E(G)$, it follows that either $b_1^+v^- \in E(G)$ or $u_hv^- \in E(G)$. If $b_1^+v^- \in E(G)$, then $T' := T - \{vv^-, b_1u_h\} + \{b_1^+v^-, l_iv\}$ either has fewer branch vertices than T (if $b_1^+ = b_3$) or else has the same number of branch vertices and leaves as T with $|V(S_{T'})| < |V(S_T)|$, so either (T1) or (T4) is violated. On the other hand, if $u_hv^- \in E(G)$, then $T' := T - \{b_1u_h, vv^-\} + \{l_iv, u_hv^-\}$ has fewer branch vertices than T, violating (T1). □

Claim 2. *The following statements hold:*

Part 1. *If $i \not\equiv 0 \pmod 3$, then $N_G(l_i) \cap V(Q_j) = \emptyset$.*

Part 2. *We have $N_G(b_1) \cap V(Q_6) \cap N_G(l_3) = \emptyset$.*

Part 3. *We have $N_G(b_1) \cap V(Q_6) \cap N_G(l_6) = \emptyset$.*

Part 4. *If $i \in [2]$, then $N_G(l_3) \cap V(Q_{3i}) \cap N_G(l_6) = \emptyset$.*

Part 5. *We have $(N_G(b_1) \cap V(Q_3))^- \cap N_G(l_3) = \emptyset$.*

Part 6. *We have $(N_G(b_1) \cap V(Q_3))^- \cap N_G(l_6) = \emptyset$.*

Part 7. *We have $N_G(l_i) \cap V(P) = \emptyset$ for each $i \in [6]$.*

Proof. To prove Part 1, suppose $v \in N_G(l_i) \cap V(Q_j)$. By symmetry, $v \neq b_{j/3}$, so $T' := T - \{b_iu_i\} + \{l_iv\}$ has the same number of branch vertices and leaves as T with $|V(S_{T'})| < |V(S_T)|$, violating (T4). To prove Part 2, suppose $v \in N_G(b_1) \cap V(Q_6) \cap N_G(l_3)$. By symmetry, $v \neq b_2$, so $T' := T - \{w_3b_3, w_6b_6\} + \{vl_3, vb_1\}$ has fewer branch vertices than T', violating (T1). To prove Part 3, suppose $v \in N_G(b_1) \cap V(Q_6) \cap N_G(l_6)$. By symmetry, $v \neq b_2$, so $T' := T - \{w_3b_3, w_6b_6\} + \{vl_6, vb_1\}$ has fewer branch vertices than T', violating (T1). To prove Part 4, let $i \in [2]$ and suppose $v \in N_G(l_3) \cap V(Q_{3i}) \cap N_G(l_6)$. Then $T' := T - \{u_3b_3, u_6b_6\} + \{vl_3, vl_6\}$ has fewer branch vertices than T, violating (T1). To prove Part 5, suppose $v \in (N_G(b_1) \cap V(Q_3))^- \cap N_G(l_3)$. Then $v^+ \in N_G(b_1) \cap V(Q_3)$, so $T' := T - \{vv^+, b_3u_3\} + \{l_3v, b_1v^+\}$ has fewer branch vertices than T, violating (T1). To prove Part 6, suppose $v \in (N_G(b_1) \cap V(Q_3))^- \cap N_G(l_6)$. Then $v^+ \in N_G(b_1) \cap V(Q_3)$, so $T' := T - \{vv^+, b_6u_6\} + \{l_6v, b_1v^+\}$ has fewer branch vertices than T, violating (T1). To prove Part 7, suppose $v \in N_G(l_i) \cap V(P)$. Now if $v \in \{b_3, b_6\}$, Lemma 3 ensures that $i \equiv 0 \pmod 3$ and $v = b_i$, so $T' := T - \{b_iw_i, b_iu_i\} + \{b_il_i, u_iw_i\}$ has fewer branch vertices than T, violating (T1). Otherwise, $b_3 \neq v \neq b_6$, so $T' := T - \{b_iu_i\} + \{vl_i\}$ has the same number of branch vertices and leaves as T, and $|S_T| = |S_{T'}|$, but P is shorter for T' than it is for T, violating (T5) and proving the claim. □

Lemma 2 ensures that $(N_G(l_h) \cap V(M_h))^-$ is disjoint from $N_G(l_i) \cap V(M_h)$ (for each $i \neq h$). Lemma 3 ensures that $(N_G(l_h) \cap V(M_h))^-$ is disjoint from $N_G(b_1) \cap V(M_h)$ for $h \not\equiv 1 (\text{mod } 3)$. Lemma 4 ensures the latter for $h \equiv 1 (\text{mod } 3)$. Lemma 5 and Claim 1 ensure that the five sets $N_G(l_i) \cap V(M_h)$ are disjoint from each other and $N_G(b_1) \cap V(M_h)$, respectively. Therefore the seven sets $(N_G(l_h) \cap V(M_h))^-$, $N_G(b_1) \cap V(M_h)$, and $N_G(l_i) \cap V(M_h)$ for $i \neq h$ are all disjoint, and by Lemmas 7 and 8, none of them contain u_h if $h \not\equiv 1$ (mod 3). Therefore:

$$
\begin{aligned}
& \sum_{v \in X} |N_G(v) \cap V(M_h)| \\
= \; & |N_G(b_1) \cap V(M_h)| + \sum_{i=1}^{6} |N_G(l_i) \cap V(M_h)| \\
= \; & |N_G(b_1) \cap V(M_h)| + |N_G(l_h) \cap V(M_h)| + \sum_{i \neq h} |N_G(l_i) \cap V(M_h)| \\
= \; & |N_G(b_1) \cap V(M_h)| + |(N_G(l_h) \cap V(M_h))^-| + \sum_{i \neq h} |N_G(l_i) \cap V(M_h)| \\
\leq \; & \begin{cases} |V(M_h)| - 1 & h \not\equiv 1 (\text{mod } 3) \\ |V(M_h)| & h \equiv 1 (\text{mod } 3). \end{cases}
\end{aligned}
$$

By Claim 2 Part 1, for $i \in [2]$, the only vertices of X that can be adjacent to Q_{3i} are l_3, l_6, and b_1. By Parts 2, 3, and 4, the three sets $N_G(l_3) \cap V(Q_6)$, $N_G(l_6) \cap V(Q_6)$, and $N_G(b_1) \cap V(Q_6)$ are disjoint. By Parts 4, 5, and 6, the three sets $N_G(l_3) \cap V(Q_3)$, $N_G(l_6) \cap V(Q_3)$, and $(N_G(b_1) \cap V(Q_3))^-$ are disjoint. Therefore:

$$
\begin{aligned}
& \sum_{v \in X} |N_G(v) \cap V(Q_6)| \\
= \; & |N_G(l_3) \cap V(Q_6)| + |N_G(l_6) \cap V(Q_6)| + |N_G(b_1) \cap V(Q_6)| \\
\leq \; & |V(Q_6)|
\end{aligned}
$$

and:

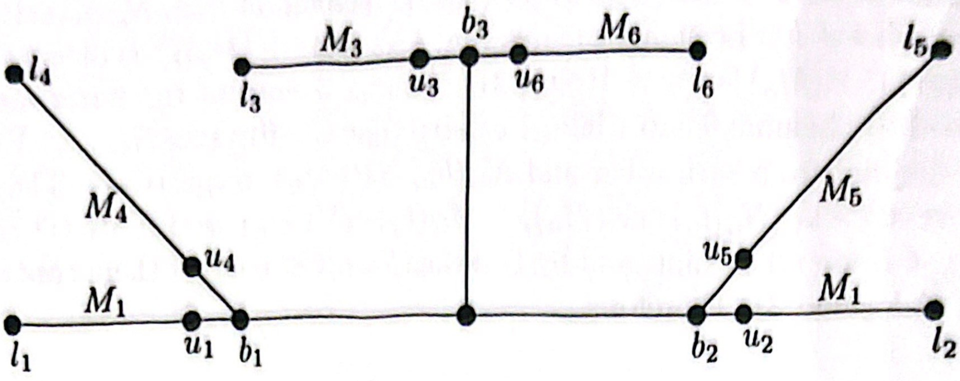

Figure 6: If T has 4 branch vertices, τ may be a claw. Each vertex labeled b_i is also called b_{i+3}.

$$\begin{aligned}
&\sum_{v\in X} |N_G(v)\cap V(Q_3)| \\
=\ & |N_G(l_3)\cap V(Q_3)| + |N_G(l_6)\cap V(Q_3)| + |N_G(b_1)\cap V(Q_3)| \\
=\ & |N_G(l_3)\cap V(Q_3)| + |N_G(l_6)\cap V(Q_3)| + |(N_G(b_1)\cap V(Q_3))^-| \\
\leq\ & |V(Q_3)|
\end{aligned}$$

By Claim 2 Part 7, b_1 is the only vertex of X that can be adjacent to any of P, so

$$\sum_{v\in X} |N_G(v)\cap V(P)| = |N_G(b_1)\cap V(P)| \leq |V(P)|.$$

Summing these inequalities gives $\sum_{v\in X} \deg_G(v) \leq n-4$, contradicting the assumption of the theorem. □

Theorem 7. *The derived tree τ is not a claw.*

Proof. By contradiction, suppose τ is a claw. We label vertices and paths as shown in Figure 6. Since $u_ib_i \in E(T)$ and $u_i \notin S_T$ for every $i \in [6]$, Lemma 1 gives that $u_iu_{i+3} \in E(G)$ for each $i \in [3]$. Furthermore, the vertex set $X := \{l_1, l_2, l_3, l_4, l_5, l_6, b_3\}$ is independent by Lemmas 2, 3, and 4.

Claim 1. *If $i \neq h$, then $N_G(l_i)\cap V(M_h)\cap N_G(b_3) = \emptyset$.*

Proof. Suppose $v \in N_G(l_i) \cap V(M_h) \cap N_G(b_3)$. By Lemma 6, we may assume that either $3|i$ or $3|h$. Now if $i \equiv 0 \not\equiv h(\text{mod } 3)$, then $T' := T - \{b_3u_3, b_3u_6, b_hu_h\} + \{vl_i, vb_3, u_3u_6\}$ has fewer branch vertices than T, violating (T1). On the other hand, if $h \equiv 0 \not\equiv i(\text{mod } 3)$, then $T' := T - \{b_3u_3, b_3u_6, b_iu_i\} + \{vb_3, vl_i, u_3u_6\}$ has fewer branch vertices than T, violating (T1). Otherwise, $h \equiv i \equiv 0(\text{mod } 3)$, so since $v \neq l_h$, there exists v^-. Since $G[v, v^-, b_3, l_i]$ is not a claw and $b_3l_i \notin E(G)$, it follows that either $v^-l_i \in E(G)$ or $v^-b_3 \in E(G)$. If $v^-l_i \in E(G)$, then $T' := T - \{vv^-, b_3u_h, b_3u_i\} + \{vb_3, u_hu_i, l_iv^-\}$ has fewer branch vertices than T, violating (T1). Otherwise, $v^-b_3 \in E(G)$, and since $G[b_3, b_3^+, u_h, v^-]$ is not a claw and $u_hb_3^+ \notin E(G)$, it follows that either $v^-u_h \in E(G)$ or $v^-b_3^+ \in E(G)$. If $v^-u_h \in E(G)$, then $T' := T - \{b_3u_h, vv^-\} + \{vl_i, v^-u_h\}$ has fewer branch vertices than T, violating (T1). Otherwise, $v^-b_3^+ \in E(G)$, so $T' := T - \{vv^-, b_3u_i\} + \{v^-b_3^+, vl_i\}$ either has fewer branch vertices than T (if $b_3^+ = x$) or else has the same number of branch vertices and leaves as T, but $|V(S_{T'})| < |V(S_T)|$, so either (T1) or (T4) is violated, so we've proven our claim. □

Claim 2. *If $i \in [6]$, then $N_G(l_i) \cap V(S_T) = \emptyset$.*

Proof. Suppose $v \in N_G(l_i) \cap V(S_T)$. By Lemma 4, $v \neq b_j$, so $T' := T - \{b_iu_i\} + \{vl_i\}$ may have fewer branch vertices than T, violating (T1), or the same number of branch vertices and leaves, violating (T4) since $|V(S_{T'})| < |V(S_T)|$. □

For any $h \in [6]$, Lemma 2 ensures that $(N_G(l_h) \cap V(M_h))^-$ is disjoint from $N_G(l_i) \cap V(M_h)$ for $i \neq h$. Lemma 3 ensures that $(N_G(l_h) \cap V(M_h))^-$ is disjoint from $N_G(b_3) \cap V(M_h)$ for $h \not\equiv 0(\text{mod } 3)$. Lemma 4 ensures that the latter are disjoint for $h \equiv 0(\text{mod } 3)$. Lemma 5 and Claim 1 ensure that the five sets $N_G(l_i) \cap V(M_h)$ with $i \neq h$ are disjoint from each other and from $N_G(b_3) \cap V(M_h)$ respectively. Therefore the seven sets $(N_G(l_h) \cap V(M_h))^-$, $N_G(b_3) \cap V(M_h)$, and $N_G(l_i) \cap V(M_h)$ for $i \neq h$ are all disjoint. Furthermore, if $3 \nmid h$, Lemmas 7 and 8 give that u_h is not in any of these sets. Therefore:

$$\sum_{v\in X}|N_G(v)\cap V(M_h)|$$

$$= |N_G(b_3)\cap V(M_h)|+\sum_{i=1}^{6}|N_G(l_i)\cap V(M_h)|$$

$$= |N_G(b_3)\cap V(M_h)|+|N_G(l_h)\cap V(M_h)|+\sum_{i\neq h}|N_G(l_i)\cap V(M_h)|$$

$$= |N_G(b_3)\cap V(M_h)|+|(N_G(l_h)\cap V(M_h))^-|+\sum_{i\neq h}|N_G(l_i)\cap V(M_h)|$$

$$\leq \begin{cases} |V(M_h)| & 3|h \\ |V(M_h)\setminus\{u_h\}| = |V(M_h)|-1 & 3\nmid h. \end{cases}$$

Meanwhile, Claim 2 gives that b_3 is the only vertex of X that can be adjacent to any vertex of S_T. Therefore

$$\sum_{v\in X}|N_G(v)\cap V(S_T)| = |N_G(b_3)\cap V(S_T)| \leq |V(S_T)\setminus\{b_3\}| = |V(S_T)|-1$$

Summing these inequalities gives $\sum_{v\in X}\deg_G(v)\leq n-5$, contradicting the assumption of the theorem. □

By Theorems 5, 6, and 7, the T we've chosen must have four branch vertices but cannot have any of the possible structures on four branch vertices, and therefore cannot exist. This is a contradiction, so Theorem 1 is proven. Thus Conjecture 1 holds when $k = 2$.

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