On the curling number of the Mycielskian of certain graphs

Sudev Naduvath

Centre for Studies in Discrete Mathematics
Vidya Academy of Science and Technology
Thalakkottukara, Thrissur, India.
sudevnk@gmail.com

Abstract

Let $S = S_1S_2S_3\ldots S_n$ be a finite string which can be written in the form $X_1^{k_1}X_2^{k_2}\ldots X_r^{k_r}$, where $X_i^{k_i}$ is the k_i copies of a non-empty string X_i and each k_i is a non-negative integer. Then, the curling number of the string S, denoted by $cn(S)$, is defined to be $cn(S) = \max\{k_i : 1 \le i \le r\}$. Analogous to this concept, the degree sequence of the graph G can be written as a string $X_1^{(k_1)} \circ X_2^{(k_2)} \circ X_3^{(k_3)} \ldots \circ X_r^{k_r}$. The compound curling number of G, denoted $cn^c(G)$ is defined to be, $cn^c(G) = \prod\limits_{i=1}^{r} k_i$. In this paper, the curling number and compound curling number of the powers of the Mycielskian of certain graphs are discussed.

Keywords: Number sequences, curling number of graphs, compound curling number of graphs.

Mathematics Subject Classification: 05C07, 05C76, 11B83.

1 Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [1, 2, 9, 11, 18]. For the terminology of curling number of number sequences and related relevant results, see [4, 5, 7, 17]. Unless mentioned otherwise, all graphs considered here are simple, finite, undirected and have no isolated vertices.

If r is a positive integer, the r-th power of G, denoted by G^r, is a graph with the same vertex set such that two vertices are adjacent in G^r if and only if the distance between them is at most r. Note that if r is the diameter of a graph G, then G^r is a complete graph (see [11]).

1.1 Mycielskian of a Graph

Consider a graph G with $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$. Then, the *Mycielski graph* or *Mycielskian* of G, denoted by $\mu(G)$ (see [13]) is the graph obtained by applying the following steps to the graph G.

(i) Take the set of new vertices $U = \{u_1, u_2, u_3, \ldots, u_n\}$ and add edges from each vertex u_i to the vertices v_j if the corresponding vertex v_i is adjacent to v_j in G,

(ii) Take another new vertex u and add edges to all elements in U.

For the ease of the notation in context of graph powers, we denote the Mycielski graph of a graph G by $\breve{G}$.

1.2 Curling Number of Graphs

Let $S = S_1S_2S_3 \ldots S_n$ be a finite string. If we partition the sequence S into two subsequences, say X, Y, we write S as the string $S = X \circ Y$. Let the sequence S can be written in the form $X_1^{(k_1)} \circ X_2^{(k_2)} \circ \ldots \circ X_r^{(k_r)}$, where $X_i^{k_i}$ is the k_i copies of a non-empty string X_i and each k_i is a non-negative integer. Then, the *curling number* of the string S, denoted by $cn(S)$, is defined to be $cn(S) = \max\{k_i : 1 \leq i \leq r\}$ (see [4]).

Given a finite non-empty graph G with the degree sequence $S = (d_1, d_2, \ldots, d_n)$, where $d_i \in \mathbb{N}$ for all $1 \leq i \leq r$. Analogous to the terminology mentioned above, the degree sequence of the graph G can be written in the form of a string as $X_1^{(k_1)} \circ X_2^{(k_2)} \circ X_3^{(k_3)} \ldots \circ X_r^{(k_r)}$. Then, the *curling number* of G, denoted by $cn(G)$, is defined to be $cn(G) = \max\{k_i : 1 \leq i \leq r\}$ (see [12]). The *compound curling number* of G, denoted $cn^c(G)$, is defined to be $cn^c(G) = \prod_{i=1}^{l} k_i$, where $1 \leq i \leq l$ (see [12]). A relevant result in this context is that the curling number and the compound curling number of a regular graph are the same and are equal to the order of that graph (see[12]).

The curling number of certain fundamental and newly introduced graph classes have been determined in [12]. Following this study, the curling number and compound curling number of graph join and different product graphs have been determined in [14]. Then, further studies on the curling number of various graphs associated with given graph classes has been done in [15] and the curling number and the compound curling number of the powers of certain graphs have been studies in [16].

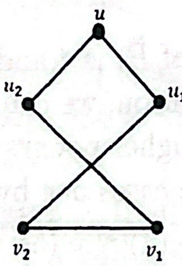
Motivated by these studies, the curling number and the compound curling number of the Mycielskian of certain fundamental graph classes are discussed in this paper.

2 Curling Number of Mycielski Graphs

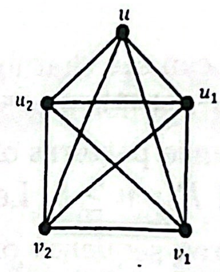
Throughout this paper, the vertex set of the given graph G is denoted by V. We denote by U, the set of newly introduces vertices corresponding to the vertices in V and w be the new vertex which is adjacent to all vertices in U. We also denote the degree of a vertex v in G by $d(v)$ and the degree of a vertex v in $\breve{G}$ by $d'(v)$.

The diameter of the Mycielskian of any graph is 4 and hence note that for any given graph G, the Mycielskian $\breve{G}^4$ is a complete graph. Hence, $\breve{G}^4$ is a $2n$-regular graph on $2n+1$ vertices. Therefore, $cn(\breve{G}^4) = cn^c(\breve{G}^4) = 2n+1$. Hence, we need to find out the curling numbers the Mycielskians and their powers up to 3.

Let us first discuss the two types of curling numbers of the powers of Mycielskians of Paths. Some initial cases need to be discussed separately as follows. Note that $\breve{P}_2 = C_5$, a regular graph on 5 vertices. Therefore, $cn(\breve{P}_2) = cn^c(\breve{P}_2) = 5$. Also, $\breve{P}_2^2$ is a complete graph and hence $cn(\breve{P}_2^2) = cn^c(\breve{P}_2^{)} = 5$ (See Figure 1).



(a) $\breve{P}_2$

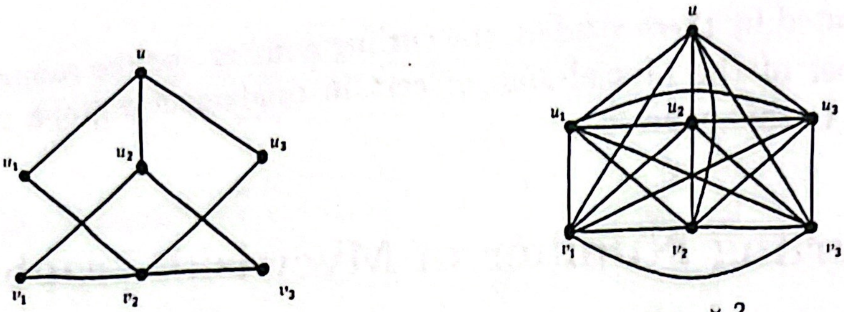


(b) $\breve{P}_2{}^2$

Figure 1

Note that the degree sequence of $\breve{P}_3$ is $(2)^{(4)} \circ (3)^{(2)} \circ (4)^{(1)}$. Therefore, the curling number of $\breve{P}_3$ is 4 and the compound curling number is 8. Also, note that the diameter of $\breve{P}_n$ is 3. Then, $\breve{P}_3^2$ is a complete graph and hence the curling number and compound curling number of $\breve{P}_3^2$ is 7 (See Figure 2).

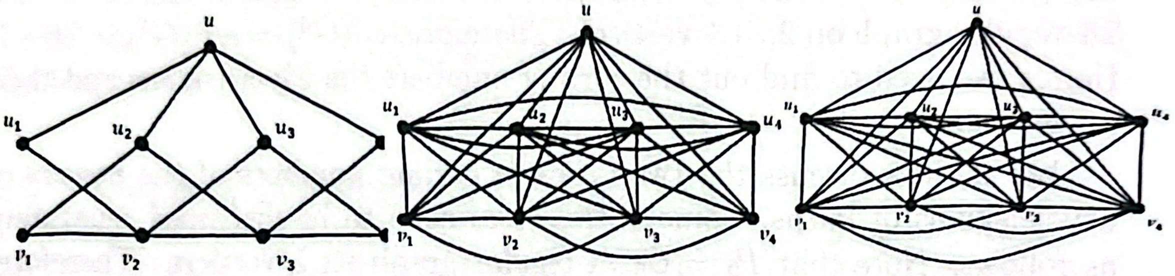
For $n = 4$, the degree sequence of $\breve{P}_4$ is $(2)^{(4)} \circ (3)^{(2)} \circ (4)^{(3)}$. Therefore, the curling number of $\breve{P}_3$ is 4 and the compound curling number is 24. The degree sequence of $\breve{P}_4^2$ can be written as $(8)^{(5)} \circ (7)^{(2)} \circ (6)^{(2)}$. Therefore, the curling number of $\breve{P}_4^2$ is 5 and its compound curling number is 20. The



(a) $\check{P}_3$ (b) $\check{P}_3{}^2$

Figure 2

cube of $\check{P}_4$ is a complete graph on 9 vertices and hence we have $cn(\check{P}_4^3) = cn^c(\check{P}_4^3) = 9$. For an illustration, see Figure 3.



(a) $\check{P}_4$ (b) $\check{P}_4{}^2$ (c) $\check{P}_4{}^3$.

Figure 3

Now, we can see that for $n \geq 5$, the diameter of $\check{P}_n$ is 4 and hence $\check{P}_n^4$ is a complete graph for $n \geq 5$. But on close examination, we can see that the degree sequence patterns of $\check{P}_n; n \leq 7$ and their higher powers are different from that of $\check{P}_n;\ n \geq 8$. Let us now discuss these cases one by one.

The degree sequence of $\check{P}_5$ is $(2)^{(4)} \circ (3)^{(3)} \circ (4)^{(3)} \circ (5)^{(1)}$ and hence $cn(\check{P}_5) = 4$ and $cn^c(\check{P}_5) = 36$. The degree sequence of $\check{P}_5^2$ is $(10)^{(5)} \circ (8)^{(4)} \circ (6)^{(2)}$ and hence $cn(\check{P}_5^2) = 5$ and $cn^c(\check{P}_5^2) = 40$. The degree sequence of $\check{P}_5^3$ is $(10)^{(9)} \circ (9)^{(2)}$ and hence $cn(\check{P}_5^3) = 9$ and $cn^c(\check{P}_5^3) = 18$.

Similarly, the degree sequence of $\check{P}_6$ is $(2)^{(4)} \circ (3)^{(4)} \circ (4)^{(4)} \circ (6)^{(1)}$ and hence $cn(\check{P}_6) = 4$ and $cn^c(\check{P}_6) = 64$. The degree sequence of $\check{P}_6^2$ is $(12)^{(1)} \circ (11)^{(2)} \circ (10)^{(4)} \circ (9)^{(2)} \circ (8)^{(2)} \circ (6)^{(2)}$ and hence $cn(\check{P}_6^2) = 4$ and $cn^c(\check{P}_6^2) = 64$. The degree sequence of $\check{P}_6^3$ is $(12)^{(9)} \circ (11)^{(2)} \circ (10)^{(2)}$ and hence $cn(\check{P}_6^3) = 9$ and $cn^c(\check{P}_6^3) = 36$.

Similarly, the degree sequence of $\check{P}_7$ is $(2)^{(4)} \circ (3)^{(5)} \circ (4)^{(5)} \circ (7)^{(1)}$ and hence $cn(\check{P}_7) = 5$ and $cn^c(\check{P}_6) = 100$. The degree sequence of $\check{P}_7^2$ is $(14)^{(1)} \circ (12)^{(2)} \circ (11)^{(2)} \circ (10)^{(5)} \circ (8)^{(2)} \circ (6)^{(2)}$ and hence $cn(\check{P}_7^2) = 5$ and $cn^c(\check{P}_7^2) = 120$. The degree sequence of $\check{P}_7^3$ is $(14)^{(9)} \circ (13)^{(2)} \circ (12)^{(2)} \circ (11)^{(2)}$

and hence $cn(\check{P}_7^3) = 9$ and $cn^c(\check{P}_7^3) = 64$.

We also note that precise closed formulae can be determined for the two types of curling numbers for different powers of the Mycielskians of paths of order at lest 8. Hence, we have the following theorem.

Theorem 2.1. *Let P_n be a path on $n \geq 8$ vertices and $\check{P}_n$ be its Mycielskian. Then,*

(i) $cn(\check{P}_n) = n - 2$ *and* $cn^c(\check{P}_n) = 4(n-2)^2$*;*
(ii) $cn(\check{P}_n^2) = n - 4$ *and* $cn^c(\check{P}_n^2) = 16(n-4)^2$*;*
(iii) $cn(\check{P}_n^3) = n + 1$ *and* $cn^c(\check{P}_n^3) = 8(n^2 - 5n - 6)$.

Proof. Part-(i) Note that for the vertices $v_i \in V$, we have $d'(v_i) = 2d(v_i)$ and for all vertices $u_i \in U$, we have $d'(u_i) = d(v_i) + 1$. Also, $d'(w) = n$. Therefore, the graph $\check{P}_n$ consists of $(n-2)$ vertices of degree 4, $(n-2)$ vertices of degree 3, 4 vertices of degree 2 and one vertex of degree n. Hence, the degree sequence of $\check{P}_n$ can be written as $(4)^{(n-2)} \circ (3)^{(n-2)} \circ (2)^{(4)} \circ (n)^{(1)}$. Therefore, $cn(\check{P}_n) = (n-2)$ and $cn^c(P_n) = 4(n-2)^2$.

Part-(ii): Here, we have to identify the degree sequence in $\check{P}_n^2$. The vertex w is adjacent to every vertex in U and V. That is, $d(w) = 2n$. Also, since between any two vertices u_i and u_j in U, there exists a path $u_i w u_j$ in $\check{P}_n$. Therefore, any two vertices in U are adjacent in $\check{P}_n^2$. Besides this, we note the following.

(a) The vertex u_1 is adjacent to the vertices v_1, v_2 and v_3 and the vertex u_n is adjacent to the vertices v_{n-2}, v_{n-1} and v_n. Therefore, $d(u_1) = d(u_n) = n + 3$.

(b) The vertex u_2 is adjacent to the vertices v_1, v_2, v_3 and v_4 and the vertex u_{n-1} is adjacent to the vertices $v_{n-3}, v_{n-2}, v_{n-1}$ and v_n. Therefore, $d(u_n) = d(u_{n-1}) = n + 4$.

(c) For $3 \leq i \leq n-2$, the vertex u_i is adjacent to the vertices $v_{i-2}, v_{i-1}, v_i, v_{i+1}$ and v_{i+2}. That is, $d(u_i) = n + 5$, for $3 \leq i \leq n-2$.

Next, we observe the adjacencies of the vertices in V as follows.

(d) The vertex v_1 is adjacent to the vertices v_2, v_3 in V and u_1, u_2, u_3 in U. Similarly, the vertex v_n is adjacent to the vertices v_{n-2}, v_{n-1} in V and u_{n-2}, u_{n-1}, u_n. Therefore, $d(v_1) = d(v_n) = 6$.

(e) The vertex v_2 is adjacent to the vertices v_1, v_3, v_4 in V and the vertices u_1, u_2, u_3, u_4 in U. Similarly, the vertex v_{n-1} is adjacent to the vertices v_{n-3}, v_{n-2}, v_n in V and the vertices in $u_{n-3}, u_{n-2}, u_{n-1}, u_n$ in U. Therefore, $d(u_n) = d(u_{n-1}) = 8$.

(f) For $3 \leq i \leq n-2$, the vertex $v_i \in V$ is adjacent to the vertices $v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}$ in V and $u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}$ in U. That is, $d(v_i) = 10$, for $3 \leq i \leq n-2$.

Therefore, the degree sequence in $\check{P}_n^2$ can be written as $(2n)^{(1)} \circ (n+5)^{(n-4)} \circ (n+4)^{(2)} \circ (n+3)^{(2)} \circ (10)^{(n-4)} \circ (8)^{(2)} \circ (6)^{(2)}$. Then, the curling number of $\check{P}_n^2$ is $(n-4)$ and the compound curling number is $16(n-4)^2$.

Part-(iii): In $\check{P}_n^3$, the vertex w is adjacent to all vertices in $U \cup V$. Also, any two vertices in U are also adjacent to each other. As the distance between a vertex in U and a vertex in V is at most 3, we have every vertex of U is adjacent to all vertices in V as well. Hence, for any vertex u in $U \cup \{w\}$, we have $d(u) = 2n$. Now, we have to find out the degree of vertices in V, which can be done as follows.

(a) The vertex v_1 is adjacent to all vertices in $U \cup \{w\}$ and to the vertices v_2, v_3 and v_4 and the vertex v_n is adjacent to all vertices in $U \cup \{w\}$ and the vertices v_{n-3}, v_{n-2} and v_{n-1}. Therefore, $d(v_1) = d(v_n) = n+4$.

(b) The vertex v_2 is adjacent to all vertices in $U \cup \{w\}$ and to the vertices v_1, v_3, v_4 and v_5. Similarly, the vertex v_n is adjacent to all vertices in $U \cup \{w\}$ and to the vertices $v_{n-4}, v_{n-3}, v_{n-2}$ and v_n. Therefore, $d(v_2) = d(v_{n-1}) = n+5$.

(c) The vertex v_3 is adjacent to all vertices in $U \cup \{w\}$ and to the vertices v_1, v_2, v_4, v_5 and v_6. Similarly, the vertex v_n is adjacent to all vertices in $U \cup \{w\}$ and to the vertices $v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}$ and v_n. Therefore, $d(v_3) = d(v_{n-2}) = n+6$.

(d) For $4 \leq i \leq n-3$, any vertex v_i is adjacent to all vertices in $U \cup \{w\}$ and to the vertices $v_{i-3}, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}$ and v_{i+3}. Hence, $d(v_i) = n+7$, for $4 \leq i \leq n-3$.

Hence, the degree sequence of $\check{P}_n^3$ can be written as $(2n)^{(n+1)} \circ (n+7)^{(n-6)} \circ (n+6)^{(2)} \circ (n+5)^{(2)} \circ (n+4)^{(2)}$. Hence, the curling number of $\check{P}_n^3$ is $n+1$ and its compound curling number is $8(n^2-5n-6)$. □

In the next theorem, we determine the curing number of the Mycielskian of cycles.

Theorem 2.2. *Let C_n be a cycle on n vertices and $\check{C}_n$ be its Mycielskian. Then,*

(i) $cn(\check{C}_n) = n$ and $cn^c(\check{C}_n) = n^2$;

(ii)

$$cn(\check{C}_n^2) = \begin{cases} 2n+1 & \textit{if } n = 3,4,5 \\ n & \textit{if } n \geq 6 \end{cases}$$

and

$$cn^c(\check{C}_n^2) = \begin{cases} 2n+1 & \textit{if } n = 3,4,5 \\ n^2; & \textit{if } n \geq 6. \end{cases}$$

(iii)

$$cn(\check{C}_n^3) = \begin{cases} 2n+1 & \textit{if } 3 \leq n \leq 7 \\ n+1 & \textit{if } n \geq 8. \end{cases}$$

and

$$cn^c(\check{C}_n^3) = \begin{cases} 2n+1 & \textit{if } 3 \leq n \leq 7 \\ n^2+n & \textit{if } n \geq 8. \end{cases}$$

Proof. Part-(i) As mentioned in the previous theorem, for the vertices $v_i \in V$, we have $d'(v_i) = 2d(v_i)$ and for all vertices $u_i \in U$, we have $d'(u_i) = d(v_i) + 1$. Also, $d'(w) = n$. Therefore, the graph $\check{C}_n$ consists of n vertices of degree 4, n vertices of degree 3, and one vertex of degree n. That is, the degree sequence of $\check{C}_n$ is $(4)^{(n)} \circ (3)^{(n)} \circ (n)^{(1)}$. Therefore, $cn(\check{C}_n) = n$ and $cn^c(C_n) = n^2$.

Part-(ii): For $n = 3,4,5$, we can see that C_n^2 is a complete graph. Hence, in this case, $cn(\check{C}_n^2) = cn^c(C_n^2) = 2n+1$. Then, let $n \geq 6$. In $\check{C}_n^2$, the vertex w is adjacent to every vertex in $U \cup V$. That is, $d(w) = 2n$. Also, since the distance between any two vertices u_i and u_j in U in $\check{P}_n$ is 2, every pair of vertices in U are adjacent in $\check{P}_n^2$. More over, each vertex $u_i \in U$ is adjacent to the vertices $v_{i-2}, v_{i-1}, v_i, v_{i+1}$ and v_{i+2}. That is, $d(u_i) = n+5$, where $1 \leq i \leq n$. Also, any vertex $v_i \in V$ is adjacent to the vertices $v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}$ in V and $u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}$ in U. That is, $d(v_i) = 10$, where $1 \leq i \leq n$. Therefore, the degree sequence of $\check{C}_n^2$ can be written as $(2n)^{(1)} \circ (n+5)^{(n)} \circ (10)^{(n)}$. Therefore, $cn(\check{C}_n^2) = n$ and $cn^c(C_n^2) = n^2$.

Part-(iii): For $n = 6,7$, we can see that C_n^3 is a complete graph. Hence, in this case, $cn(\check{C}_n^3) = cn^c(C_n^3) = 2n+1$. Now, let $n \geq 8$. In $\check{C}_n^3$, vertex w is adjacent to all vertices in $U \cup V$. Since the distance between any two vertices in U is 2 and the distances between a vertex in U, we note that all vertices in $U \cup \{w\}$ are adjacent to each other. Moreover, every vertex in V is adjacent to all vertices of $U \cup \{w\}$. Hence, for any vertex u in $U \cup \{w\}$, we have $d(w) = 2n$. In addition to the vertices in $U \cup \{w\}$, a vertex $v_i \in V$ is adjacent to the vertices $v_{i-3}, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}$ and

v_{i+3}. Hence, $d(v_i) = n+7$ for all $v_i \in V$. Therefore, the degree sequence of $\check{C}_n^3$ is given by $(2n)^{(n+1)} \circ (n+7)^{(n)}$. Therefore, $cn(\check{C}_n^2) = n+1$ and $cn^c(C_n^2) = n^2+n$. □

A *fan graph* F_{n+1} is the graph obtained by drawing edges from all vertices of a path P_n to an external vertex (see [2]). That is, $F_{n+1} = P_n + K - 1$. Then, we have the following result.

Proposition 2.3. *For a fan graph F_{n+1}, we have*

(i) $cn(\check{F}_{n+1}) = n$ and $cn^c(\check{F}_{n+1}) = 4(n^2 - 2n)$;

(ii) $cn(\check{F}_{n+1}^2) = cn^c(\check{F}_{n+1}^2) = 2n+3$.

Proof. Part-(i): Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n and v be the central vertex in W_{n+1} and let $u_1, u_2, \ldots, u_n$ and u be the corresponding vertices in U and w be the vertex which is adjacent to all vertices in U. Then, we have $d'(w) = n+1$, $d'(u) = n+1$, $d'(v) = 2n$, while $d'(v_1) = d'(v_n) = 4$, $d'(u_1) = d'(u_n) = 3$, and for all $2 \le i \le n-1$, we have $d(v_i) = 6$ and $d(u_i) = 4$. That is, the degree sequence of $\check{F}_{n+1}^2$ is given by $(n+1)^{(2)} \circ (2n)^{(1)} \circ (6)^{(2)} \circ (4)^{(n)} \circ (3)^{(2)}$. Hence, we have $cn(\check{F}_{n+1}) = n$ and $cn^c(\check{F}_{n+1}) = 4(n^2 - 2n)$.

Part-(ii): Note that the diameter of the fan graph F_{n+1} is 2 and hence F_{n+1}^2 is a complete graph on $n+1$ vertices and $\check{F}_{n+1}^2$ is a complete graph on $2n+3$ vertices. Hence, $cn(\check{F}_{n+1}^2) = cn^c(\check{F}_{n+1}^2) = 2n+3$. □

A wheel graph, denoted by W_{n+1} is the graph obtained by adding edges from all vertices of a cycle C_n to en external vertex (see [2]). That is, $W_{n+1} = C_n + K_1$. The following proposition discusses the two curling numbers of a wheel graph.

Proposition 2.4. *For a wheel graph W_{n+1}, we have*

(i) $cn(\check{W}_{n+1}) = n$ and $cn^c(\check{W}_{n+1}) = 2n^2$;

(ii) $cn(\check{W}_{n+1}^2) = cn^c(\check{W}_{n+1}^2) = 2n+3$.

Proof. Part-(i): Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n and v be the central vertex in W_{n+1} and let $u_1, u_2, \ldots, u_n$ and u be the corresponding vertices in U and w be the vertex which is adjacent to all vertices in U. Then, $d'(w) = n+1$, $d'(u) = n+1$ and $d'(v) = 2n$, while $d'(v_i) = 6$ and $d'(u_i) = 4$. Therefore, the degree sequence of $\check{W}_{n+1}$ is given by $(2n)^{(1)} \circ (n+1)^{(2)} \circ (6)^{(n)} \circ 4^{(n)}$. Therefore, $cn(\check{W}_{n+1}) = n$ and $cn^c(\check{W}_{n+1}) = 2n^2$.

Part-(ii): We know that the diameter of the wheel graph W_{n+1} is 2 and hence $\breve{W}^2_{n+1}$ is a complete graph on $2n+3$ vertices. Hence, $cn(\breve{W}^2_{n+1}) = cn^c(\breve{W}^2_{n+1}) = 2n+3$. □

The curling number and the compound curling number of the Mycielskians of complete graphs can be found out as explained in the following theorem.

Theorem 2.5. *For a complete graph K_n, we have*

(i) $cn(\breve{K}_n) = n+1$ *and* $cn^c(\breve{K}_n) = n^2+n$;

(ii) $cn(\breve{K}^2_n) = cn^c(\breve{K}^2_n) = 2n+1$.

Proof. Part-(i): Let $v_1, v_2, \ldots, v_n$ be the vertices of the complete graph K_n and let $u_1, u_2, \ldots, u_n$ be the corresponding newly introduced vertices in U in $\breve{K}_n$ and w be the vertex which is adjacent to all vertices in U in $\breve{K}_n$. Then, $d'(w) = n$, $d'(u_i) = n$ and $d'(v_i) = 2(n-1)$. Therefore, the degree sequence of $\breve{K}_n$ is given by $(n)^{(n+1)} \circ (2n-2)^{(n)}$. Therefore, $cn(\breve{K}_n) = n+1$ and $cn^c(\breve{K}_n) = n^2+n$.

Part-(ii): We know that the diameter of the graph $\breve{K}_n$ is 2 and hence $\breve{K}^2_n$ is a complete graph on $2n+1$ vertices. Hence, $cn(\breve{K}^2_n) = cn^c(\breve{K}^2_n) = 2n+1$. □

The two curling numbers of the Mycielski graphs of given complete bipartite graphs can be found out as follows.

Theorem 2.6. *For a complete graph $K_{m,n}$, we have*

(i) $cn(\breve{K}_{m,n}) = \max\{m,n\}$ *and* $cn^c(\breve{K}_{m,n}) = m^2n^2$;

(ii) $cn(\breve{K}^2_{m,n}) = cn^c(\breve{K}^2_{m,n}) = 2m+2n+1$.

Proof. Let (X,Y) be the bipartition of the complete bipartite graph $K_{m,n}$ such that $|X| = m$ and $|Y| = n$. Also, let X' and Y' be the sets of newly introduced vertices in $\breve{K}_{m,n}$ corresponding to the vertices in X and Y respectively in $K_{m,n}$. Also let w be the new vertex in $\breve{K}_{m,n}$ which is adjacent to all vertices in $X' \cup Y'$. Then, the vertices in X are adjacent to the vertices in $Y \cup Y'$ and the vertices in Y are adjacent to the vertices in $X \cup X'$. Therefore, every vertex in X has degree $2n$ and every vertex in Y has degree $2m$.

Now, note that the vertices in X' is adjacent to the vertices in $Y \cup \{w\}$ and the vertices in Y' is adjacent to the vertices in $X \cup \{w\}$. Therefore, every vertex in X' has degree $n+1$ and every vertex in Y' has degree $m+1$. Also, note that the degree of the vertex w is $m+n$.

Therefore, the degree sequence of $\breve{K}_{m,n}$ is given by $(2m)^{(n)} \circ (2n)^{(m)} \circ (m+1)^{(n)} \circ (n+1)^{(m)} \circ (m+n)^{(1)}$. Hence, the curling number of $\breve{K}_{m,n}$ is $\max\{m,n\}$ and its compound curling number is m^2n^2.

Part-(ii): We know that the diameter of the graph $\breve{K}_{m,n}$ is 2 and hence $\breve{K}^2_{m,n}$ is a complete graph on $2(m+n)+1$ vertices. Hence, $cn(\breve{K}^2_{m,n}) = cn^c(\breve{K}^2_{m,n}) = 2(m+n)+1$. □

An *n-sun* or a *trampoline* is a chordal graph G on $2n$ vertices, where $n \geq 3$, whose vertex set can be partitioned into two sets $V_1 = \{v_1, v_2, v_3, \ldots, v_n\}$ and $V_2 = \{x_1, x_2, x_3, \ldots, x_n\}$ such that V_2 is an independent set of G and x_j is adjacent to v_i if and only if $j = i$ or $j = i+1 \pmod{n}$. A *complete sun*, denoted by S_n, is a sun G where the induced subgraph $\langle V_1 \rangle$ is complete (see [2]). The following theorem discusses the curling numbers of the Mycielski graphs of the complete sun graphs S_n.

Theorem 2.7. *For a complete sun graph S_n, we have*

(i) $cn(\breve{S}_n) = n$ and $cn^c(\breve{S}_n) = n^4$;

(ii) $cn(\breve{S}^2_n) = 2n+1$ and $cn^c(\breve{S}^2_n) = n^2(2n+1)$;

(iii) $cn(\breve{S}^3_n) = cn^c(\breve{S}^3_n) = 4n+1$.

Proof. Part-(i): Clearly, $\breve{S}_n$ is a graph on $4n+1$ vertices. Let $V = V_1 \cup V_2 = \{v_1, v_2, v_3, \ldots, v_n, x_1, x_2, x_3, \ldots, x_n\}$ be the vertex set of S_n and $U = U_1 \cup U_2$ be the set of vertices in $\breve{S}_n$ corresponding to the vertices in V, where the vertex set $U_1 = \{u_1, u_2, u_3, \ldots, u_n\}$ corresponds to V_1 and the set $U_2 = \{y_1, y_2, y_3, \ldots, y_n\}$ corresponds to V_2 and w be the vertex which is adjacent to every vertex in U in $\breve{S}_n$. Then, for $1 \leq i \leq n$, we have $d'(v_i) = 2(n+1), d'(x_i) = 4, d'(u_i) = n+1, d'(y_i) = 3$ and $d'(w) = 2n$. That is, the degree sequence of $\breve{S}_n$ is given by $(2n+1)^{(n)} \circ (n+1)^{(n)} \circ (4)^{(n)} \circ (3)^{(n)} \circ (2n)^{(1)}$. Hence, $cn(\breve{S}_n) = n$ and $cn^c(\breve{S}_n) = n^4$.

Part-(ii): Here, we determine the pattern in $\breve{S}^2_n$ as follows. Note that the vertex w will be adjacent to all other $4n$ vertices $\breve{S}^2_n$. That is, $d'(w) = 4n$.

Now, every vertex in U is adjacent to each other and every vertex in V_1 will be adjacent to all vertices in V. Therefore, for $1 \leq i \leq n$, $d(u_i) = 4n$. Now, any vertex $y_i \in U_2$ is adjacent to w and all other vertices in U and all vertices in V_1. Moreover, y_i is adjacent to the vertices x_{i-1}, x_i and x_{i+1} in V_2. Therefore, for $1 \leq i \leq n$, we have $d'(y_i) = 2n+3$.

It can be noted that all vertices in V_1 is adjacent to all vertices in $U \cup \{w\}$. Moreover, each vertex $v_i \in V_1$ is adjacent to all vertices in V_2. Therefore, for $1 \leq i \leq n$, each vertex v_i is adjacent to all other vertices in $\breve{S}^2_n$ and hence we have $d'(v_i) = 4n$. Now, each vertex $x_i \in V_2$ is adjacent to all vertices in V_1 and all vertices in U_1, the vertices y_{i-1}, y_i, and y_{i+1} in U_2 and the vertices x_{i-1} and x_{i+1} in V_2. That is, $d'(x_i) = 2n+5$, for all $1 \leq i \leq n$.

Hence, the degree sequence of $\breve{S}_n^2$ can be written as $(4n)^{(2n+1)} \circ (2n+5)^{(n)} \circ (2n+3)^{(n)}$. Therefore, $cn(\breve{S}_n^2) = 2n+1$ and $cn^c(\breve{S}_n^2) = n^2(2n+1)$.

Part-(iii): We know that the diameter of the graph $\breve{S}_n$ is 3 and hence $\breve{S}_n^3$ is a complete graph on $4n+1$ vertices. Hence, $cn(\breve{S}_n^3) = cn^c(\breve{S}_n^3) = 4n+1$. □

3 Conclusion

In this paper, the two types of curling numbers of the Mycielskians of certain fundamental graph classes have been determined. More problems in this area are still open. There are several graph classes for which the curling numbers are still to be investigated. Problems on the curling numbers of certain graphs that are derived from given or known graph classes, including the Mycielskians, are yet to be settled. All these facts highlight a wide scope for further investigations in this area.

Acknowledgement

The author would like to dedicate this work to his research supervisor Prof. (Dr.) K. A. Germina as a tribute and honour for the two decades of her outstanding research career in the field of graph theory.

References

[1] J. A. Bondy, U. S. R. Murty, **Graph theory with applications**, Macmillan Press, London, 1976.

[2] A. Brandstädt, V. B. Le and J. P. Spinrad, **Graph classes: A survey**, SIAM, Philadelphia, 1999.

[3] S. R. Canoy. Jr., M. A. Labendia, *Polynomial representation and degree sequence of a graph*, Int. J. Math. Anal., 8(29)(2014), 1445-1455., DOI:10.12988/ijma.2014.46154.

[4] B. Chaffin, J. P. Linderman, N. J. A. Sloane, A. R. Wilks, *On curling numbers of integer sequences*, J. Integer Seq., 16(2013), Article-13.4.3, 1-31.

[5] B. Chaffin, N. J. A. Sloane, *The curling number conjecture*, Preprint, http://arxiv.org/abs/0912.2382.

[6] G. Chartrand, L. Lesniak, **Graphs and digraphs**, CRC Press, 2000.

[7] N. J. Fine, and H. S. Wilf, *Uniqueness theorems of periodic sequences*, Proc. Amer. Math. Soc., **16** (1965), 109 - 114.

[8] R. Frucht, F. Harary, (1970). *On the corona of two graphs*, Aequationes Math., 4(3), 322-325.

[9] J. T. Gross, J. Yellen, **Graph theory and its applications**, CRC Press, 2006.

[10] R. Hammack, W. Imrich and S. Klavzar, **Handbook of product graphs**, CRC Press, 2011.

[11] F. Harary, **Graph theory**, New Age International, Delhi., 2001.

[12] J. Kok, N. K. Sudev, K. P. Chithra, *On curling number of certain graphs*, Southeast Asian Bull. Math., 2016, to appear.

[13] W. Lin, J. Wu, P. C. B. Lam, G. Gu, *Several parameters of generalized Mycielskians*, Discrete Appl. Math., **154**(8)(2006), 1173-1182, DOI:10.1016/j.dam.2005.11.001.

[14] N. K. Sudev, C. Susanth, K. P. Chithra, S. J. Kalayathankal and J. Kok, *Some new results on the curling number of graphs*, J. Combin. Math. Combin. Comput., to appear.

[15] C. Susanth, S. J. Kalayathankal, N. K. Sudev, K. P. Chithra and J. Kok, *A study on the curling number of graph classes*, preprint, arXiv:1512.01096.

[16] C. Susanth, S. J. Kalayathankal, N. K. Sudev, K. P. Chithra and J. Kok, *Curling numbers of certain graph powers*, preprint, arXiv:1509.00220.

[17] N. J. A. Sloane, *Eight hateful sequences*, pre-print, arXiv:0805.2128v1 [math.GM]

[18] D. B. West, **Introduction to graph theory**, Pearson Education Inc., Delhi, 2001.