Roman, Italian, and 2-Domination

William F. Klostermeyer
School of Computing
University of North Florida
Jacksonville, FL 32224-2669
wkloster@unf.edu

Gary MacGillivray*
Department of Mathematics and Statistics
University of Victoria, P.O. Box 3060 STN CSC
Victoria, BC, CANADA V8W 3R4
gmacgill@math.uvic.ca

Abstract

Motivated by finding a way to connect the Roman domination number and 2-domination number, which are in general not comparable, we consider a parameter called the Italian domination number (also known as the Roman $\{2\}$−domination number). This parameter is bounded above by each of the other two. Bounds on the Italian domination number in terms of the order of the graph are shown. The value of the Italian domination number is studied for several classes of graphs. We also compare the Italian domination number with the 2-domination number.

1 Context and framework

The general topic of this paper is domination in finite, simple graphs, a thorough survey on which can be found in [18]. We begin by stating the area-specific definitions needed to put our work in perspective.

A subset $D \subseteq V$ is a *dominating set* if $|N[x] \cap D| \geq 1$ for each $x \in V$, and is a *double dominating set* if $|N[x] \cap D| \geq 2$ for each $x \in V$. The notation $N[x]$ denotes the *closed neighborhood of* x: it is the set $N(x) \cup \{x\}$, where $N(x) = \{v : xv \in E\}$ is the *open neighborhood of* x. Observe that a graph which has isolated vertices cannot have a double dominating set. The minimum

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cardinality amongst all dominating sets of G is the *domination number*, $\gamma(G)$. The minimum cardinality amongst all double dominating sets of G is the *double domination number*, $\gamma_{\times 2}(G)$. Double domination is a special case of k-tuple domination, which was introduced in [17] (also see [16]). A subset $D \subseteq V$ is a *total dominating set* if $|N(x) \cap D| \geq 1$ for each $x \in V$

An equivalent definition of a dominating set is a set $D \subseteq V$ such that each $x \in V - D$ is adjacent to a vertex in D. A *2-dominating set* is a set $D \subseteq V$ such that each $x \in V - D$, is adjacent to two vertices in D. Observe that 2-dominating sets are defined for all graphs. The minimum cardinality amongst all 2-dominating sets of G is the *2-domination number*, $\gamma_2(G)$. 2-domination is a special case of k-domination, which was introduced in [11] (also see [8, 13, 14]).

A *Roman dominating function* is a function $f : V \to \{0, 1, 2\}$ such that every vertex x for which $f(x) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{x \in V} f(x)$. The minimum weight of a Roman dominating function on a graph G is called the *Roman domination number*, denoted $\gamma_R(G)$. One may view Roman domination as graph labeling problem in which each vertex labeled 0 must be adjacent to at least one vertex labeled 2. Roman dominating sets in graphs were first studied in [27], and have been studied in a number of subsequent papers, for example [8]. Several variations of Roman domination have been considered in the literature, for example [22, 24, 26].

The main focus of this paper is a parameter which is a variation of both 2-domination and Roman domination: an *Italian dominating function (IDF)* is a function $f : V \to \{0, 1, 2\}$ such that for each vertex x such that $f(x) = 0$, $\sum_{v \in N(x)} f(v) \geq 2$. The weight of an Italian dominating function is the value $f(V) = \sum_{x \in V} f(x)$. The minimum weight of an Italian dominating function on the graph G is called the *Italian domination number of* G, and denoted as $\gamma_I(G)$. This same concept was recently studied in [7], where it was called Roman $\{2\}$−domination and what we call IDFs are called Roman $\{2\}$−dominating functions in [7]. Italian domination in trees was considered in [21]. The main results in [7] involve comparisons between γ_I and the total domination number, weak Roman domination number, and 2-rainbow domination number.

A generalization of double domination can be obtained by imposing the condition $\sum_{v \in N(x)} f(v) \geq 2$ on all vertices x. We call the minimum weight of such function the *generalized Italian domination number*, and denote it by $\gamma_{2 \times I}$.

Let G be a graph with no isolated vertices. Then, from comparing the definitions above, we have:

$$\gamma(G) \leq \gamma_I(G) \leq \gamma_2(G) \leq \gamma_{\times 2}(G)$$

and

$$\gamma(G) \leq \gamma_I(G) \leq \gamma_R(G) \leq 2\gamma(G).$$

To illustrate, for $m \geq 3$, the complete bipartite graph $K_{3,m}$ has $\gamma = 2, \gamma_I = 3, \gamma_R = \gamma_{\times 2} = 4$.

After introducing some more terminology in Section 2, in Section 3, we bound the Italian domination number in terms of the order of the graph in the cases where the minimum degree δ is 1, 2, or 3. Quartic graphs are also considered. The proof of the first bound in Section 3 suggests considering cographs. We do so in Section 4, and then contrast the results by considering general diameter 2 graphs. In Section 5, we characterize some classes of trees for which $\gamma_I = 2\gamma$. Finally, in Section 6, we explore Italian and 2-domination, focusing on a class a graphs where equality between the two parameters holds.

2 Terminology

A graph is a *Roman graph* if $\gamma_R(G) = 2\gamma(G)$, see [8]. Analogously, a graph is called an *Italian graph* if $\gamma_I(G) = 2\gamma(G)$. Stars and double-stars (i.e., two disjoint stars, each with at least three vertices, with the center vertices of each joined by an edge) are examples of Italian graphs. Obviously, if G is Italian, then it is Roman. The converse is false, for example P_5 is a Roman graph which is not Italian.

A graph is called an *I1 graph* if the range of every minimum weight Italian dominating function is the set $\{0, 1\}$. A graph is called an *I2 graph* if the range of every minimum weight Italian dominating function is the set $\{0, 2\}$. Both C_4 and P_5 are I1 graphs. Notice that no I1 graph can be Roman. Observe that a graph cannot be both Italian and I1. For a graph G to be Italian, it satisfies $\gamma_I(G) = 2\gamma(G)$; a minimum-weight Italian dominating function for G can be formed by assigning $f(v) = 2$ for each vertex v in a minimum dominating set. K_2 is Italian, but not I2: let $f(v) = 1$ for both of the vertices. A graph is called an *I1a graph* if range of some minimum weight Italian dominating function is the set $\{0, 1\}$. The path on 6 vertices is an example of an I1a graph that is not I1.

The following proposition is immediate from the definitions.

Proposition 1 *For all G, $\gamma_I(G) = \gamma_2(G)$ if and only if G is I1a.*

For $m \geq 3$, the star $K_{1,m}$ is an I2 graph, and also an Italian graph. Clearly every I2 graph is Roman. However, the converse is false. The graph $K_{4,4}$ is

Roman graph, but is not I2. By the same token, if G is I2, it is necessarily Italian. Both P_3 and C_3 are examples of Italian graphs that are not I2.

3 Upper bounds

3.1 Bounds in terms of order

It is well-known that the domination number of any connected graph with n vertices is at most $n/2$, and that the connected graphs which achieve equality are the corona of a connected graph G with respect to K_1, for example, see [18]. If D is a dominating set of G and X is the set of degree one vertices, then $D \cup X$ is a 2-dominating set of the corona of G and K_1. Hence the 2-domination number and Italian domination number of such graphs are both at most $\frac{3n}{4}$. The graphs $K_{1,n-1}$ show that $n-1$ is the best possible general upper bound on the 2-domination number of a connected graph with $n \geq 3$ vertices.

Theorem 2 *For all connected graphs G with $n \geq 3$ vertices, $\gamma_I(G) \leq \frac{3n}{4}$.*

Proof. It suffices to prove the statement for an arbitrary spanning tree T of G. The proof is by induction on the number of vertices in T. It is easy to verify the statement if T has either three or four vertices. Choose an edge e of T and consider the trees T_1 and T_2 that are formed when e is deleted. If both T_1 and T_2 have at least three vertices, then the result follows from induction. So we must only consider the case when there is no such edge e. This implies the diameter of T is at most four and the following are the possibilities. If the diameter of T is two, then T is a star and thus $\gamma_I(G) \leq \frac{3n}{4}$. If the diameter of T is three, then T is a double-star and thus $\gamma_I(G) \leq \frac{3n}{4}$ (with P_4 being the extremal case).

If the diameter of T is four, then if T is P_5, in which case $\gamma_I(G) = 3$. Otherwise, T has a vertex of degree at least three and therefore has at least six vertices. Then either there is an edge e such that when e is deleted both T_1 and T_2 have at least three vertices (in which case we are done), or else every edge e is such that $T - e$ has a component with at most two vertices. In this case, let v be a vertex in T of minimum eccentricity. First suppose the degree of v is at least three. Let $f(v) = 2$, $f(u) = 1$ for each leaf u of T that is not adjacent to v, and $f(w) = 0$ for each other vertex. This shows that $\gamma_I(T) \leq \frac{3n}{4}$. Finally, suppose the degree of v is two. Let the neighbors of v be x, y. If x and y both have degree at least three, then set $f(x) = 2$, $f(y) = 2$ and let all other vertices be labeled with 0. This is an IDF satisfying $\gamma_I(T) \leq \frac{3n}{4}$. Otherwise, since T is not a P_5, one of x and y, say x, has degree at least three, and thus y has degree two. Let z be the other vertex adjacent to y. Since T is a tree, the degree of z is one. Then let $f(x) = 2, f(v) = 1$, and $f(z) = 1$ and label all other vertices with 0. This is an IDF satisfying $\gamma_I(T) \leq \frac{3n}{4}$. □

As suggested in the discussion above, there are infinitely many graphs with $\gamma_I(G) = \frac{3n}{4}$. Suppose H is the corona of a connected graph G with respect to K_1, and let G' be the corona of H with respect to K_1. There is a 2-dominating set of G' consisting of a dominating set of H and the degree one vertices. This 2-dominating set has $3n/4$ vertices. We claim that $\gamma_I(G') = \frac{3n}{4}$, from which is follows that the 2-dominating set just mentioned is minimum. Observe that G' consists of $n/4$ induced P_4's, each of which contains exactly one vertex of G. By the structure of G', any Italian dominating function must have weight at least three on each of these P_4's.

Hansberg and Volkmann [15] used the probabilistic method to show that the 2-domination number of any connected graph G satisfies

$$\gamma_2(G) \leq \frac{n(1 + \ln(\delta + 1))}{\delta + 1}.$$

Similar bounds hold for Roman domination [8, 19] and double domination [16]. Hence as the minimum degree increases, γ_I is bounded above by smaller and smaller fractions of n.

Favaron, Hansberg and Volkmann [10] showed that if G has minimum degree $\delta \geq 2$, then $\gamma_2 \leq \frac{\delta n}{2\delta - 1}$. When $\delta = 2$ this bound agrees with the one in [8] and is better than bound above arising from the probabilistic method. The latter bound is better for all larger values of δ.

We next consider Italian domination in graphs of minimum degree $\delta \geq 2$. It is known in this case that $\gamma_R(G) \leq 8n/11$, see [6]. The family of graphs achieving that bound is described in [6]. It is easy to see that each graph in this family of graphs is I1 and satisfies $\gamma_I(G) = 6n/11$. The theorem of Favaron, Hansberg and Volkmann cited above gives $\gamma_2 \leq \frac{2n}{3}$ for graphs with $\delta \geq 2$. It follows immediately that $\gamma_I \leq \frac{2n}{3}$ for such graphs. We now describe infinitely many examples where this bound is sharp. Let k be a positive integer. Begin with a path $v_1, v_2, \ldots, v_k$. For each vertex v_i add vertices u_i, w_i and edges v_iu_i, v_iw_i, u_iw_i. The resulting graph has $3k$ vertices and $\gamma_I(G) = 2k$.

We shall give a proof of the upper bound for the Italian domination number of graphs with $\delta \geq 2$ which is simpler than the proof of the corresponding results for 2-domination. For any dominating set $D \subseteq V$ $x \in D$, we say that $v \in V - D$ is an *external private neighbor* of x if v is adjacent to x but to no other vertex in D.

Theorem 3 *Let G be a graph with $n \geq 3$ vertices and $\delta(G) \geq 2$. Then $\gamma_I(G) \leq 2n/3$.*

Proof. Let D be a minimum dominating set of G. Then we may assume that $|D| > n/3$, otherwise simply label each vertex in D with a 2. In particular, choose D such that each vertex in D has at least one external private neighbor, which is possible as shown in [2]. Note that $|D| \leq n/2$. Let $B = V - D$. We construct an Italian dominating function f such that there is a partition of V into sets such that each set S in this partition satisfies $f(S) \leq 2|S|/3$.

If $v \in D$ has two or more external private neighbors, let $f(v) = 2$, otherwise let $f(v) = 1$. Now consider a vertex $u \in B$ such that $\sum_{x \in N(u)} f(x) < 2$. Such a u exists since $|D| > n/3$; note that for such a vertex u, since D is a dominating set, $\sum_{x \in N(u)} f(x) = 1$. Let $w \in B$ be a neighbor of u and denote u's neighbor in D as x. We examine two cases.

Case 1. Suppose w is not an external private neighbor of any $v \in D$. Then let $f(w) = 1$. Taken together u, w, x are three vertices with total weight two.

Case 2. Suppose w is an external private neighbor of a vertex $v \in D$. If v has three or more external private neighbors with weight 0, then assign $f(w) = 1$. In this case w still has at least two external private neighbors with weight 0 and we can assign w to the part with u, x as in Case 1.

On the other hand, suppose v has two external private neighbors with weight 0. Let z be the external private neighbor of v other than w. Then change $f(v) = 0$ and let $f(w) = f(z) = 1$. Then u, u, w, v, z are five vertices with total weight three. □

For graphs of minimum degree 3, both the theorem of Favaron, Hansberg and Volkmann [10] and the theorem of Hansberg and Volkmann [19] give $\gamma_2 \leq 0.6n$. We show that this bound can be improved.

Theorem 4 *Let G be a graph with n vertices and minimum degree $\delta \geq 3$. Then $\gamma_2(G) \leq n/2$.*

Proof. It is known that every graph has a spanning bipartite subgraph containing at least half the edges incident with each vertex, see Exercise 1.5.8 from Bondy and Murty [3].

Suppose G has minimum degree $\delta \geq 3$ and let the bipartite subgraph as above have bipartition (A, B). Each vertex in A has at least 2 neighbors in B, and vice-versa. Thus each of A and B is a 2-dominating set. Since one of them has size at most $n/2$, the result follows. □

Corollary 5 *Let G be a graph with n vertices and minimum degree $\delta \geq 3$. Then $\gamma_I(G) \leq n/2$.*

An infinite family of cubic graphs with $\gamma_I = \gamma_2 = n/2$ can be constructed as follows. Let $X = K_4 - e$ with degree two vertices x and y. Let $k \geq 2$, and $X_1, X_2, \ldots, X_k$ be disjoint copies of X. Let s_i and t_i be the degree 2 vertices in X_i. For $i = 1, 2, \ldots, k$, join t_i to s_{i+1}, where addition is modulo k. Let G be a graph arising from this construction. An IDF of G must have weight at least 2 in each copy of X: either both degree three vertices have a 1, or one of them has a zero and has a weight of 2 in its neighborhood. Hence $\gamma_I(G) = \gamma_2(G) = n/2$.

3.2 Bounds for regular graphs

In this subsection, we present some further order-related upper bounds, focusing on cubic and quartic graphs. As we shall see, the results will primarily be for 2-domination, with conjectures posed for Italian domination. The Petersen graph has 2-domination number 4 and thus is a cubic graph with $\gamma_2(G) < n/2$. We next describe an infinite family of triangle-free cubic graphs where equality holds in Theorem 4.

The *circulant graph* $C_n(x_1, x_2, x_3)$ is the graph with n vertices $v_0, v_1, \ldots, v_{n-1}$ with v_i adjacent to each vertex $v_{i \pm x_j}, j \in \{1, 2, 3\}$, where addition is done mod n.

Proposition 6 *Let G be the circulant graph $C_{8k}(1, 4k, 8k-1)$. Then $\gamma_2(G) = 4k$.*

Proof. Since G has a Hamilton cycle C, it is clear that $\gamma_2(G) \leq 4k$. We say two vertices are *opposites* if they are not adjacent on C but are adjacent in G. Let D be a minimum-weight 2-dominating set of G. To see that $\gamma_2(G) = 4k$, we shall partition C into $2k$ parts, each consisting of four consecutive vertices. Number the parts around C as $S_0, S_1, \ldots, S_{2k-1}$. Index the vertices within any part by $0, 1, 2, 3$ in clockwise order.

Two vertices in S_i are called *ends* if they are adjacent and one of them has degree one in the subgraph induced by S_i. Let S_0 be a part with vertices v_0, v_1, v_2, v_3. It is easy to see that at least one vertex of S_0 must be in D, else v_2 is not 2-dominated. We claim that either (i) two vertices of S_0 must be contained in D or (ii) if exactly one vertex of S_0 is in D, then there are three vertices from D in S_k. So suppose exactly one vertex of S_0 is in D, w.l.o.g., let us say it is v_2 (it must be either v_1 or v_2). Then $v_0, v_1 \notin D$ and in order for v_0, v_1 to be 2-dominated, their opposites must be in D. Observe that these opposites, u_0, u_1, are contained in S_k and are ends in S_k. If S_k contains three vertices in D, we are done. Otherwise, suppose $|S_k \cap D| = 2$. Then it must be that the other two ends in S_k (i.e., u_2, u_3) are not in D and it follows that their opposites must be in D. Since $2k$ is even, the opposite of u_3 is v_3 and since $v_3 \notin D$, D cannot be a 2-dominating set. □

Proposition 7 *Let G be the circulant graph $C_{4k}(1, 2k, 4k-1)$ with k odd. Then $\gamma_2(G) = 2k$.*

Proof. Assume $k > 1$, else G is K_4 and the result is trivial. Let D be a minimum weight 2-dominating set of G and suppose it has fewer than $2k$ vertices. Partition the vertices into groups of four as above; label these parts $S_0, S_1, \ldots, S_{k-1}$ around the cycle. Construct auxiliary graph H with vertices corresponding to the parts of G with two vertices in H adjacent if any vertices in the corresponding part of G are opposites of one another.

As in the proof of Proposition 6, if a part $S_0 = \{v_0, v_1, v_2, v_3\}$ contains exactly one vertex from D it must be one of v_1, v_2. Suppose, w.l.o.g., that $v_2 \in D$, which implies the opposites of v_0, v_1 are in D. Note that for part S_0, the opposites of v_0, v_1 lie in one part (namely, $S_{\lfloor \frac{k}{2} \rfloor}$), and the opposites of v_2, v_3 lie in a different part (namely, $S_{\lceil \frac{k}{2} \rceil}$). Since k is odd, we have that $H = C_k$. Specifically, H contains the cycle $S_0, S_{\lfloor \frac{k}{2} \rfloor}, S_{k-1}, S_{\lfloor \frac{k}{2} \rfloor - 1}, \ldots, S_{\lceil \frac{k}{2} \rceil}, S_0$.

If part $S_{\lfloor \frac{k}{2} \rfloor}$ does not contain at least three vertices in D (again, we are trying to show an average of two vertices per part), then two adjacent vertices from $S_{\lfloor \frac{k}{2} \rfloor}$ are in D and the other two ends from this part must have their opposites in D. Continuing this chain of reasoning, we are essentially following a path from part to part, searching for a part with three vertices. Since $H = C_k$, if we never find a part with three vertices from D, we eventually wind up back at S_0 (which contains exactly one vertex from D) and we have a contradiction.

Suppose we find a part X with at least three vertices from D along this path of parts in H. Maybe there is some other part Y with one vertex from D that we have yet to visit while walking along this Hamilton cycle of parts that started at S_0. Then either the opposites of the two vertices from Y that are not in D lie in X, in which case X must have four vertices from D, so the average of at least two vertices per part applies, or the opposites are further along our Hamilton cycle of parts (i.e., further from S_0 walking in the direction we are walking). In which case we keep going along the path, eventually winding up

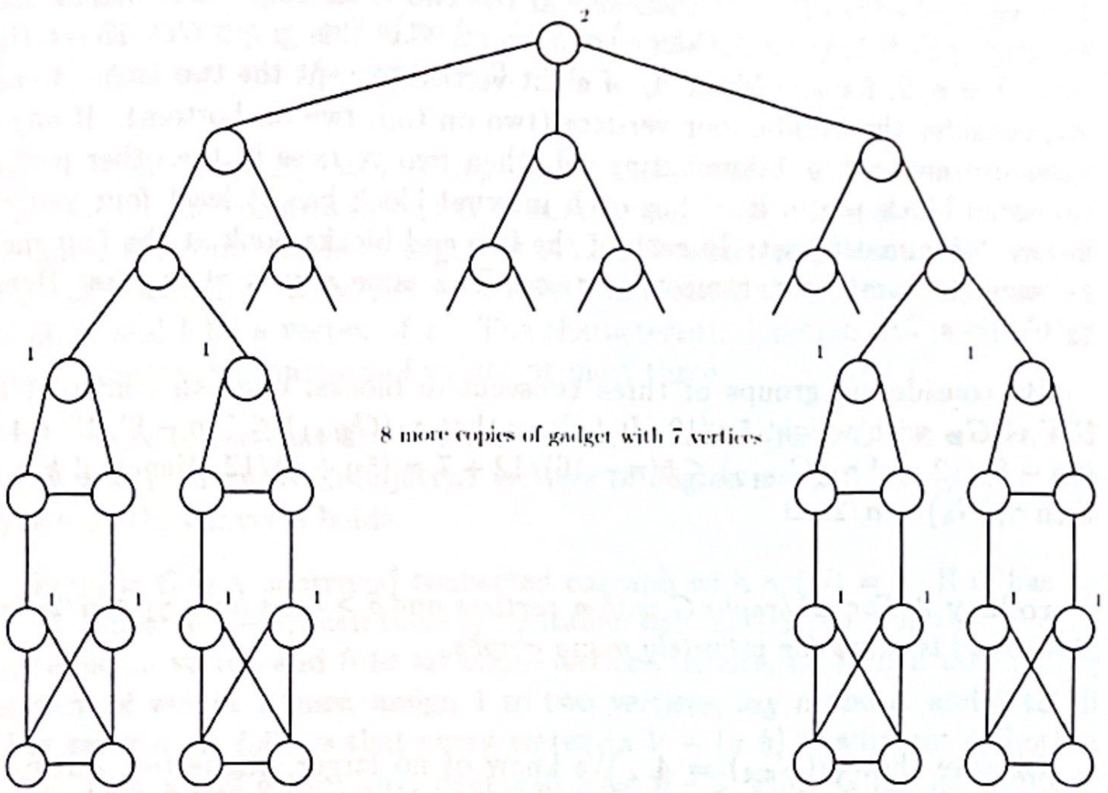

Figure 1: Cubic Graph.

at S_0, a contradiction, or finding a part with three vertices. □

We make the following conjecture.

Conjecture 1 *$\gamma_I(G) = n/2$ for $G = C_{4k}(1, 2k, 4k-1)$ and for $G = C_{8k}(1, 4k, 8k-1)$.*

A cubic graph with $\gamma_I(G) < \gamma_2(G)$ is shown in Figure 1. Note that this graph has odd-length cycles.

We next consider 4-regular graphs.

Proposition 8 *There exist infinitely many bipartite 4-regular graphs G with n vertices, $\gamma_2(G) = n/2$, and $\gamma_I(G) < \gamma_2(G)$.*

Proof. Take $k > 1$ copies of $K_{4,4}$. Call them $A_1, A_2, \ldots, A_k$. Imagine these as k blocks of eight vertices ordered left to right, with edges going from top to bottom. For $i = 1, 2, \ldots, k-1$, delete the last vertical edge of A_i and the first vertical edge of A_{i+1}, then add in the two cross edges that makes these to subgraphs belong to the same component. Call this graph G_k. To see that $\gamma_2(G_k) = n/2$, for any block A_i of eight vertices except the two ends, A_1 and A_k, consider the middle four vertices (two on top, two on bottom). If any of these are not in the 2-dominating set, then two vertices in the other part of the same block are in it. Thus each internal block has at least four vertices in any 2-dominating set. In each of the two end blocks, look at the four most extreme (leftmost or rightmost) vertices. The same argument applies. Hence $\gamma_2(G_k) = n/2$.

By considering groups of three consecutive blocks, one can construct an IDF of G_{3t} with weight $5n/12$. It follows that $\gamma_I(G_{3t+1}) \leq 5(n-8)/12 + 4 = (5n+8)/12$ and $\gamma_I(G_{3t+2}) \leq 5(n-16)/12 + 7 = (5n+4)/12$. Hence, if $k > 1$, then $\gamma_I(G_k) < n/2$. □

Corollary 9 *For all graphs G with n vertices and $\delta \geq 4$ we have $\gamma_2 \leq n/2$, and this bound is sharp for infinitely many graphs.*

Observe that $\gamma_I(K_{4,4}) = 4$. We know of no larger, connected, 4-regular graphs with $\gamma_I = n/2$ and suspect there are none.

Proposition 8 generalizes. Instead of each block being $K_{4,4}$, use $K_{t,t}$. The resulting graph is t-regular with $n = 2kt$ and $\gamma_2 = 4k = 2n/t$.

It was shown in [7] that if G is a connected graph, then $\gamma_I(G) \geq 2n/(\Delta+2)$. Some of the regular graphs shown above demonstrate that γ_I can be much larger than $2n/(\Delta+2)$.

4 Diameter two graphs

The proof of Theorem 2 suggests that paths of length four can cause the Italian domination number to be large. A *cograph* is a graph with no induced P_4, see [4]. Equivalently, cographs are precisely the graphs that can be constructed from K_1 using the operations of disjoint union and complementation. The last operation in the construction of a non-trivial connected cograph must be complementation, hence any connected cograph has diameter at most two and domination number at most two. The Roman domination number of a connected cograph is known to be one of the integers 2, 3, 4, and the situations in which each value occurs are characterized in [25]. We prove an analogous result for Italian domination.

Theorem 10 *Let G be a non-trivial connected cograph with n vertices. Then $\gamma_I(G) \leq 3$. Further, $\gamma_I = 2$ if and only if G has a vertex of degree $n-1$, or two non-adjacent vertices of degree $n-2$.*

Proof. Clearly $\gamma_I(K_2) = 2$. Suppose $n \geq 3$. Since G is connected, the last operation in the construction of G is complementation. Suppose G is formed by taking the complement of cographs A and B, which need not be connected. Without loss of generality, A has at least two vertices. Let D be a dominating set of A, and b be a vertex of B. The characteristic function of $D \cup \{b\}$ is an Italian dominating function of weight at most three.

It is easy to see that if the nontrivial connected cograph G has a vertex of degree $n-1$, or two non-adjacent vertices of degree $n-2$, then $\gamma_I = 2$. We show that the converse holds.

Suppose G is a nontrivial connected cograph with $\gamma_I(G) = 2$. If G has no vertex of degree $n-1$, then there is no Italian dominating function that assigns 2 to a single vertex and 0 to all other vertices. Hence, an Italian dominating function of weight 2 must assign 1 to two vertices, say a and b, and 0 to all other vertices. It follows that every vertex in $V - \{a,b\}$ is adjacent to both a and b. Thus a and b each have degree at least $n-2$. Since G has no vertex of degree $n-1$, the vertices a and b are non-adjacent and therefore have degree exactly $n-2$. □

The complement of $P_3 \cup P_3$ is an example of a cograph with $\gamma_I = 3$ and $\gamma_R = 4$.

The Italian domination number of a diameter 2 graph can be greater than 3. For example, the unique planar graph of diameter two and domination number three, see [12, 23], can be seen to have $\gamma_I(G) = 5$ and $\gamma_R(G) = 6$ (to see the former, put 1's on the corner vertices and middle vertex of Figure 1 of [12] and 0's on all other vertices). The Cartesian product of K_t and K_t has diameter 2 and domination number t. Suppose we draw this graph so that each of t K_t's are drawn as horizontal rows of t vertices, one beneath the other (so the t^2 vertices are laid out in a grid). If there is a "row" with no vertex in the dominating set, then there must be an element of the dominating set in each "column". Thus the Italian domination number is at least t. The set of vertices on the "diagonal" is a 2-dominating set, so $\gamma_I = t$. One can therefore observe that the Roman domination number of this graph is $2t$.

It is shown in [19] that $\gamma(G) \leq \lfloor \frac{n}{4} \rfloor + 1$ for any diameter two graph. The only examples they demonstrate achieving this bound have $n \in \{8, 10\}$. Each of these examples with $n = 8$, see Figure 1 [19], has $\gamma(G) = 3$ and $\gamma_I(G) \leq \gamma_2 \leq 4$.

The examples with $n = 10$ are 4-regular graphs. We prove that the examples with $n = 10$ are not Italian.

Proposition 11 *Let G be a 4-regular, diameter two graph with $n = 10$ vertices. Then $\gamma(G) \leq 3$ and $\gamma_I(G) \leq 5$.*

Proof. The bound on $\gamma(G)$ is obvious. Let v be a vertex and let $A = V(G) - N[v]$. Note that $|A| = 5$. If each vertex in A has at least two neighbor in $N(v)$, then $\gamma_I(G) \leq 4$ (label each vertex in $N(v)$ with 1 and all other vertices with 0). Otherwise, let $w \in A$ have only one neighbor in A (it has at least one, since the diameter of G is two). Then w has three neighbors in A. Let $u \in A$ be the vertex in A that is not adjacent to w and let $x \in N(v)$ be adjacent to w.

Suppose u has at least two neighbors in $N(v)$. Then the following is an IDF of weight five: label w with 2, x and v with 0, the other three vertex in $N(v)$ with 1. Otherwise, suppose u has one neighbor in $N(v)$. If that neighbor is x, then an IDF of weight five is: label u, w with 1, x and v with 0, the other three vertex in $N(v)$ with 1. If that neighbor is not x, then an IDF of weight five is: label u, w, v with 1, x and u's neighbor in A with 0, the other two vertex in $N(v)$ with 1. □

We claim that there are only finitely many diameter two graphs with $\gamma(G) = \lfloor \frac{n}{4} \rfloor + 1$. For any diameter two graph, $\gamma(G) \leq \delta(G)$ as the open neighborhood of any vertex is a dominating set. It is also well-known that when $\delta(G) > 1$, $\gamma(G) \leq n[1 + \ln(\delta(G) + 1)]/(\delta(G) + 1)$, see [1]. Let $\delta(G) = n/k$ for some integer $k \geq 1$. Then for any k, once n is sufficiently large, the second upper bound gives a better bound on the domination number than $\lfloor \frac{n}{4} \rfloor + 1$. Thus for any fixed k, there are only finitely many graphs G for which $\gamma(G) < n/k$.

The bound on the domination number cited above can be used to obtain a bound on the domination number of diameter 2 graphs which is better than $\lfloor \frac{n}{4} \rfloor + 1$ once n is large enough.

Proposition 12 *If G has diameter two, then $\gamma(G) \leq \sqrt{n}\ln(n)$.*

Proof. If the minimum degree $\delta \leq \sqrt{n}\ln(n)$, then the neighbors of a vertex of minimum degree form a dominating set of the required size. If $\delta \leq 10$, then $\lfloor \frac{n}{4} \rfloor + 1 \leq \sqrt{n}\ln(n)$, and the result follows from the bound of Hellwig and Volkmann [19]. Hence, assume $\delta \geq \sqrt{n}\ln(n) \geq 10$.

Then,

$$\gamma \leq n(1 + \ln(\sqrt{n}\ln(n) + 1))/(\sqrt{n}\ln(n) + 1)$$

$$
\begin{aligned}
&\le n(1+\ln(\sqrt{n}\ln(n)))/(\sqrt{n}\ln(n))\\
&\le \sqrt{n}(1+\ln(\sqrt{n}\ln(n)))/\ln(n)\\
&= \sqrt{n}(1+\ln(\sqrt{n})+\ln(\ln(n))/ln(n)\\
&\le \sqrt{n}(1+\ln(\sqrt{n})+\ln(\sqrt{n})/ln(n)\\
&= \sqrt{n}(1+\ln(n))/\ln(n)\\
&\le \sqrt{n}\ln(n),
\end{aligned}
$$

where the last inequality uses $n \geq \delta + 1 \geq 11$. □

5 Trees

5.1 Tree Fundamentals

A *stem* in a tree is a vertex of degree at least two that is adjacent to at least one leaf. A *strong stem* is adjacent to at least two leaves. A *weak stem* is adjacent to exactly one leaf. The result stated in Proposition 13 also appears in [7], but we include a self-contained proof to aid in what follows.

Proposition 13 $\gamma_I(P_n) = \lceil \frac{n+1}{2} \rceil$.

Proof. We first show $\gamma_I(P_n) \leq \lceil \frac{n+1}{2} \rceil$. For n odd, let $f(v) = 1$ for every other vertex, including the leaves. For n even, let $f(v) = 1$ for every other vertex, starting with the leftmost leaf, except that $f(v) = 2$ for the rightmost stem.[1]

That $\gamma_I(P_n) \geq \lceil \frac{n+1}{2} \rceil$ follows by a simple induction on n, given the base cases for $n \leq 4$, which are easy to verify by inspection. □

Corollary 14 *P_n with n odd and $n \neq 3$ is an I1 graph and these are the only paths that are I1.*

Proposition 15 *P_2, P_3, P_6 are the only Italian paths and no paths are I2.*

Proof. If P_n were Italian, it would be that $\lceil \frac{n+1}{2} \rceil = \lceil \frac{2n}{3} \rceil$, which is not possible when $n > 6$. That no path is I2 follows from Proposition 13. □

[1]This is not necessarily the only minimum-weight Italian dominating function for even length paths. P_6 has a minimum-weight Italian domination function with $f(v) = 2$ for each stem. Likewise, all paths can be seen to be I1a.

$$
\begin{aligned}
&\leq n(1+\ln(\sqrt{n}\ln(n)))/(\sqrt{n}\ln(n)) \\
&\leq \sqrt{n}(1+\ln(\sqrt{n}\ln(n)))/\ln(n) \\
&= \sqrt{n}(1+\ln(\sqrt{n})+\ln(\ln(n))/ln(n) \\
&\leq \sqrt{n}(1+\ln(\sqrt{n})+\ln(\sqrt{n})/ln(n) \\
&= \sqrt{n}(1+\ln(n))/\ln(n) \\
&\leq \sqrt{n}\ln(n),
\end{aligned}
$$

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[1]This is not necessarily the only minimum-weight Italian dominating function for even length paths. P_6 has a minimum-weight Italian domination function with $f(v) = 2$ for each stem. Likewise, all paths can be seen to be I1a.

Therefore, the only Italian paths are P_2, P_3, P_6.

Theorem 16 *Let T be a tree with $n > 1$ vertices. Then $\gamma(T) < \gamma_I(T)$.*

Proof. The proof is by induction on n. It is easy to verify the statement if T has fewer than three vertices. Let $n \geq 3$. Let u be a leaf of T having maximum eccentricity and v the stem adjacent to u. If v is a strong stem, then v is in every minimum dominating set of T and there exists a minimum-weight Italian dominating function of T with $f(v) = 2$. The result follows by induction in this case. Otherwise, v is a weak stem and $T - \{u, v\}$ is a tree with $\gamma(T - \{u, v\}) < \gamma_I(T - \{u, v\})$. Since there is a minimum-weight dominating set of T containing v and a minimum-weight Italian dominating function of T with $f(v) = 1$, the result follows by induction. □

5.2 Italian Trees

Since $\gamma_I(T) \leq \gamma_R(T)$ for all trees T, the Italian trees are a subset of the Roman trees. We hoped to modify the construction of [20] to yield a construction that generates all Italian trees. However, any such construction seems to require conditions more complex than those in [20]. For example, let $T_1 = K_{1,3}$. Apply construction $\mathcal{T}_2$ from [20] with tree P_3 three times, once to each leaf of T_1. The domination number of the resulting tree is four, but the Italian domination number is seven, even though each tree generated during this construction is Italian until the last one. Hence in this section, we focus our attention to some special classes of trees.

A *spider* S is a tree formed by taking k paths and attaching them together at a common vertex. In other words, a spider is a subdivision of a star: k paths with $l_1, l_2, \ldots, l_k$ vertices, called the *legs* of the spider which are joined at a common vertex v. When we refer to the number of vertices of a leg, we do not include v in that count. If $k = 2$, S is a path. For spiders that are not paths, the unique vertex of degree greater than two is called the *central* vertex.

Theorem 17 *Let S be a spider with $k \geq 3$ legs. Then S is Italian if and only if either (i) each l_i is equal to 1 (i.e., S is a star) or (ii) S is one of the two types of spiders shown in Figure 2.*

Proof. It is easy to verify that each of the spiders of type (i) and (ii) are Italian.

For the other direction, let v be the unique vertex of S of degree k. Suppose to the contrary that S is Italian and not of type (i) or (ii). We distinguish two cases.

Case 1. Suppose that no leg has more than two vertices. If only one leg has two vertices, then S is not Italian as $\gamma(S) = 2$ and $\gamma_I(S) = 3$. So suppose at least two legs have two vertices. If no leg has one vertex, then let $f(x) = 1$ for each leaf of S and $f(v) = 1$. This shows S is not Italian as $\gamma(S) = k$ and $\gamma_I(S) = k + 1$. Now suppose at least one leg has one vertex. Then let $f(x) = 1$ for each leaf of S that is on a leg with two vertices (say there are q of these) and let $f(v) = 2$. The $\gamma(S) = q + 1$ and $\gamma_I(S) = q + 2$.

Case 2. Suppose that some leg has at least three vertices. We consider two subcases.

Case 2.1 Suppose v is a stem (so some leg has one vertex). First suppose that S has a leg with two vertices. Then by letting $f(v) = 2$, $f(x) = 1$, where x is the leaf on the leg with two vertices and then assigning weights per Proposition 13 for all other paths in the subgraph induced by $V(S) - \{N[v] \cup x\}$, we produce an IDF of weight less than $2\gamma(S)$. This is true because $\gamma(S) \geq 2 + \sum \gamma(P_i)$, over all P_i, where the P_i are the paths in the subgraph induced by $V(S) - \{N[v] \cup x\}$, and the weight of this IDF is at most $3 + 2\gamma(P_i)$.

Thus we may suppose that S has no legs with two vertices. If any leg has six vertices, then let $f(v) = 2$, let $f(x) = 1$ for three vertices x on the leg with six vertices (the leaf and vertex distance two and four from the leaf) and then assigning weights per Proposition 13 for all other non-trivial paths in the subgraph induced by $V(S) - N[v]$, we get an IDF of weight less than $2\gamma(S)$. This is true because $\gamma(S) \geq 3 + \sum \gamma(P_i)$, over all P_i, where the P_i are the paths in the subgraph induced by $V(S) - \{N[v] \cup x\}$, and the weight of this IDF it most $5 + 2\gamma(P_i)$.

If any leg has either five, or more than six vertices, then assign weights to vertices on that leg so that it is not an Italian weighting, assign $f(v) = 2$, and then assign weights per Proposition 13 for all other non-trivial paths in the subgraph induced by $V(S) - N[v]$. In this manner we get an IDF of weight less than $2\gamma(S)$.

To complete this case, observe that if one leg has one vertex and all $q > 1$ remaining legs have three vertices, then S is not Italian, as $\gamma(S) = q + 1$ and $\gamma_I(S) = 2q + 1$.

Case 2.2 Suppose v is not a stem. Partition S into $k - 1$ paths, one of which

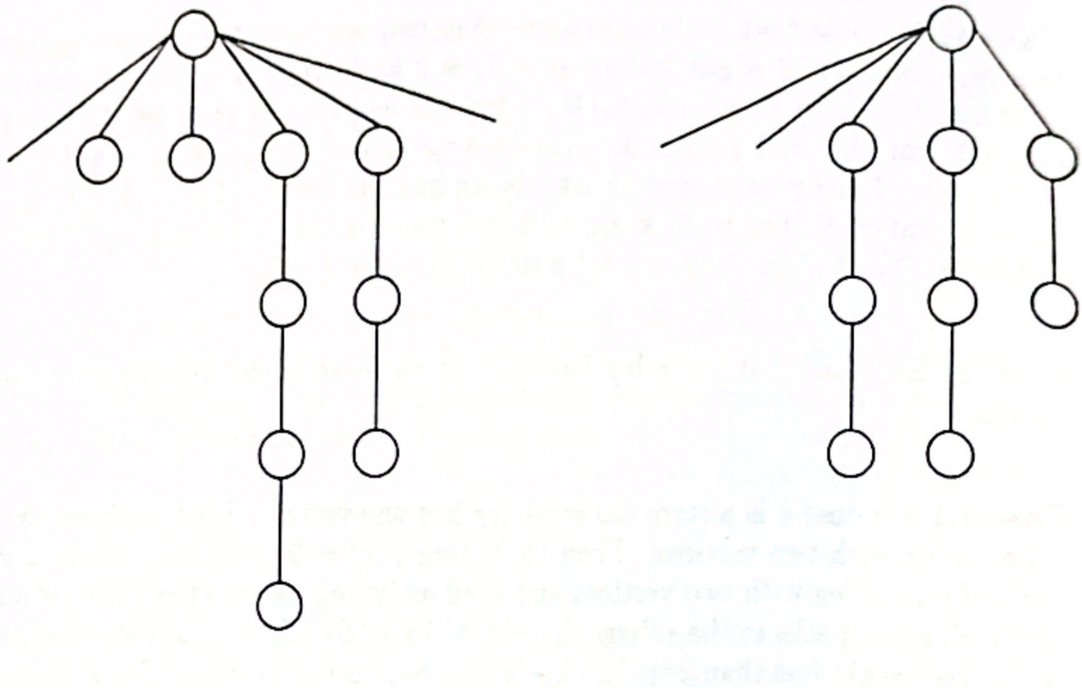

Figure 2: Italian spiders. Type (a) has $t \geq 1$ leaves and $q \geq 1$ legs with 3 or 4 vertices each (and at least one has 4). Type (b) has $q \geq 2$ legs with 3 vertices each and one leg with two vertices.

combines v and two legs into a single path. If each of these $k-1$ paths is such that it has $2\gamma = \gamma_I$ (i.e., the induced subgraph of each such path is Italian), it seems hypothetically possible that S is Italian; but we show that, in fact, S is not Italian. Since v it not a stem, the path containing v must be a P_6 in order to be Italian. If one of the $k-1$ paths has $2\gamma > \gamma_I$, then clearly S is not Italian. If one of the other paths is a P_2, then that path can be merged with the P_6 into a spider S' containing eight vertices with $\gamma(S') = 3$ and $\gamma_I(S') = 5$ and it follows that S is not Italian. If one of the other paths is a P_6, then that can be merged with the P_6 containing v into a spider S' with $\gamma(S') = 4$ and $\gamma_I(S') = 7$ and it follows that S is not Italian. Thus, all the other paths must be P_3's and S is of type (ii.b). Hence the proof. □

A *super-caterpillar* is a tree in which all vertices of degree greater than two lie on a path known as the *spine*. A *caterpillar* is a super-caterpillar in which each vertex is at most distance one from some vertex on the spine. Stated another way, a super-caterpillar is a subdivision of a caterpillar. Refer to a vertex of degree greater than two in a super-caterpillar as a *branch* vertex. A *spider partition* of super-caterpillar C is a partition of $V(C)$ into parts $S_1, \ldots, S_k$ such that each S_i induces a spider. A spider partition of C is said to be minimum if the sum of the Italian domination numbers of the spiders is minimum over all spider partitions of C. Observe that for any minimum spider partition of C, the central vertices of every spider lie on the spine of C, though not every vertex on the spine of C is the central vertex of a spider in the partition. It is easy to see that every super-caterpillar has a spider partition.

Theorem 18 *Let C be a super-caterpillar with $n \geq 3$ vertices. Then C is Italian if and only if every minimum spider partition of C consists of Italian spiders.*

Proof. First suppose that C is Italian. We use induction on the length of the spine, s. If s=1, then C is a spider, so assume $s > 1$. We also assume that C has more than one branch vertex, otherwise it is a spider. Let v be a branch vertex of maximum eccentricity. Then v is the central vertex in a spider in every minimum spider partition of C. Let S be such a partition and S_1 the spider containing v (we abuse notation and use S_1 to refer to both this set of vertices and the subgraph induced by these vertices). Let C' be the super-caterpillar induced by $V(C) - S_1$. We claim that S_1 is Italian. Suppose to the contrary that S_1 is not Italian and so $2\gamma(S_1) > \gamma_I(S_1)$. We know that $2\gamma(C') \geq \gamma_I(C')$ and that $\gamma_I(C) \leq \gamma_I(C') + \gamma_I(S_1)$. If $\gamma(C) = \gamma(C') + \gamma(S_1)$, we are done (since this would imply C is not Italian), so suppose this is not the case. That is, suppose $\gamma(C) = \gamma(C') + \gamma(S_1) - 1$. This implies there is a leaf w in the spider S_1 (w is not a leaf in C, i.e., it is the unique neighbor of v on the spine of C) that is adjacent to a vertex $x \in V(C')$ such that x is in a minimum dominating set D' of C' and w is in a minimum dominating set of S_1, but they do not appear together in any minimum dominating set of C. Since C is Italian, label each vertex of D', including x, with a 2 and this is a minimum weight IDF of C'. Then w can effectively be "moved" from S_1 to C' without changing the Italian domination number of C'. In so doing, $\gamma(S_1 - w) = \gamma(S_1) - 1$ and $\gamma_I(S_1 - w) = \gamma_I(S_1) - 1$, thereby forming a spider partition of weight less than S, a contradiction. Therefore S_1 is Italian and it follows by induction that each spider from S that lies in C' is Italian.

Now suppose that every minimum spider partition of C consists of Italian spiders. We claim that C is Italian. To the contrary, suppose every minimum spider partition of C consists of Italian spiders and that C is not Italian. Let C be a minimum counterexample; obviously, C has at least two branch vertices. As above, let S_1 be a spider (from a minimum spider partition S) centered at a branch vertex v of maximum eccentricity. By assumption, S_1 is Italian. Let C' be the super-caterpillar induced by $V(C) - S_1$. Then C' is an Italian super-caterpillar, else it is a smaller counterexample.

Let f be a minimum weight IDF of C. Since C is not Italian, there must be some vertex $z \in C$ such that $f(z) = 1$. Since C is a smallest counterexample, we may assume that $z \in S_1$ and in fact $z = v$ (otherwise either S_1 or C' is a smaller counterexample). Thus there is a $w \in S_1$ that is adjacent to v and to a vertex $y \in V(C')$ and such that $f(w) = 0$ [2]. If $f(y) = 2$, then we may move w to the part S_2 of S containing y (in which case S_1 is a smaller counterexample), so therefore $f(y) = 1$. There are now two possibilities. If the super-caterpillar induced by $V(C') \cup w$ is Italian, it follows that $f^*(y) = 2$ for some minimum weight IDF f^* of C and therefore S_1 is a smaller counterexample. On the other hand, suppose the super-caterpillar induced by $V(C') \cup w$ is not Italian. Then a minimum weight IDF f_2 of $V(C') \cup w$ can be combined with a minimum weight IDF of S_1 to form an IDF of weight less than f, a contradiction. □

6 2-domination

It follows from Theorem 17 that the spiders with $\gamma_I(G) = \gamma_2(G)$ are paths, the spiders of type (ii.b) from Figure 17, and the spiders of type (ii.a) from Figure 17 that have at most two legs with one vertex.

A *cactus* is a graph in which any edge belongs to at most one cycle. The cactus graphs G with $\gamma(G) = \gamma_2(G)$ were characterized in [14] as those that are formed by taking any non-trivial tree, replacing each edge with two parallel edges and subdividing each edge (note that $\delta(G) = 2$ for these graphs). Therefore, these are exactly the cactus graphs with $\gamma(G) = \gamma_I(G)$.

There are infinitely many cactus graphs with $\delta(G) = 1$ and $\gamma_I(G) < \gamma_2(G)$, so we focus on those with $\delta(G) = 2$.

Theorem 19 *Let G be a cactus with $\delta(G) = 2$. Then G is I1a and thus $\gamma_I(G) = \gamma_2(G)$.*

Proof. Assume without loss of generality that G is connected. Let C be an end-block of G with cut-vertex u. Let $G' = G - \{V(C) - u\}$. We distinguish two cases.

Case 1. Suppose $\delta(G') \geq 2$. If there is a minimum-weight IDF f of G' such that $f(u) = 1$, then we proceed by induction, since any cycle is I1a and the inductive hypothesis ensures that the range of f is $\{0, 1\}$. The result then follows from Proposition 1. So suppose every minimum-weight IDF f of G' has $f(u) = 0$. $V(G) - V(G')$ induces a path P with $k \geq 2$ vertices. A minimum-weight labeling

[2]Note that w may be a branch vertex of degree three, but it has no neighbors with non-zero labels from f other than v, y, so we may assume for simplicity that the degree of w is two.

of any path can be achieved with $\lceil \frac{k}{2} \rceil$ 1's on alternate vertices (starting with the first).

Case 1.1 Suppose P has two vertices (i.e., C is a cycle of length three). Then any minimum-weight IDF needs total weight at least two on the vertices of C and this can be achieved with $f(u) = 0$ and labeling the two vertices of P with 1's.

Case 1.2 Suppose P has $k > 2$ vertices (i.e., C is a cycle of length $k+1$). If we label u with 0, any minimum-weight IDF needs total weight at least $\lceil \frac{k+1}{2} \rceil$ on the vertices of P. If we label u with a label greater than 0, then we can label C with total weight $\lceil \frac{k+1}{2} \rceil$ and the total weight of this labeling is at most one more than that of f (when restricted to G'). In fact, a label of 1 on u suffices in this case. In either event, we can achieve a minimum-weight I1a IDF of G of weight $\lceil \frac{k+1}{2} \rceil$ more than the minimum-weight IDF of G'.

Case 2. Suppose that $\delta(G') < 2$. If G' is I1a, then in any such labeling of G' it must be that $f(u) = 1$, sine the degree of u ins G' is 1. This labeling can easily be extended to an I1a labeling of G.

On the other hand, suppose G' is not I1a. Let f' be a minimum-weight IDF of G'. Let u's neighbor in G' be w. Then G' can be partitioned into a cactus G^* with $\delta(G^*) \geq 2$ and a path P^* with $k \geq 1$ vertices, including u. There are three possibilities.

First suppose $P^* = uw$. Then the degree of w in G is two. Then we can find an I1a labeling f^* of G^*. Clearly, the weight of f' is at least two more than the weight of f^*. Thus we may assume that the that the label assigned by f' to w is a 2 and the label assigned to u is 0. Modify this labeling so that the label assigned to u is 1, that assigned to w is 0, and that assigned to the other neighbor of w is 1. This is an I1a labeling of G' and this labeling can be extended to an I1a labeling of G.

Next suppose $P^* = u$. Therefore the degree of w in G' is greater than two. Thus there is a minimum-weight I1a IDF of $G' - u$ in which the label assigned to w is either 0 or 1. This can then be extended to a minimum-weight I1a IDF of G by labeling the cycle containing u with 1's on alternating vertices, starting with one of the vertices adjacent to u.

Finally, suppose the length of P^* is $k > 1$. Then in any minimum-weight IDF of G, the sum of the weights of the vertices on $P^* - u$ must be at least $\lfloor \frac{k}{2} \rfloor$. If equality is achieved in some IDF, then we can find an I1a labeling of G:

start with an I1a labeling of $G^{\bullet}$, label $\lfloor \frac{k}{2} \rfloor$ vertices of $P^{\bullet}$ with 1's (including the furthest vertex from u), and label alternate vertices of C with 1's.

If this is not possible, then k is odd and there is no minimum-weight I1a IDF of $G^{\bullet}$ with a 1 on the vertex $z \in V(G^{\bullet})$ that is adjacent to $P^{\bullet}$. Then either (i) every minimum-weight IDF of $G^{\bullet}$ has a 0 on z, in which case every minimum-weight IDF of G requires weight $\lceil \frac{k}{2} \rceil$ on the vertices of $P^{\bullet} - u$ and this can be achieved by alternating 1's on the vertices of $P^{\bullet}$ and extending that around C to complete an I1a labeling of G or (ii) some minimum-weight IDF of $G^{\bullet}$ has a label of 2 on z. Let z_1 be the vertex on $P^{\bullet}$ adjacent to z and z_2 its neighbor on $P^{\bullet}$. Let H be the subgraph of G induced by $V(G^{\bullet}) \cup \{z_1, z_2\}$. Let $H^{\bullet}$ be the graph H plus edge zz_2. Then $H^{\bullet}$ is a cactus of minimum degree two. Note that $\gamma_I(H^{\bullet}) = \gamma_I(G^{\bullet})$. But since $H^{\bullet}$ is a cactus of minimum degree two smaller than G, it is I1a, via Italian dominating function h. Either function h is an I1a IDF of H, or else $h(z_1) = h(z) = 1, h(z_2) = 0$ in which case modify that to $h(z_2) = h(z) = 1, h(z_1) = 0$. The modified h can be extended an I1a labeling of G. □

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