Signed edge k-independence in graphs

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Abstract

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. If $k \geq 2$ is an integer, then the signed edge k-independence function of G is a function $f : E \rightarrow \{-1, 1\}$ such that $\sum_{e' \in N[e]} f(e') \leq k - 1$ for each $e \in E$. The weight of a signed edge k-independence function f is $\omega(f) = \sum_{e \in E} f(e)$. The signed edge k-independence number $\alpha'_{ks}(G)$ of G is the maximum weight of a signed edge k-independence function of G. In this paper we initiate the study of the signed edge k-independence number and we present bounds for this parameter. In particular, we determine this parameter for some classes of graphs.

Keywords: Signed edge k-independence function; Signed edge k-independence number

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1 Terminology and introduction

For terminology and notation on graph theory not defined here, the reader is referred to [2, 3, 6]. Let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$, and the size $|E|$ of G is denoted by $m = m(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *complement* $\overline{G}$ of G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. We write K_n for the *complete graph* of order n, C_n for a *cycle* of length n and P_n for a *path* of length $n-1$. The line graph of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ee' \in E(L(G))$ when $e = uv$ and $e' = vw$ in G. It is easy to see that $L(K_{1,n}) = K_n$, $L(C_n) = C_n$ and $L(P_n) = P_{n-1}$.

Two edges e_1, e_2 of G are called *adjacent* if they are distinct and have a common end-vertex. The *open neighborhood* $N_G(e) = N(e)$ of an edge $e \in E$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N_G[e] = N[e] = N(e) \cup \{e\}$. The *edge-degree* of a edge $e \in E$ is $\deg_G(e) = \deg(e) = |N(e)|$. The *minimum* and *maximum edge-degree* of a graph G are denoted by $\delta_e = \delta_e(G)$ and $\Delta_e = \Delta_e(G)$, respectively. For a function $f : E \rightarrow \mathbb{R}$ and a subset E' of E we define $f(E') = \sum_{e \in E'} f(e)$. If $E' = N_G[e]$ for some $e \in E$, then we denote $f(E')$ by $f[e]$. For each vertex $v \in V(G)$ we also define $f(v) = \sum_{e \in E(v)} f(e)$, where $E(v)$ is the set of all edges at vertex v.

Recall that a set $D \subseteq V$ is a dominating set of G if every vertex of $V - D$ has a neighbor in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. A set $E' \subseteq E$ is an edge dominating set of G if every edge of $E - E'$ is adjacent to an edge in E'. The edge domination number $\gamma'(G)$ is the minimum cardinality of an edge dominating set in G.

A *signed dominating function* (abbreviated SDF) on G is defined as a function $f : V \longrightarrow \{-1, 1\}$ such that $f(N[v]) = \sum_{x \in N[v]} f(x) \geq 1$ for every $v \in V$. The weight of an SDF f on a graph G is $w(f) = \sum_{v \in V} f(v)$. The *signed domination number* $\gamma_s(G)$ of G is the minimum weight of an SDF on G. The signed domination number was introduced by Dunbar et al. [1].

A function $f : E \longrightarrow \{-1, 1\}$ is called a *signed edge dominating function* (SEDF) of G, if $f(N[e]) \geq 1$ for each edge e of G. The minimum of the values $f(E)$, taken over all signed edge dominating functions f of G, is called the *signed edge domination number* of G and is denoted by $\gamma_s'(G)$. In [7], B. Xu introduced this concept.

Let $k \geq 2$ be an integer. A function $f : V \longrightarrow \{-1, 1\}$ is called a *signed k-independence function* (SkIF) of G, if $f(N[v]) \leq k - 1$ for each vertex v of G. The minimum of the values $f(V)$, taken over all signed k-independence functions f of G, is called the *signed k-independence number* of G and is denoted by $\alpha_s^k(G)$. The special case $k = 2$ was introduced by Zelinka in [8] and in [5] Volkmann introduced this concept for $k \geq 2$.

A *signed edge k-independence function* (SEkIF) of G is a function $f : E \to \{-1, 1\}$ such that $\sum_{e' \in N[e]} f(e') \leq k - 1$ for each $e \in E$. The weight of a signed edge k-independence function f is $\omega(f) = \sum_{e \in E} f(e)$. The signed edge k-independence number $\alpha'_{ks}(G)$ of G is the maximum weight of a signed edge k-independence function of G. The signed edge k-independence function f of G with $\omega(f) = \alpha'_{ks}(G)$ is called a $\alpha'_{ks}(G)$-function. If $k \geq q + 1$, then obviously $\alpha'_{ks}(G) = q$. Therefore we assume throughout this paper that $k \leq q$.

Throughout this paper, if f is an $\alpha'_{ks}(G)$-function, then we let P and S denote the set of those edges in G which are assigned under f the values 1 and -1, respectively, and we let $|P| = p$, and $|S| = s$. Thus $\omega(f) = |P| - |S| = m - 2s = 2p - m$.

In this paper we initiate the study of the signed edge k-independence number and we present bounds for this parameter. In particular, we determine this parameter for some classes of graphs.

The proof of the following results can be found in [8].

Proposition A. [8] Let C_n be a cycle of order $n \geq 3$. Then

$$\alpha_s^2(C_n) = \begin{cases} \frac{n}{3} & n \equiv 0 \pmod 3 \\ \frac{n-4}{3} & n \equiv 1 \pmod 3 \\ \frac{n-2}{3} & n \equiv 2 \pmod 3). \end{cases}$$

Proposition B. [8] Let P_n be a path of order $n \geq 3$. Then

$$\alpha_s^2(P_n) = \begin{cases} \frac{n}{3} & n \equiv 0 \pmod 3 \\ \frac{n-4}{3} & n \equiv 1 \pmod 3 \\ \frac{n-2}{3} & n \equiv 2 \pmod 3). \end{cases}$$

Proposition C. [8] If $n \geq 2$ is an integer, then

$$\alpha_s^2(K_n) = \begin{cases} 0 & n \equiv 0 \pmod 2 \\ 1 & n \equiv 1 \pmod 2). \end{cases}$$

The proof of the following result is straightforward and therefore omitted.

Observation 1. For any nonempty graph G of order $n \geq 2$, then $\alpha'_{2s}(G) = \alpha'_{2s}(L(G))$.

By the above Observation and Propositions A, B, C we have

Corollary 2. For $n \geq 1$, $\alpha'_{2s}(K_{1,n}) = 0$ when n is even, and $\alpha'_{2s}(K_{1,n}) = 1$ when n is odd.

Corollary 3. Let P_n be a path of order $n \geq 3$. Then $\alpha'_{2s}(P_n) = \frac{n-3}{3}$ when $n \equiv 0 \pmod 3$, $\alpha'_{2s}(P_n) = \frac{n-1}{3}$ when $n \equiv 1 \pmod 3$, and $\alpha'_{2s}(P_n) = \frac{n-5}{3}$ when $n \equiv 2 \pmod 3$.

Corollary 4. Let C_n be a cycle of order $n \geq 3$. Then $\alpha'_{2s}(C_n) = \frac{n}{3}$ when $n \equiv 0 \pmod 3$, $\alpha'_{2s}(C_n) = \frac{n-4}{3}$ when $n \equiv 1 \pmod 3$, and $\alpha'_{2s}(C_n) = \frac{n-2}{3}$ when $n \equiv 2 \pmod 3$.

2 Bounds on the signed edge k-independence number

The subdivision star $K_{1,t}$ for $t \geq 1$, is called a healthy spider S_t.

Proposition 5. For every integer $t \geq 1$, there exists a simple connected graph G such that $\alpha'_{ks}(G) \geq t(k-1)$.

Proof. Let $G' = S_t$ be a healthy spider for some $t \geq 1$. Assume that G is the graph obtained from G' by adding $k-1$ pendant edges at each suport vertex of G'. Define $f : E(G) \to \{-1, 1\}$ by $f(e) = 1$ if e is a pendant edge and $f(e) = -1$ otherwise. Obviously, f is an SEkIF on G of weight $t(k-1)$ and this completes the proof. □

Theorem 6. Let $k \geq 2$ be an integer, and let G be a graph of size m. If $\Delta_e = \max\{\deg(u) + \deg(v) - 2 \mid uv \in E(G)\}$, then

$$2k - 2 - m \leq \alpha'_{ks}(G) \leq m - 2\left\lceil \frac{\Delta_e + 2 - k}{2} \right\rceil.$$

Proof. Let $e \in E(G)$ be an edge of maximum edge-degree $\Delta_e(G) = \Delta_e$. Assume that f is a $\alpha'_{ks}(G)$-function. First let $f(e) = 1$, that is, $e \in P$. The condition $f[e] \leq k-1$ leads to the inequality $|N(e) \cap P| - |N(e) \cap S| + f(e) \leq k-1$. Thus $|N(e) \cap P| - |N(e) \cap S| \leq k-2$. Also since e is a edge of maximum degree, we have $|N(e) \cap P| + |N(e) \cap S| = \Delta_e$. Combining the last two inequalities, we deduce that $s \geq |N(e) \cap S| \geq \lceil \frac{\Delta_e + 2 - k}{2} \rceil$, and this yields to

$$\alpha'_{ks}(G) = m - 2s \leq m - 2\left\lceil \frac{\Delta_e + 2 - k}{2} \right\rceil.$$

Assume second that $f(e) = -1$ and so $e \in S$. As $f[e] \leq k-1$ and $\deg(e) = \Delta_e$, we obtain $|N(e) \cap P| - |N(e) \cap S| \leq k$ and $|N(e) \cap P| + |N(e) \cap S| = \Delta_e$. Combining these two inequalities, we conclude that $s \geq |N(e) \cap S| + 1 = \frac{2|N(e) \cap S| + 2}{2} \geq \frac{\Delta_e + 2 - k}{2}$, and thus $s \geq \lceil \frac{\Delta_e + 2 - k}{2} \rceil$. Thus this yields to

$$\alpha'_{ks}(G) = m - 2s \leq m - 2\left\lceil \frac{\Delta_e + 2 - k}{2} \right\rceil.$$

For the lower bound define $f : E(G) \to \{-1,1\}$ by $f(e_1) = f(e_2) = \cdots = f(e_{k-1}) = 1$ for an arbitrary set of $k-1$ edges $E' = \{e_1, e_2, \ldots, e_{k-1}\}$ and $f(e) = -1$ for each edge $e \in E(G) - E'$. Obviously, f is a signed edge k-independence function on G of weight $2k-2-m$ and thus $\alpha'_{ks}(G) \geq 2k-2-m$. □

Corollary 7. Let $k \geq 2$ be an integer, and let G be a graph of size m. Then $\alpha'_{ks}(G) \leq m - 2\lceil \frac{\Delta+\delta-k}{2} \rceil$.

Proof. Let $v \in V(G)$ be a vertex of maximum degree $\Delta(G)$, and $e = uv$ such that $u \in N(v)$. Since $\Delta_e \geq \deg(v) + \deg(u) - 2 \geq \Delta + \delta - 2$, Theorem 6 implies that

$$\alpha'_{ks}(G) \leq m - 2\left\lceil \frac{\Delta_e + 2 - k}{2} \right\rceil \leq m - 2\left\lceil \frac{\Delta + \delta - k}{2} \right\rceil .$$

□

The following example demonstrates that the upper bounds in Theorem 6 and Corollary 7 are sharp.

Example 8. Let $k \geq 2$ be an integer, and let G be a star $K_{1,t}$ such that $t \geq k$. Let u be the central vertex and $u_1, u_2, \ldots, u_t$ the leaves of the star G. It is easy to see that $\Delta_e + 2 = \Delta + \delta = t + 1$ and $m = t$.

Assume first that $t-k-1$ is even. Define the function $f : E(G) \to \{-1,1\}$ by $f(uu_1) = \cdots = f(uu_{\frac{t+k-1}{2}}) = 1$ and $f(e) = -1$ otherwise. Then $f[uu_i] = \frac{t+k-1}{2} - \frac{t-k+1}{2} = k-1$ for $1 \leq i \leq t$. Therefore f is a signed edge k-independence function on G with $\omega(f) = f[uu_1] = k-1$. Hence Theorem 6 implies that

$$k - 1 \leq \alpha'_{ks}(G) \leq m - 2\left\lceil \frac{\Delta_e + 2 - k}{2} \right\rceil = k - 1,$$

and thus $\alpha'_{ks}(G) = k-1$ in the case.

Now let $t-k-1$ is odd. then $t-k \geq 2$. Define the function $f : E(G) \to \{-1,1\}$ by $f(uu_1) = \cdots = f(uu_{\frac{t+k-2}{2}}) = 1$ and $f(e) = -1$ otherwise. Then $f[uu_i] = \frac{t+k-2}{2} - \frac{t-k+2}{2} = k-2$ for $1 \leq i \leq t$. Therefore f is a signed edge k-independence function on G with $\omega(f) = f[uu_1] = k-2$. Hence Theorem 6 implies that

$$k - 2 \leq \alpha'_{ks}(G) \leq m - 2\left\lceil \frac{\Delta_e + 2 - k}{2} \right\rceil = k - 2,$$

and thus $\alpha'_{ks}(G) = k-2$ in the case.

Corollary 9. Let $k \geq 2$ be an integer, and let G be a graph of size m. Then $\alpha'_{ks}(G) = m$ if and only if $\Delta_e \leq k-2$.

Proof. First let $\Delta_e(G) \leq k-2$. Then define the function function $f : E(G) \to \{-1, 1\}$ by $f(e) = 1$ for each edge $e \in E(G)$. It is easy to see that f is a signed edge k-independence function on G of weight m and thus $\alpha'_{ks}(G) = m$.

Conversely, assume that $\alpha'_{ks}(G) = m$. If $\Delta_e(G) \geq k-1$, then Theorem 6 leads to the contradiction $m = \alpha'_{ks}(G) \leq m-2$. Therefore $\Delta_e(G) \leq k-2$, and this completes the proof. □

Let E_o and E_e be the edge sets of odd and even degree and $|E_o| = m_o$ and $|E_e| = m_e$, respectively.

Theorem 10. Let $k \geq 2$ be an even integer, and let G be a graph of size m. If $\Delta_e = \max\{\deg(u) + \deg(v) - 2 \mid uv \in E(G)\}$ and $\delta_e = \min\{\deg(u) + \deg(v) - 2 \mid uv \in E(G)\}$, then

$$\alpha'_{ks}(G) \leq \frac{m(k + \Delta_e - \delta_e - 1) - m_o}{\Delta_e + 1}.$$

Proof. Let f be an $\alpha'_{ks}(G)$-function. Assume that $P = \{e \in E(G) \mid f(e) = 1\}$ and $S = \{e \in E(G) \mid f(e) = -1\}$ such that $|P| = p$ and $|S| = s$. If $\deg(e)$ is odd, then $|N[e]|$ is even. Therefore $f[e] \leq k-1$ and k even imply that $f[e] \leq k-2$ for $e \in E_o$. Hence

$$\begin{aligned}\sum_{e \in E(G)} f[e] &= \sum_{e \in E_o} f[e] + \sum_{e \in E_e} f[e] \\ &\leq m_o(k-2) + (m - m_o)(k-1) \\ &\leq m(k-1) - m_o.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{e \in E(G)} f[e] &= \sum_{e \in E(G)} f(e) + \sum_{e \in E(G)} f(N(e)) \\ &= 2p - m + \sum_{e \in P} \deg(e) - \sum_{e \in S} \deg(e) \\ &= 2p - m + \sum_{e \in E(G)} \deg(e) - 2\sum_{e \in S} \deg(e).\end{aligned}$$

Thus

$$\begin{aligned}m(k-1) - m_o &\geq 2p - m + \sum_{e \in E(G)} \deg(e) - 2\sum_{e \in S} \deg(e) \\ &= 2p - m + m\delta_e - 2(m-p)\Delta_e.\end{aligned}$$

Therefore

$$2p \leq \frac{m(k - \delta_e + 2\Delta_e) - m_o}{\Delta_e + 1},$$

and so

$$\alpha'_{ks}(G) \leq 2p - m \leq \frac{m(k - \delta_e + \Delta_e - 1) - m_o}{\Delta_e + 1}.$$

This complete the proof. □

Corollary 11. Let $k \geq 2$ be an even integer, and let G be an r-edge-regular graph of size m. Then

$$\alpha'_{ks}(G) \leq \frac{m(k-1) - m_o}{r+1}.$$

In the case that k is odd, we obtain the next results analogously to Theorem 10 and Corollary 11.

Theorem 12. Let $k \geq 3$ be an odd integer, and let G be a graph of size m. If $\Delta_e = \max\{\deg(u) + \deg(v) - 2 \mid uv \in E(G)\}$ and $\delta_e = \min\{\deg(u) + \deg(v) - 2 \mid uv \in E(G)\}$, then

$$\alpha'_{ks}(G) \leq \frac{m(k + \Delta_e - \delta_e - 1) - m_e}{\Delta_e + 1}.$$

Corollary 13. Let $k \geq 3$ be an odd integer, and let G be an r-edge-regular graph of size m. Then

$$\alpha'_{ks}(G) \leq \frac{m(k-1) - m_e}{r+1}.$$

Example 8 shows that Corollaries 11 and 13 and therefore Theorems 10 and 12 are sharp.

Theorem 14. Let $k \geq 2$ be an integer, and let G be a graph of size m. If $\Delta_e = \max\{\deg(u) + \deg(v) - 2 \mid uv \in E(G)\}$ and $\delta_e = \min\{\deg(u) + \deg(v) - 2 \mid uv \in E(G)\}$, then

$$\alpha'_{ks}(G) \leq \frac{(2\Delta_e + k - \delta_e - 2\lceil \frac{\delta_e - k + 2}{2} \rceil)m}{\delta_e + k + 2\lceil \frac{\delta_e - k + 2}{2} \rceil}.$$

Proof. Let f be a $\alpha'_{ks}(G)$-function of G. Define $P = \{e \in E \mid f(e) = 1\}$ and $S = \{e \in E \mid f(e) = -1\}$. Let $|P| = p$ and $|S| = s = m - p$. Since $f[e] \leq k - 1$, we have $|N(e) \cap P| \leq |N(e) \cap S| + k - 2$ when $e \in P$ and $|N(e) \cap P| \leq |N(e) \cap S| + k$ when $e \in S$. Thus we conclude that

$$\delta_e \leq \deg(e) = |N(e) \cap P| + |N(e) \cap S| \leq 2|N(e) \cap S| + k - 2,$$

and so $|N(e) \cap S| \geq \lceil \frac{\delta_e - k+2}{2} \rceil$ for each $e \in P$. Hence we deduce that

$$\sum_{e \in P} |N(e) \cap S| \geq p \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil = (m - s) \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil ,$$

and

$$\sum_{e \in S} |N(e) \cap P| \leq \sum_{e \in S} (|N(e) \cap S| + k) = \sum_{e \in S} |N(e) \cap S| + ks.$$

We know that $\sum_{e \in P} |N(e) \cap S| = \sum_{e \in S} |N(e) \cap P|$, and so $\sum_{e \in S} |N(e) \cap S| \geq p \lceil \frac{\delta_e - k+2}{2} \rceil - ks$. Furthermore, we have

$$\begin{aligned}
\sum_{e \in P} |N(e) \cap S| + \frac{1}{2} \sum_{e \in P} |N(e) \cap P| &= \frac{1}{2} \sum_{e \in P} |N(e) \cap S| + \frac{1}{2} \sum_{e \in P} |N(e) \cap S| \\
&\quad + \frac{1}{2} \sum_{e \in P} |N(e) \cap P| \\
&= \frac{1}{2} \sum_{e \in P} |N(e)| + \frac{1}{2} \sum_{e \in P} |N(e) \cap S| \\
&\geq \frac{p\delta_e}{2} + \frac{p}{2} \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil .
\end{aligned}$$

Therefore

$$\begin{aligned}
\Delta_e m &\geq \sum_{e \in P} |N(e) \cap S| + \sum_{e \in P} |N(e) \cap P| + \sum_{e \in S} |N(e) \cap S| + \sum_{e \in S} |N(e) \cap P| \\
&\geq p\delta_e + p \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil + p \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil - k(m - p) \\
&= p(\delta_e + k + 2 \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil) - km.
\end{aligned}$$

Hence

$$p \leq \frac{(\Delta_e + k)m}{\delta_e + k + 2\lceil \frac{\delta_e - k+2}{2} \rceil},$$

and so

$$\alpha'_{ks}(G) \leq 2p - m = \frac{(2\Delta_e + k - \delta_e - 2\lceil \frac{\delta_e - k+2}{2} \rceil)m}{\delta_e + k + 2\lceil \frac{\delta_e - k+2}{2} \rceil},$$

and this is the desired bound. □

Example 8 shows that Theorem 14 is sharp.

Theorem 15. Let $k \geq 2$ be an integer, and let G be a graph of size m. If $\delta_e + 2 - k \geq 0$, then

$$\alpha'_{ks}(G) \leq m+k+\left\lceil \frac{\delta_e - k + 2}{2} \right\rceil - \sqrt{(k + \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil)^2 + 4m \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil}.$$

Proof. Let f be a $\alpha'_{ks}(G)$-function of G. Define $P = \{e \in E \mid f(e) = 1\}$ and $S = \{e \in E \mid f(e) = -1\}$. Let $|P| = p$ and $|S| = s = m - p$. Since $f[e] \leq k-1$, we have $|N(e) \cap P| \leq s+k-2$ when $e \in P$ and $|N(e) \cap P| \leq s+k$ when $e \in S$. Thus we conclude that

$$\delta_e \leq \deg(e) = |N(e) \cap P| + |N(e) \cap S| \leq 2|N(e) \cap S| + k - 2,$$

and so $|N(e) \cap S| \geq \lceil \frac{\delta_e - k + 2}{2} \rceil$ for each $e \in P$. Hence we deduce that

$$\sum_{e \in P} |N(e) \cap S| \geq p \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil.$$

We know that $\sum_{e \in P} |N(e) \cap S| = \sum_{e \in S} |N(e) \cap P|$, and so $\sum_{e \in S} |N(e) \cap P| \leq s(s+k)$. Thus

$$p \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil \leq \sum_{e \in P} |N(e) \cap S| = \sum_{e \in S} |N(e) \cap P| \leq s(s+k),$$

and

$$m = s + p \leq s + \frac{s(s+k)}{\lceil \frac{\delta_e - k + 2}{2} \rceil}.$$

This implies that

$$s^2 + s(k + \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil) - m \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil \geq 0$$

and so

$$s \geq \frac{-1}{2}(k + \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil) + \sqrt{\frac{1}{4}(k + \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil)^2 + m \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil}.$$

This yields to

$$\alpha'_{ks}(G) \leq m - 2s \leq m + k + \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil - \sqrt{(k + \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil)^2 + 4m \left\lceil \frac{\delta_e - k + 2}{2} \right\rceil},$$

and this is the desired bound. □

Corollary 16. Let $k \geq 2$ be an integer, and let G be a graph of size m. If $\delta_e + 2 - k \geq 1$, then

$$\alpha'_{ks}(G) \leq m + k + 1 - \sqrt{(k+1)^2 + 4m}.$$

Proof. Using the notation introduced in the proof of Theorem 15, $m \leq s + \frac{s(s+k)}{\lceil \frac{\delta_e - k + 2}{2} \rceil}$. Since $\delta_e + 2 - k \geq 1$, we have $m \leq s + s(s+k)$. Thus

$$s \geq \frac{-(k+1) + \sqrt{(k+1)^2 + 4m}}{2},$$

and so

$$\alpha'_{ks}(G) = m - 2s \leq m + k + 1 - \sqrt{(k+1)^2 + 4m}.$$

This complete the proof. □

We close this section by establishing a relationship between the signed edge k-independence number and the edge domination number of a graph.

Theorem 17. Let $k \geq 2$ be an integer, and let G be a graph of size m and minimum edge-degree $\delta_e \geq k - 1$. Then

$$\alpha'_{ks}(G) + 2\gamma'(G) \leq m.$$

Proof. Let f be any $\alpha'_{ks}(G)$-function. Since $f[e] \leq k-1$ for every $e \in E(G)$, the hypothesis $\delta_e \geq k-1$, shows that each edge of P is adjacent to at least one edge of S. This implies that S is an edge dominating set of G and thus $\gamma'(G) \leq s$. Hence $\alpha'_{ks}(G) = m - 2s \leq m - 2\gamma'(G)$ or, equivalently, $\alpha'_{ks}(G) + 2\gamma'(G) \leq m$. □

Proposition D. *For $n \geq 2$, $\gamma'(P_n) = \lceil \frac{n-1}{3} \rceil$.*

It follows from Proposition D and Corollary 3, that $\alpha'_{2s}(P_n) + 2\gamma'(P_n) = m$. Therefore Theorem 17 is sharp for $k = 2$.

3 Trees

A vertex of degree one is called a leaf, and its neighbor is called a stem. For $r, s \geq 1$, a double star $S(r, s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other to s leaves.

Proposition 18. For $r, s \geq 1$,

$$\alpha'_{2s}(S(r,s)) = \begin{cases} 1 & r + s \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $S(r,s)$ be a double star whose central vertices are u, v with r pendant edges uu_i and s pendant edges vv_i. Clearly, $S(1,1) = P_4$ and $\alpha'_{2s}(P_4) = 1$, by Corollary 3. Without loss of generality, suppose that $r \geq s$ and $r \geq 2$. Let f be a $\alpha'_{2s}(S(r,s))$-function. Since $f[uv] \leq 1$, we have $\omega(f) = f[uv] \leq 1$.

First let $r+s$ be even. If r, s are odd, then define $h : E(S(r,s)) \rightarrow \{-1,1\}$ by $h(uv) = -1$, $h(uu_i) = (-1)^{i+1}$ for $1 \leq i \leq r$ and $h(vv_i) = (-1)^{i+1}$ for $1 \leq i \leq s$. Obviously, h is an SE2IF of $S(r,s)$ of weight 1 and hence $\alpha'_{2s}(S(r,s)) \geq 1$ in this case. If r, s are even, then define $h(uv) = 1$, $h(uu_i) = (-1)^i$ for $1 \leq i \leq r$ and $h(vv_i) = (-1)^i$ for $1 \leq i \leq s$. Obviously, h is an SE2IF of $S(r,s)$ of weight 1 and hence $\alpha'_{2s}(S(r,s)) \geq 1$ in this case. Theorem 6 implies $\alpha'_{2s}(S(r,s)) \leq 1$ when $r+s$ is even and thus $\alpha'_{2s}(S(r,s)) = 1$ when $r+s$ is even.

Now let $r+s$ be odd. Without loss of generality, suppose that r is even and s is odd. Define $h(uv) = -1$, $h(uu_i) = (-1)^i$ for $1 \leq i \leq r$ and $h(vv_i) = (-1)^{i+1}$ for $1 \leq i \leq s$. Obviously, h is an SE2IF of $S(r,s)$ of weight 0 and hence $\alpha'_{2s}(S(r,s)) \geq 0$ in this case. Again Theorem 6 shows that $\alpha'_{2s}(S(r,s)) \leq 0$ when $r+s$ is odd and thus $\alpha'_{2s}(S(r,s)) = 0$ when $r+s$ is odd and this completes the proof. □

Theorem 19. Let T be a tree of order $n \geq 2$. Then

$$\alpha'_{2s}(T) \geq 0,$$

and this bound is sharp.

Proof. The proof is by induction on n. If $\mathrm{diam}(T) \leq 3$, then T is a star or a double star and we have $\alpha'_{2s}(T) \geq 0$ by Corollary 2 and Proposition 18. Hence the statement holds for all trees T with $\mathrm{diam}(T) \leq 3$. Assume that T is an arbitrary tree of order $n \geq 5$ and $\mathrm{diam}(T) \geq 4$. Let $v_1 v_2 \ldots v_D$ be a diametral path in T chosen to maximize $\deg_T(v_2)$ and root T at v_D. First let $\deg_T(v_2) \geq 3$, and $v_1, w \in N(v_2) - \{v_3\}$. Assume that $T' = T - \{v_1, w\}$, and let g be a $\alpha'_{2s}(T')$-function. Define $f : E(T) \rightarrow \{-1,1\}$ by $f(v_1v_2) = -1$, $f(wv_2) = 1$ and $f(e) = g(e)$ otherwise. Obviously, f is an SE2IF of T of weight $\omega(g)$ and hence $\alpha'_{2s}(T) \geq \omega(f) = \omega(g) \geq 0$. Now if $\deg_T(v_2) = 2$, then let $T' = T - \{v_1, v_2\}$, and let g be a $\alpha'_{2s}(T')$-function. Define $f : E(T) \rightarrow \{-1,1\}$ by $f(v_1v_2) = 1$, $f(v_2v_3) = -1$ and $f(e) = g(e)$ otherwise.

Assume first that v_3 is a stem, and u is a leaf adjacent to v_3. Then

$$f[v_2v_3] = g[uv_3] + f(v_2v_3) + f(v_1v_2) = g[uv_3] \leq 1.$$

Therefore f is an SE2IF of T of weight $\omega(g)$ and hence $\alpha'_{2s}(T) \geq \omega(f) = \omega(g) \geq 0$.

Assume second that $\deg_T(v_3) = 2$. Then $f[v_2v_3] = g(v_3v_4) + f(v_2v_3) + f(v_1v_2) \leq 1$. So f is an SE2IF of T of weight $\omega(g)$ and hence $\alpha'_{2s}(T) \geq \omega(f) = \omega(g) \geq 0$.

Assume third that v_3 is not a stem and that $u_1, u_2, \ldots, u_t \neq v_2, v_4$ are neighbors of v_3 with $t \geq 1$. The choice of v_2 leads to $\deg_T(u_i) = 2$ for $1 \leq i \leq t$. Let w_i be a leaf adjacent to u_i for $1 \leq i \leq t$. If $g(u_iv_3) = -1$ for $1 \leq i \leq t$, then

$$f[v_2v_3] = g(v_3v_4) + \sum_{i=1}^{t} g(u_iv_3) + f(v_2v_3) + f(v_1v_2) \leq 0.$$

Therefore f is an SE2IF of T of weight $\omega(g)$ and hence $\alpha'_{2s}(T) \geq \omega(f) = \omega(g) \geq 0$. So assume next that, without loss of generality, $g(u_1v_3) = 1$. This implies $g(u_1w_1) = -1$. Now define the function $h : E(T') \to \{-1, 1\}$ by $h(u_1w_1) = 1$, $h(u_1v_3) = -1$ and $h(e) = g(e)$ otherwise. Then h is also a $\alpha'_{2s}(T')$-function. It follows that

$$f[v_2v_3] = h[u_1v_3] - h(u_1w_1) + f(v_2v_3) + f(v_1v_2) = h[u_1v_3] - h(u_1w_1) \leq 0.$$

Thus f is an SE2IF of T of weight $\omega(h)$ and hence $\alpha'_{2s}(T) \geq \omega(f) = \omega(h) \geq 0$. This completes the proof. □

Corollary 20. If T is a tree of order $n \geq 2$, then

$$\alpha'_{2s}(T) \geq \begin{cases} 1 & n \equiv 0 \pmod 2 \\ 0 & n \equiv 1 \pmod 2. \end{cases}$$

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