Some Combinatorial Identities about Daehee Sequences

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Abstract In this paper, we investigate and obtain the properties of higher-order Daehee sequences by using generating functions. In particular, by means of the method of coefficients and generating functions, we establish some identities involving higher-order Daehee numbers, generalized Cauchy numbers, Lah numbers, Stirling numbers of the first kind, unsigned Stirling numbers of the first kind, generalized harmonic polynomials and the numbers $P(r, n, k)$.

Keywords Higher-order Daehee numbers; Generating functions; Generalized Cauchy numbers; Lah numbers; Stirling numbers of the first kind; Generalized Harmonic polynomials.

1. Introduction

The Daehee numbers and polynomials studied in this paper are a new combinatorial sequences defined by Taekyun Kim, which appear in various areas in mathematics and have been classically investigated in many directions. Recently, many papers have been devoted to the study of Daehee number identities by various methods. For example, Taekyun Kim and Dae San Kim give various identities of the higher-order Daehee numbers and polynomials arising from umbral calculus (see in [6]). T.Kim use invariant p-adic integral deduced identities associated with Daehee numbers (see in [3]). Dae San Kim, Taekyun Kim, Takao Komatsu and Jong-Jin Seo use Sheffer sequence obtained identities about Barnes-type Daehee polynomials (see in [8]) and so on. In this paper, we investigate some properties of high order Daehee

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polynomials associated with special sequences by using the generating functions and the method of coefficient and we deduced some identities involving high order Daehee numbers and polynomials.

The generating functions are of fundamental importance in combinatorial mathematics, especially in counting. In the present paper, we proved some properties of high order Daehee polynomials by generating functions and the method of coefficient and we established a series of identities about high order Daehee numbers, generalized Cauchy numbers, Lah numbers, Stirling numbers of the first kind, unsigned Stirling numbers of the first kind, generalized Harmonic polynomials and the combinatorial numbers $P(r,n,k)$. The generating functions of some special combinatorial sequences play an important role in mathematical physics and in many branches of Mathematics.

Throughout this paper, we need the following notation:

$[t^n]f(t)$ denotes the coefficient of $[t^n]$ in $f(t)$, where

$$f(t) = \sum_{n=0}^{\infty} f_n t^n. \tag{1}$$

If $f(t)$ and $g(t)$ are formal series, the following relations hold in [13]:

$$[t^n](af(t) + bg(t)) = a[t^n]f(t) + b[t^n]g(t). \tag{2}$$

$$[t^n]tf(t) = [t^{n-1}]f(t). \tag{3}$$

$$[t^n]f(t)g(t) = \sum_{k=0}^{n}[t^k]f(t)[t^{n-k}]g(t). \tag{4}$$

$$[t^n]f(g(t)) = \sum_{k=0}^{\infty}[y^k]f(y)[t^n]g^k(t). \tag{5}$$

Let $r \in \mathbb{N}$, $x \in \mathbb{R}$, the high order Daehee polynomials $D_n^{(r)}(x)$ are defined by the following generating function (see [1] – [6])

$$\sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!} = (\frac{(1+t)\ln(1+t)}{t})^r(1+t)^x. \tag{6}$$

Setting $r \in \mathbb{N}$, $x = 0$, we get the generating function of the high order Daehee numbers $D_n^{(r)}$ (see [1], [4], [5]):

$$\sum_{n=0}^{\infty} D_n^{(r)}\frac{t^n}{n!} = (\frac{(1+t)\ln(1+t)}{t})^r. \tag{7}$$

The classical Daehee numbers D_n defined by the following generating function:

$$\sum_{n=0}^{\infty} D_n \frac{t^n}{n!} = \frac{(1+t)\ln(1+t)}{t}, \tag{8}$$

with $x = 0$, $r = 1$, $D_n^{(1)}(0) = D_n$.

Let $-t$ instead of t in (6), (7), (8) we get

$$\sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{(-t)^n}{n!} = (\frac{(1-t)\ln(1-t)}{-t})^r (1-t)^x. \tag{9}$$

$$\sum_{n=0}^{\infty} D_n^{(r)} \frac{(-1)^n t^n}{n!} = (\frac{(1-t)\ln(1-t)}{-t})^r. \tag{10}$$

$$\sum_{n=0}^{\infty} D_n \frac{(-1)^n t^n}{n!} = \frac{(1-t)\ln(1-t)}{-t}. \tag{11}$$

The particular combinatorial numbers are defined by following generating functions:

The Stirling numbers of the first kind $s(n,k)$ are defined by

$$\sum_{n \geq k} s(n,k) \frac{t^n}{n!} = \frac{\ln^k(1+t)}{k!}. \tag{12}$$

Let $-t$ instead of t, we get the generating function of the unsigned Stirling numbers of the first kind $\mathfrak{s}(n,k)$

$$\sum_{n \geq k} \mathfrak{s}(n,k) \frac{(-t)^n}{n!} = \frac{\ln^k(1-t)}{k!}. \tag{13}$$

The unsigned Stirling numbers of the first kind $\mathfrak{s}(n,k)$ and Stirling numbers of the first kind $s(n,k)$ are have the following relation:

$$\mathfrak{s}(n,k) = |s(n,k)| = (-1)^{n+k} s(n,k), \tag{14}$$

with $n \geq k$ can be replaced with $n \geq 0$.

For $\alpha \in \mathbb{Z} \geq 0$, α-Cauchy numbers $C_n^{(\alpha)}$ ([14]) are defined by the following generating function:

$$\sum_{n=0}^{\infty} C_n^{(\alpha)} \frac{t^n}{n!} = \frac{t(1+t)^{\alpha-1}}{\ln(1+t)}. \tag{15}$$

Let $\alpha = 1$, we get the Cauchy numbers of the first kind c_n are defined by following generating function ([14]):

$$\sum_{n=0}^{\infty} c_n \frac{t^n}{n!} = \frac{t}{\ln(1+t)}. \tag{16}$$

Let $\alpha = 0$, we get the Cauchy numbers of the second kind $\hat{c}_n$ are defined by following generating function ([14]):

$$\sum_{n=0}^{\infty} \hat{c}_n \frac{t^n}{n!} = \frac{t}{(1+t)\ln(1+t)}. \tag{17}$$

Lah numbers $L(n,k)$ defined by following generating function:

$$\sum_{n\geq k} L(n,k)\frac{t^n}{n!} = \frac{(\frac{-t}{1+t})^k}{k!}, \tag{18}$$

with $n \geq k$ can be replaced with $n \geq 0$.

For $r \in \mathbb{N}^+$, $z \in \mathbb{R}$, generalized Harmonic polynomials $H_n^{(r)}(z)$ ([15]) are defined by following generating function:

$$\sum_{n=0}^{\infty} H_n^{(r)}(z)t^n = \frac{[-\ln(1-t)]^{r+1}}{t(1-t)}(1-t)^z. \tag{19}$$

Classical Harmonic numbers H_n are defined by following generating function:

$$\sum_{n=1}^{\infty} H_n t^n = \frac{-\ln(1-t)}{(1-t)}, \tag{20}$$

with $H_0^{(r)}(z) = 1$, let $r = 0,\ z = 0$, we get: $H_{n-1}^{(0)}(0) = H_n\ (n \geq 1)$.

For $r,\ k \in \mathbb{N}$, when $n > r + k$, the combinatorial numbers $P(r,n,k)$ defined by the following generating function:

$$\sum_{n=0}^{\infty} \binom{n+k}{k} P(r, n+k, k)t^n = \frac{-\ln(1-t)^r}{(1-t)^{k+1}}. \tag{21}$$

2. Properties of the generalized Daehee numbers

In this section, we obtain the properties of the Daehee polynomials. Meanwhile, we give some basic properties of the high order Daehee numbers.

Theorem 2.1 Suppose $r_i \in \mathbb{N}$, $i \in [m]$, $m \in \mathbb{N}^+$, for the generalized Daehee polynomials, we have the following property:

$$\begin{aligned} &D_n^{(r_1+\cdots+r_m)}(x_1+\cdots+x_m) \\ &= \sum_{n_1+\cdots+n_m=n} \binom{n}{n_1,\cdots n_m} D_{n_1}^{(r_1)}(x_1)\cdots D_{n_m}^{(r_m)}(x_m). \end{aligned} \tag{22}$$

Proof By (6), we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} D_n^{(r_1+\cdots+r_m)}(x_1+\cdots+x_m)\frac{t^n}{n!} \\ &= (\frac{(1+t)\ln(1+t)}{t})^{r_1+r_2+\cdots+r_m}(1+t)^{x_1+x_2+\cdots+x_m} \\ &= \sum_{n_1=0}^{\infty} D_{n_1}^{(r_1)}(x_1)\frac{t^{n_1}}{n_1!}\cdots\sum_{n_m=0}^{\infty} D_{n_m}^{(r_m)}(x_m)\frac{t^{n_m}}{n_m!} \\ &= \sum_{n=0}^{\infty}\sum_{n_1+\cdots+n_m=n}\binom{n}{n_1,\cdots n_m} D_{n_1}^{(r_1)}(x_1)\cdots D_{n_m}^{(r_m)}(x_m)\frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of the last equation, we have the identities (22).

Corollary 2.1 For $x_1 = \cdots = x_m = 0$ in (22), we obtain the following properties of the high order Daehee numbers:

$$\begin{aligned} &D_n^{(r_1+r_2\cdots+r_m)} \\ &= \sum_{n_1+n_2+\cdots+n_m=n}\binom{n}{n_1,n_2,\cdots n_m} D_{n_1}^{(r_1)} D_{n_2}^{(r_2)}\ldots D_{n_m}^{(r_m)}. \end{aligned} \tag{23}$$

For $r_i \in \mathbb{N}$, $i \in [m]$, $m = 1$, the above formula is reduced to the theorem 3 in reference [4]:

$$D_n^{(r)} = \sum_{n_1+n_2+\cdots+n_m=n}\binom{n}{n_1,n_2,\cdots n_m} D_{n_1}D_{n_2}\cdots D_{n_m}. \tag{24}$$

Let $r_i \in \mathbb{N}$, $i \in [m]$, $m = 2$ in (22), we get

$$D_n^{(r+s)}(x+y) = \sum_{k=0}^{n}\binom{n}{k} D_k^{(r)}(x) D_{n-k}^{(s)}(y). \tag{25}$$

Setting $r = 1$, $s = 0$ in (25), we get

$$D_n(x+y) = \sum_{k=0}^{n}\binom{n}{k} D_{n-k}(y)_k. \tag{26}$$

Let $s=0$ in (25), we get

$$D_n^{(r)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}(y)_{n-k}D_k^{(r)}(x). \tag{27}$$

Let $y=0$ in (27), we get

$$D_n^{(r)}(x)=\sum_{k=0}^{n}\binom{n}{k}(x)_{n-k}D_k^{(r)}. \tag{28}$$

Let $x=y=0$ in (25), we get

$$D_n^{(r+s)}=\sum_{k=0}^{n}\binom{n}{k}D_k^{(r)}D_{n-k}^{(s)}. \tag{29}$$

Similarly, we obtain the following basic properties by using generating functions

$$D_n^{(r+s)}(x+1)-D_n^{(r+s)}(x)=nD_{n-1}^{(r+s)}(x). \tag{30}$$

Let $r=1$, $s=0$ in (30), we get

$$D_n(x+1)-D_n(x)=nD_{n-1}(x). \tag{31}$$

We can also obtain the derivative of generalized Daehee polynomials

$$[D_n^{(r)}(x)]'=nD_{n-1}^{(r+1)}(x-1). \tag{32}$$

3. Identities involving high order Daehee numbers

Now, we consider the generating functions of the high order Daehee numbers and we establish the identities involving high order Daehee numbers, generalized Cauchy numbesr, Lah numbers, Stirling numbesr of the first kind, unsigned Stirling numbers of the first kind, generalized Harmonic polynomials and the combinatorial numbers $P(r,n,k)$.

Theorem 3.1 For $r\in\mathbb{N}$, let $i,\ k,\ l\in\mathbb{Z}\geq 0$, we get

$$\begin{aligned}&\sum_{i=0}^{n-k}\binom{n}{k}\binom{n-k}{i}(-1)^i(i+k-1)_iD_{n-k-i}^{(r+k)}\\&=\sum_{l=0}^{n}\binom{n}{l}D_l^{(r)}s(n-l,k).\end{aligned} \tag{33}$$

$$\sum_{i=0}^{n-k}\binom{n}{k}\binom{n-k}{i}(-1)^{n-i}(k+i-1)_i D_{n-k-i}^{(r+k)}$$
$$=\sum_{l=0}^{n}\binom{n}{l}(-1)^l D_l^{(r)}s(n-l,k). \quad (34)$$

Proof Form the generating functions of D_n^r and $s(n,k)$, we have

$$\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}D_l^{(r)}s(n-l,k)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}D_n^{(r)}\frac{t^n}{n!}\sum_{n=0}^{\infty}s(n,k)\frac{t^n}{n!}$$
$$=(\frac{(1+t)\ln(1+t)}{t})^r\frac{\ln^k(1+t)}{k!}$$
$$=(\frac{(1+t)\ln(1+t)}{t})^{k+r}(1+t)^{-k}\frac{t^k}{k!}$$
$$=\sum_{n=0}^{\infty}D_n^{(k+r)}\frac{t^n}{n!}\sum_{n=0}^{\infty}(-1)^n(k+n-1)_n\frac{t^n}{n!}\frac{t^k}{k!}$$
$$=\sum_{n=k}^{\infty}\sum_{i=0}^{n-k}\binom{n}{k}\binom{n-k}{i}(-1)^i(i+k-1)_i D_{n-k-i}^{(r+k)}\frac{t^n}{n!}.$$

Then (33) holds. The proof of (34) are similar to that of (33).

Theorem 3.2 For $r\in\mathbb{N}$, let $k,\ l,\ m,\ j\in\mathbb{Z}\geq 0$, we get

$$\sum_{l+m+j=n}\binom{n}{l,m,j}(k)_j D_l^{(r)}s(m,k)=\binom{n}{k}D_{n-k}^{(r+k)}. \quad (35)$$

$$\sum_{l+m+j=n}\binom{n}{l,m,j}(-1)^{l+j}(k)_j D_l^{(r)}s(m,k)=\binom{n}{k}(-1)^n D_{n-k}^{(r+k)}. \quad (36)$$

Proof From (7) and (12), we have

$$\sum_{n=0}^{\infty}\sum_{l+m+j=n}\binom{n}{l,m,j}(k)_j D_l^{(r)}s(m,k)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{l+m+j=n}\frac{n!}{l!m!j!}\frac{k!}{(k-j)!}D_l^{(r)}s(m,k)\frac{t^n}{n!}$$
$$=\sum_{l=0}^{\infty}D_l^{(r)}\frac{t^l}{l!}\sum_{m=0}^{\infty}s(m,k)\frac{t^m}{m!}\sum_{j=0}^{\infty}\binom{k}{j}t^j$$
$$=(\frac{\ln(1+t)(1+t)}{t})^r\frac{[\ln(1+t)]^k}{k!}(1+t)^k$$

$$= (\frac{\ln(1+t)(1+t)}{t})^{r+k}\frac{t^k}{k!}$$

$$= \sum_{n=k}^{\infty}\binom{n}{k}D_{n-k}^{(r+k)}\frac{t^n}{n!}.$$

Then (35) holds. The proof of (36) are similar to that of (35).

Corollary 3.1 In the special case when $r=1$, Eq. (35) and (36) are reduced to

$$\sum_{l+m+j=n}\binom{n}{l,m,j}(k)_j D_l s(m,k) = \binom{n}{k}D_{n-k}^{(k+1)}. \tag{37}$$

$$\sum_{l+m+j=n}\binom{n}{l,m,j}(-1)^{l+j}(k)_j D_l s(m,k) = \binom{n}{k}(-1)^n D_{n-k}^{(k+1)}. \tag{38}$$

Theorem 3.3 For $i,\ l,\ \alpha \in \mathbb{Z} \geq 0,\ r \in \mathbb{N}$, we have

$$\sum_{l=0}^{n}\binom{n}{l}D_l^{(r)}C_{n-l}^{(\alpha)} = \sum_{i=0}^{n}\binom{n}{i}(\alpha)_i D_{n-i}^{(r-1)}. \tag{39}$$

Proof From (7) and (15), we have

$$\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}D_l^{(r)}C_{n-l}^{(\alpha)}\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}C_n^{(\alpha)}\frac{t^n}{n!}\sum_{n=0}^{\infty}D_n^{(r)}\frac{t^n}{n!}$$

$$= \frac{t(1+t)^{\alpha-1}}{\ln(1+t)}(\frac{(1+t)\ln(1+t)}{t})^r$$

$$= \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1}(1+t)^{\alpha}$$
$$= \sum_{n=0}^{\infty} D_n^{(r-1)}\frac{t^n}{n!}\sum_{n=0}^{\infty}(\alpha)_n\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(\alpha)_i D_{n-i}^{(r-1)}\frac{t^n}{n!}.$$

Then, the identity can be obtained immediately.

Corollary 3.2 By setting $\alpha = 1$, we have the identity between the higher-order Daehee number with the Cauchy number of the first kind

$$\sum_{l=0}^{n}\binom{n}{l}D_l c_{n-l} = D_n^{(r-1)} + nD_{n-1}^{(r-1)}. \tag{40}$$

Corollary 3.3 By setting $\alpha = 0$, we have the identity between high order Daehee number with the Cauchy number of the second kind

$$\sum_{l=0}^{n}\binom{n}{l}D_l^{(r)}\hat{c}_{n-l} = D_n^{(r-1)}. \tag{41}$$

Theorem 3.4 Let $\alpha \in \mathbb{Z} \geq 0$, $r \in \mathbb{N}$, $l,\ m,\ j \in \mathbb{Z} \geq 0$, then we have

$$\sum_{l+m+j=n}\binom{n}{l,m,j}C_j^{(\alpha)}D_m^{(r)}L(j,\alpha) = (-1)^{\alpha}\binom{n}{\alpha}D_{n-\alpha}^{(r-1)}. \tag{42}$$

Proof From (7), (15) and (18), we have

$$\sum_{n=0}^{\infty}\sum_{l+m+j=n}\binom{n}{l,m,j}C_j^{(\alpha)}D_m^{(r)}L(j,\alpha)\frac{t^n}{n!}$$
$$= \sum_{l=0}^{\infty}C_l^{(\alpha)}\frac{t^l}{l!}\sum_{m=0}^{\infty}D_m^{(r)}\frac{t^m}{m!}\sum_{j=0}^{\infty}L(j,\alpha)\frac{t^j}{j!}$$
$$= \frac{t(1+t)^{\alpha-1}}{\ln(1+t)}\left(\frac{(1+t)\ln(1+t)}{t}\right)^{r}\frac{(-1)^{\alpha}t^{\alpha}}{(1+t)^{\alpha}\alpha!}$$
$$= \left(\frac{(1+t)\ln(1+t)}{t}\right)^{r-1}\frac{(-1)^{\alpha}t^{\alpha}}{\alpha!}$$
$$= \sum_{i=0}^{\infty}D_i^{(r-1)}\frac{t^{i+\alpha}}{i!\alpha!}(-1)^{\alpha}$$
$$= \sum_{n=\alpha}^{\infty}(-1)^{\alpha}\binom{n}{\alpha}D_{n-\alpha}^{(r-1)}\frac{t^n}{n!}.$$

We can easily get the desired result.

Corollary 3.5 Let $\alpha = 1$, the following formula holds true:

$$\sum_{l+m+j=n} \binom{n}{l,m,j} c_l D_m^{(r)} L(j,1) = -nD_{n-1}^{(r-1)}. \tag{43}$$

Theorem 3.5 Let $r \in \mathbb{N}$, $k \in \mathbb{N}^+$, $i,\ l \in \mathbb{Z} \geq 0$, $z \in \mathbb{R}$, then we have

$$\begin{aligned} &\sum_{i=0}^{n-k} (n)_k \binom{n-k}{i} (-1)^{n-k-i} (i+k+1-z)_i D_{n-k-i}^{(k+r+1)} \\ &= \sum_{l=0}^{n} (n)_l (-1)^{n-l} H_l^{(k)}(z) D_{n-l}^{(r)}. \end{aligned} \tag{44}$$

Proof From equation (10) and (19), we have

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{l=0}^{n} (n)_l (-1)^{n-l} H_l^{(k)}(z) D_{n-l}^{(r)} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} H_n^{(k)}(z) t^n \sum_{n=0}^{\infty} D_n^{(r)} \frac{t^n}{n!} \\ &= \frac{(-\ln(1-t))^{k+1}}{t(1-t)} (1-t)^z \left(\frac{(1-t)\ln(1-t)}{-t}\right)^r \\ &= \left(\frac{(1-t)\ln(1-t)}{-t}\right)^{r+k+1} (1-t)^{-(k+2-z)} t^k \\ &= \sum_{n=0}^{\infty} D_n^{(r+k+1)} (-1)^n \frac{t^n}{n!} \sum_{n=0}^{\infty} (n+k+1-z)_n \frac{t^n}{n!} t^k \\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{n-k} (n)_k \binom{n-k}{i} (-1)^{n-k-i} (i+k+1-z)_i D_{n-k-i}^{(k+r+1)} \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of the last equation, we have the identities.

Corollary 3.6 Let $r = 1$, $z = 0$, we get

$$\begin{aligned} &\sum_{i=0}^{n-k} (n)_k \binom{n-k}{i} (-1)^{n-k-i} (i+k+1)_i D_{n-k-i}^{(k+2)} \\ &= \sum_{l=0}^{n} (n)_l (-1)^{n-l} H_l^{(k)}(0) D_{n-l}. \end{aligned} \tag{45}$$

Corollary 3.7 Classical Harmonic number with classical Daehee number have following identity

$$\sum_{i=0}^{n-1} n!(i+1)(n-1)_i D_{n-1-i}^{(2)} = \sum_{l=0}^{n}(n)_l(-1)^{n-l}H_l D_{n-l}. \tag{46}$$

Theorem 3.6 Let $r \in \mathbb{N}$, $k \in \mathbb{N}^+$, $l,\ m,\ j \in \mathbb{Z} \geq 0$, $z \in \mathbb{R}$, then we have

$$\begin{aligned}&\sum_{l+m+j=n}\binom{n}{l,m,j}(-1)^{m+j}l!H_l^{(k)}(z)D_m^{(r)}(k+2)_j\\&=(-1)^k(n)_k D_{n-k}^{(k+r+1)}(z).\end{aligned} \tag{47}$$

Proof It is obvious that

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{l+m+j=n}\binom{n}{l,m,j}(-1)^{m+j}l!H_l^{(k)}(z)D_m^{(r)}(k+2)_j\frac{t^n}{n!}\\
&=\sum_{l=0}^{\infty}H_l^{(k)}(z)t^l\sum_{m=0}^{\infty}D_m^{(r)}(-1)^m\frac{t^m}{m!}\sum_{j=0}^{\infty}\binom{k+2}{j}(-1)^j t^j\\
&=\frac{(-\ln(1-t))^{k+1}}{t(1-t)}(1-t)^z\left(\frac{(1-t)\ln(1-t)}{-t}\right)^r(1-t)^{k+2}\\
&=\left(\frac{(1-t)\ln(1-t)}{-t}\right)^{k+r+1}t^k(1-t)^z\\
&=\sum_{n=k}^{\infty}(-1)^{n-k}(n)_k D_n^{(k+r+1)}(z)\frac{t^n}{n!}.\end{aligned}$$

Hence (47) hold.

Corollary 3.8 For $r=1$, $z=0$, the following relationship holds true:

$$\begin{aligned}&\sum_{l+m+j=n}\binom{n}{l,m,j}(-1)^{m+j}l!H_l^{(k)}(0)D_m(k+2)_j\\&=(-1)^k(n)_k D_{n-k}^{(k+2)}.\end{aligned} \tag{48}$$

Theorem 3.7 Let $r,\ m,\ k \in \mathbb{N}$, $i,\ j \in \mathbb{Z} \geq 0$, then we have

$$\begin{aligned}&\sum_{j=0}^{n}(-1)^{n-j}(n)_j\binom{j+k}{k}P(r,j+k,k)D_{n-j}^{(m)}\\&=\sum_{i=0}^{n-r}(-1)^{n-r}(n)_r(i+k+m)_i\binom{n-r}{i}D_{n-r-i}^{(m+r)}.\end{aligned} \tag{49}$$

Proof It is obvious that

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^{n-j}(n)_j\binom{j+k}{k}P(r,j+k,k)D_{n-j}^{(m)}\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\binom{j+k}{k}P(r,j+k,k)t^n\sum_{n=0}^{\infty}D_n^{(m)}\frac{(-t)^n}{n!}$$

$$=\frac{-\ln^r(1-t)}{(1-t)^{k+1}}(\frac{(1-t)\ln(1-t)}{-t})^m$$

$$=(\frac{(1-t)\ln(1-t)}{-t})^{m+r}(1-t)^{-(r+k+1)}(-1)^{r-1}t^r$$

$$=\sum_{n=0}^{\infty}D_n^{(m+r)}\frac{(-1)^n t^n}{n!}\sum_{n=0}^{\infty}(-1)^{n+r-1}(r+k+n)_n\frac{t^n}{n!}t^r$$

$$=\sum_{n=r}^{\infty}\sum_{i=0}^{n-r}(-1)^{n-1}(n)_r\binom{n-r}{i}(i+r+k)_i D_{n-r-i}^{(m+r)}\frac{t^n}{n!}.$$

Thus, we arrive at the desired result.

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