The Maximum Number of Disjoint Paths in Faulty Enhanced Hypercubes

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Abstract: The maximum number of internal disjoint paths between any two distinct nodes of faulty enhanced hypercube $Q_{n,k}(1 \leq k \leq n-1)$ are considered in a more flexible approach. Using the structural properties of $Q_{n,k}(1 \leq k \leq n-1)$, $\min\{d_{Q_{n,k}-V'}(x), d_{Q_{n,k}-V'}(y)\}$ disjoint paths connecting two distinct vertices x and y in an $n-$dimensional faulty enhanced hypercube $Q_{n,k} - V'(n \geq 8, k \neq n-2, n-1)$ are conformed when $|V'|$ is at most $n-1$. Meanwhile, it is proved that there exists $\min\{d_{Q_{n,k}-V'}(x), d_{Q_{n,k}-V'}(y)\}$ internal disjoint paths between x and y in $Q_{n,k} - V'(n \geq 8, k \neq n-2, n-1)$, under the constraints that (1) The number of faulty vertices is no more than $2n-3$; (2) every vertex in $Q_{n,k} - V'$ is incident to at least two fault-free vertices. This results generalize the results of folded hypercube FQ_n which is a special case of $Q_{n,k}$, and have improved the present results with further theoretical evidence of the fact that $Q_{n,k}$ has excellent node-fault-tolerance when used as a topology of large scale computer networks, thus remarkably improve the performance of the interconnect networks.
key words: Enhanced hypercubes; Fault-toleranct; Internal disjoint paths;

1 Introduction

Assume that G is a network with connectivity κ. This means that there exists a subset F of processors in G such that $\kappa = \min\{|F| : G-F \text{ is disconnected}\}$. That is, κ is the smallest number of processors such that if κ nodes of F is deleted then there does not exist any path in G to connect some pair of two vertices. There are three kinds of internal node-disjoint paths, i.e., one-to-one, one-to-nodes, and nodes-to-nodes which concern with the concept of connectivity of networks. One-to-one means that there are κ internal node-disjoint paths between any two distinct nodes with Menger's theorem [1]. Many one-to-one internal disjoint paths have been designed in literature. F.Cao et al [2] has investigated the disjoint paths in Pyramid

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networks. K.Day al [3] and M.Dietzfelbinger et al [4] have proposed the one-to-one parallel paths for Star networks. [5] discussed disjoint paths in WK-Recursive networks. W.Yang et al have studied the strong Menger connectivity with conditional faults of folded hypercubes [6]. One-to-many internal node-disjoint paths means that there exist κ internal disjoint pathes from one node to another κ distinct nodes, which was first investigated in [7], where the Information Dispersal Algorithm (IDA) was proposed for the hypercubes. One-to-many internal node-disjoint paths were studied in networks such as hypercube ([7]), generalized hypercube ([8]), folded hypercube ([9][10]). Many-to-many internal node-disjoint paths connect two sets of nodes in G (see [11]). Suppose that a network G has a set V' of fault processors, the numbers of the non-faulty neighbors of nodes x and y are the degrees $d_{G-V'}(x)$ of x and $d_{G-V'}(y)$ of y in $G - V'$ respectively.

Usually, two models are used for fault-tolerance analysis. One is standard fault model in which the distribution of faulty edges and faulty vertices is not restricted. The other is the conditional fault model in which each fault-free vertex must be adjacent to at least two fault-free nodes. This paper aims at constructing the maximum number of internal disjoint paths between x and y under standard and conditional fault models

The hypercube is one of the most famous interconnection network architectures yet developed for multiprocessor system and large computation in industrial because of its ideal properties. There are many attentions payed to the properties and fault-tolerance of hypercube ([12]-[14]). To improve the performance of hypercubes, many variants of hypercubes have been proposed, for instance, folded hypercube ([15]- [18]), enhanced hypercube ([19]- [23]), generalized hypercube ([8]) and so on. Folded hypercube is a special case of enhanced hypercube. This paper focus on the enhanced hypercube.

The remainder of this paper is organized as follows. Section 2 gives the basic definitions and existing results in literature which will be used in our discussion. The maximum internal disjoint paths under standard and conditional fault-tolerant enhanced hypercube are presented in Section 3. Finally, some concluding remarks are given in Section 4.

2 Preliminaries

A network is usually modeled by a connected graph $G = (V, E)$, where V denotes the set of processors and E denotes the set of communication links between processors. Two vertices $x, y \in V$ are adjacent if they are incident with a common edge. The set of vertices $N_G(v) = \{u : uv \in E\}$ is called the neighbor set of vertex v in G, that is, the set of adjacent nodes of v. $d(v) = |N_G(v)|$ is called the degree of vertex v in G when no loop occurs. Let $A \subseteq V$, $N_G(A)$ denotes the vertex set $\bigcup_{v\in A} N_G(v) \setminus A$ and $C_G(A) = N_G(A) \bigcup A$. Let $\theta_G(m)$ denote the minimum number of vertices that are adjacent to a vertex set of m vertices in G. If each vertex is adjacent to k vertices, the graph G is called k−regular. A path is a sequence of adjacent vertices, with the original vertex v_0 and end vertex v_m, represented as $P(v_0, v_m) = v_0v_1v_2 \dots v_m$, where all the vertices $v_0, v_1, v_2, \dots, v_m$ are distinct except the case that the path is a cycle where $v_0 = v_m$. Two paths are internal node-disjoint (or node disjoint) if and only if they don't have any vertices in common except their ends. The length of a path P is defined as the number of edges contained in P. The distance $d_G(x, y)$ between any two nodes x and y is the length of a shortest path of joining x and y. The length of a shortest cycle is defined as the girth of graph G, denoted by $g(G)$. A graph G is connected if and only if any two vertices of G can be joined by a path. G is bipartite if the vertex set V can be partitioned into two subsets V_1 and V_2, such that every edge in G joins a vertex in V_1 with a vertex in V_2. A graph G is bipartite if and only if G contains no odd cycle. Two graphs G_1 and G_2 are isomorphic, denoted as $G_1 \cong G_2$, if there is a one to one mapping f from $V(G_1)$ to $V(G_2)$ such that $xy \in E(G_1)$ if and only if $f(x)f(y)) \in E(G_2)$.

An n-dimensional hypercube, denoted by Q_n, has 2^n vertices represented by the vertex set $V(Q_n) = \{x_1x_2 \cdots x_n : x_i = 0 \text{ or } 1, 1 \le \text{i} \le \text{n}\}$, where two vertices $x_1x_2 \cdots x_n$ and $y_1y_2 \cdots y_n$ are adjacent if and only if $\sum_{i=1}^{n} |x_i - y_i| = 1$. Let $x, y \in Q_n$, it is easy to see that there is a shortest routing from x to y if and only if at each step of the routing one bit of x's strings is complemented to match the corresponding bit in y's strings. The Hamming distance between x and y, denoted by $h(x, y) = \sum_{i=1}^{n} |x_i - y_i|$, is the number of different bits between the corresponding strings of x and y. Obviously, $d_{Q_n}(x, y) = h(x, y)$. The weight of a vertex x is defined as $w(x) = \sum_{i=1}^{n} x_i$ (or the number of 1's in x).

Definition. Enhanced hypercube $Q_{n,k} = (V, E)$ for $2 \le k \le n - 1$ is an undirected simple graph with the vertices set V (or $V(Q_{n,k})$)and the edge set of E (or $E(Q_{n,k})$). $V = \{x_1x_2 \cdots x_n : x_i = 0\text{or } 1, 1 \le i \le n\}$, in fact, $V(Q_n) = V(Q_{n,k})$. Two vertices $x = x_1x_2 \cdots x_n$ and y are connected by an edge of E if and only if y satisfies one of the following two conditions:

(1) $y = x^i = x_1x_2 \cdots x_{i-1}\bar{x}_ix_{i+1} \dots x_n$, $1 \le i \le n$, or

(2) $y = \bar{x} = x_1x_2 \cdots x_{k-1}\bar{x}_k\bar{x}_{k+1} \dots \bar{x}_n$, where $\bar{x}_i = 1 - x_i$.

Denote $E_c = \{x\bar{x} : x \in V\}$ the set of complementary edges, $E_i = \{xx^i : x \in V\}$ the set of all i−dimensional edges. Thus we have $E(Q_{n,k}) = E(Q_n) \cup E_c$.

According to the above definition, $Q_{n,k}$ is $(n + 1)$-regular and has 2^n vertices and $(n + 1)2^{n-1}$ edges, and it contains Q_n as its subgraph. It has 2^{n-1} more links than Q_n. If $k = 1$, $Q_{n,k}$ is the well-known folded hypercube denoted by FQ_n. If $k = n$, $Q_{n,k}$ is reduced as Q_n. This paper mainly consider $2 \le k \le n - 1$.

When n and k have the same parity, $Q_{n,k}$ is a bipartite graph with containing no odd cycle , for example, $Q_{4,2}$ is a bipartite graph with bipartition $V_1 = \{x : w(x) \text{ is even}\}$ and $V_2 = \{x : w(x) \text{ is odd}\}$. When n and k have different parity, $Q_{n,k}$ is not bipartite graph.

$Q_{n,k}$ can be partitioned into two subgraphs along some component $i(1 \le i \le n)$. We use $Q^{i0}_{n-1,k}$ and $Q^{i1}_{n-1,k}$ to denote the two subgraphs respectively. For convenience, $Q_{n,k}$ can be expressed as $Q^{i0}_{n-1,k} \uplus Q^{i1}_{n-1,k}$ From the definition of the partition, we can conclude that when $i < k$, $Q^{i0}_{n-1,k}$ and $Q^{i1}_{n-1,k}$ are isomorphic to $(n-1)$-dimensional enhanced hypercube; when $i \geq k$, $Q^{i0}_{n-1,k}$ and $Q^{i1}_{n-1,k}$ are isomorphic to $(n-1)$-dimensional hypercube. A vertex $x = x_1x_2\cdots x_n$ belong to $Q^{i0}_{n-1,k}$ if and only if the i-th position $x_i = 0$; Similarly, x belongs to $Q^{i1}_{n-1,k}$ if and only if the i-th position $x_i = 1$.

3 Main Results

The following lemmas are benefit for us.

Lemma 1 ([24], [25]) Let S be a vsubset of $V(Q_n)$ with $|S| = m$, then

$$\theta_{Q_n}(m) = \begin{cases} -\frac{1}{2}m^2 + (n - \frac{1}{2})m + 1, & 1 \leq m \leq n+1, \\ -\frac{1}{2}m^2 + (2n - \frac{3}{2})m - n^2 + 2, & n+2 \leq m \leq 2n. \end{cases}$$

Lemma 2 ([26]) Let $n \geq 4$ and $F \subseteq V(Q_n)$. Then the following holds.

(i) If $|F| < \theta_{Q_n}(m)$ and $1 \leq m \leq n-3$, then $Q_n - F$ contains exactly one large component of order at least $2^n - |F| - (m-1)$.

(ii) If $|F| < \theta_{Q_n}(m)$ and $n-2 \leq m \leq n+1$, then $Q_n - F$ contains exactly one large component of order at least $2^n - |F| - (n+1)$.

(iii) If $|F| < \theta_{Q_n}(m)$ and $n+2 \leq m \leq 2n-4$, then $Q_n - F$ contains exactly one large component of order at least $2^n - |F| - (m-1)$.

Lemma 3 ([27]) $Q_{n,k}$ is a bipartite graph if and only if n, k have the same parity.

Lemma 4 ([27]) When n, k have different parity, $Q_{n,k}$ contains odd cycle, and the smallest odd cycle contains exactly one complementary edge and the length of $n - k + 2$.

Lemma 3 and lemma 4 lead to lemma 5.

Lemma 5 The girth of $Q_{n,k}$ is $g(Q_{n,k}) = 4$ for $n \geq 3, 2 \leq k \leq n-2$, and $g(Q_{n,n-1}) = 3$.

Lemma 6 Let x, y be any two vertices in $Q_{n,k}$ for $n \geq 4$, then one of the following holds.

(i) $x, y \in V(Q_{n,k})$ for $2 \leq k \leq n-4$, then x and y have exact two common neighbors if they have.

(ii) $x, y \in V(Q_{n,n-3})$

If $\overline{x} \in N_{Q_{n,n-3}}(x) \bigcap N_{Q_{n,n-3}}(y)$, then x, y have exact two common neighbors.

If $\overline{x} \notin N_{Q_{n,n-3}}(x) \bigcap N_{Q_{n,n-3}}(y)$, then x and y have exact two common neighbors if they have.

(iii) $x, y \in V(Q_{n,n-2})$

$$\begin{cases} x = x_1 \cdots x_i \cdots x_j \cdots x_n, \\ y \neq x_1 \cdots x_{i-1}\overline{x_i}x_{i+1} \cdots x_{j-1}\overline{x_j}x_{j+1} \cdots x_n, \end{cases}$$

where $\{i, j\} \subseteq \{n-2, n-1, n\}$, then x and y have exact two common neighbors if they have.

(iv) $x, y \in V(Q_{n,n-2})$

$$\begin{cases} x = x_1 \cdots x_i \cdots x_j \cdots x_n, \\ y = x_1 \cdots x_{i-1}\overline{x_i}x_{i+1} \cdots x_{j-1}\overline{x_j}x_{j+1} \cdots x_n, \end{cases}$$

where $\{i, j\} \subseteq \{n-2, n-1, n\}$, then x and y have exact four common neighbors $\{x^{n-2}, x^{n-1}, x^n, \overline{x}\}$.

(v) $x, y \in V(Q_{n,n-1})$

If $y \notin \{x^{n-1}, x^n\}$, then x, y have exact two common neighbors if they have.

If $y \in \{x^{n-1}, x^n\}$, then x, y have exact two common neighbors $\{\overline{x}, \overline{y}\}$.

Proof: Let $x = x_1 \cdots x_i \cdots x_j \cdots x_n$, $y = x_1 \cdots x_{i-1}\overline{x_i}x_{i+1} \cdots x_{j-1}\overline{x_j}x_{j+1} \cdots x_n$. Then x, y have some common neighbors.

(i) x and y have exact two common neighbors $x^i = x_1 \cdots x_{i-1}\overline{x_i}x_{i+1} \cdots x_j \cdots x_n$ and $x^j = x_1 \cdots x_i \cdots x_{j-1}\overline{x_j}x_{j+1} \cdots x_n$.

(ii) $x, y \in V(Q_{n,n-3})$

If $\overline{x} \in N_{Q_{n,n-3}}(x) \bigcap N_{Q_{n,n-3}}(y)$, by lemma 4, there exists a smallest odd cycle with length 5 passing through vertices x and y. Now x, y have exact two common neighbors $\overline{x}$ and $\overline{y}$.

If $\overline{x} \notin N_{Q_{n,n-3}}(x) \bigcap N_{Q_{n,n-3}}(y)$, then x and y have exact two common neighbors if they have, the proof is similar to (i).

(iii) The proof is similar to (i).

(iv) $x, y \in V(Q_{n,n-2})$, then $\overline{x} = x_1 x_2 \cdots x_{n-3}\overline{x}_{n-2}\overline{x}_{n-1}\overline{x}_n$. x and y have exact four common neighbors x^{n-2}, x^{n-1}, x^n and $\overline{x}$.

If $y = x_1 \cdots \overline{x}_{n-2}\overline{x}_{n-1}x_n$, there are four internal disjoint paths of length two between x and y as follows.

$$x \rightarrow x^{n-2} = x_1 \cdots \overline{x}_{n-2}x_{n-1}x_n \rightarrow x_1 \cdots \overline{x}_{n-2}\overline{x}_{n-1}x_n = y$$

$$x \rightarrow x^{n-1} = x_1 \cdots x_{n-2}\overline{x}_{n-1}x_n \rightarrow x_1 \cdots \overline{x}_{n-2}\overline{x}_{n-1}x_n = y$$

$$x \rightarrow x^n = x_1 \cdots x_{n-2}x_{n-1}\overline{x}_n \rightarrow \overline{x^n} = x_1 \cdots \overline{x}_{n-2}\overline{x}_{n-1}x_n = y$$

$$x \rightarrow \overline{x} = x_1 \cdots x_{n-3}\overline{x}_{n-2}\overline{x}_{n-1}\overline{x}_n \rightarrow (\overline{x})^n = x_1 \cdots \overline{x}_{n-2}\overline{x}_{n-1}x_n = y$$

If $y = x_1 \cdots x_{n-2}\overline{x}_{n-1}\overline{x}_n$, the four paths of length two as:

$$x \rightarrow x^{n-2} = x_1 \cdots \overline{x}_{n-2}x_{n-1}x_n \rightarrow \overline{x^{n-2}} = x_1 \cdots x_{n-2}\overline{x}_{n-1}\overline{x}_n = y$$

$$x \rightarrow x^{n-1} = x_1 \cdots x_{n-2}\overline{x}_{n-1}x_n \rightarrow x_1 \cdots x_{n-2}\overline{x}_{n-1}\overline{x}_n = y$$

$$x \rightarrow x^n = x_1 \cdots x_{n-2}x_{n-1}\overline{x}_n \rightarrow x_1 \cdots x_{n-2}\overline{x}_{n-1}\overline{x}_n = y$$

$$x \rightarrow \overline{x} = x_1 \cdots x_{n-3}\overline{x}_{n-2}\overline{x}_{n-1}\overline{x}_n \rightarrow (\overline{x})^{n-2} = x_1 \cdots x_{n-2}\overline{x}_{n-1}\overline{x}_n = y$$

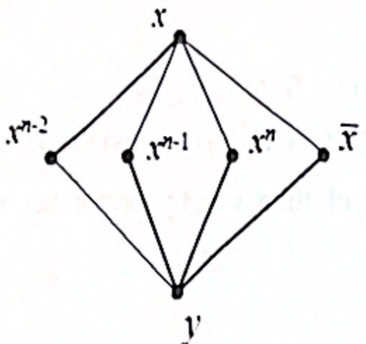

Figure 1: The illustration for the proof of lemma 6 (iv)

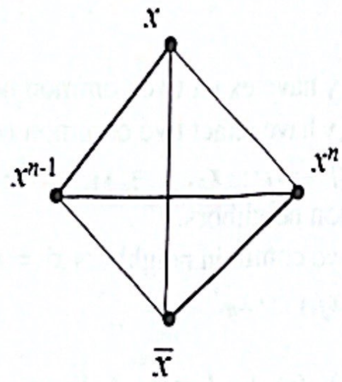

Figure 2: The illustration for $y = x^{n-1}$ or $y = x^n$ in lemma 6 (v)

If $y = x_1 \cdots \overline{x}_{n-2} x_{n-1} \overline{x}_n$, the four paths with length two as:

$$
\begin{aligned}
x &\rightarrow x^{n-2} = x_1 \cdots \overline{x}_{n-2} x_{n-1} x_n \rightarrow x_1 \cdots \overline{x}_{n-2} x_{n-1} \overline{x}_n = y \\
x &\rightarrow x^{n-1} = x_1 \cdots x_{n-2} \overline{x}_{n-1} x_n \rightarrow \overline{x^{n-1}} = x_1 \cdots \overline{x}_{n-2} x_{n-1} \overline{x}_n = y \\
x &\rightarrow x^{n} = x_1 \cdots x_{n-2} x_{n-1} \overline{x}_n \rightarrow x_1 \cdots \overline{x}_{n-2} x_{n-1} \overline{x}_n = y \\
x &\rightarrow \overline{x} = x_1 \cdots x_{n-3} \overline{x}_{n-2} \overline{x}_{n-1} \overline{x}_n \rightarrow (\overline{x})^{n-1} = x_1 \cdots \overline{x}_{n-2} x_{n-1} \overline{x}_n = y
\end{aligned}
$$

(v) $x, y \in V(Q_{n,n-1})$, then $\overline{x} = x_1 x_2 \cdots x_{n-2} \overline{x}_{n-1} \overline{x}_n$.

If $y \notin \{x^{n-1}, x^n\}$, the proof is similar to (i) and x, y have exact two common neighbors.

If $y = x^{n-1} = x_1 \cdots x_{n-2} \overline{x}_{n-1} x_n$ (or $y = x^n = x_1 \cdots x_{n-1} \overline{x}_n$), then $\overline{y} = x^n$ (or $\overline{y} = x^{n-1}$), Therefore x, y have exact two common neighbors $\{\overline{x}, \overline{y}\}$.

Fig.2 illustrates the relationship between x, x^{n-1}, x^n and $\overline{x}$ in $Q_{n,n-1}$.

The proof is finished.

Lemma 6 means that if two vertices x, y have some common neighbors, then number of common neighbors is two or four when $k = n - 2$, otherwise number of common neighbors is two.

Lemma 7 Let $Q_{n,k}$ be an enhanced hypercube with $n \geq 4, 2 \leq k \leq n-1$, and X be a subset of vertices with $|X| = 2$. Then

(i) When $k \neq n-2, n-1$, $|N_{Q_{n,k}}(X)| \geq 2n$.

(ii) When $k = n-2, n-1$ $|N_{Q_{n,k}}(X)| \geq 2n-2$.

Proof: Suppose $X = \{x, y\} \subset V(Q_{n,k})$. The proof will be given according to the distance between x and y.

Case 1. $d(x,y) = 1$

Case 1.1 $2 \leq k \leq n-2$, then $|N_{Q_{n,k}}(X)| = 2(n+1) - 2 = 2n$.

Case 1.2 $k = n-1, y \notin \{x^{n-1}, x^n\}$, then $|N_{Q_{n,n-1}}(X)| = 2(n+1) - 2 = 2n$.

Case 1.3 $k = n-1, y \in \{x^{n-1}, x^n\}$

By lemma 6 (v), $\overline{x} \in N_{Q_{n,n-1}}(x) \bigcap N_{Q_{n,n-1}}(x^{n-1}) \bigcap N_{Q_{n,n-1}}(x^n)$ and x, y have two common adjacent vertices, therefore $|N_{Q_{n,n-1}}(X)| = 2(n+1) - 2 = 2n$.

Case 2. $d(x,y) = 2$

Case 2.1 $k \geq 2, k \neq n-2$, then $|N_{Q_{n,k}}(X)| = 2(n+1) - 2 = 2n$.

Case 2.2 $k = n-2$

$x = x_1 \cdots x_i \cdots x_j \cdots x_n$, $y = x_1 \cdots x_{i-1}\overline{x_i}x_{i+1} \cdots x_{j-1}\overline{x_j}x_{j+1} \cdots x_n$ and $\{i, j\} \not\subset \{n-2, n-1, n\}$. Then by lemma 6 (ii), $|N_{Q_{n,n-2}}(X)| = 2(n+1) - 2 = 2n$.

Case 2.3 $k = n-2$

$x = x_1 \cdots x_i \cdots x_j \cdots x_n$, $y = x_1 \cdots x_{i-1}\overline{x_i}x_{i+1} \cdots x_{j-1}\overline{x_j}x_{j+1} \cdots x_n$ and $\{i, j\} \subset \{n-2, n-1, n\}$. Then by lemma 6 (iv), $|N_{Q_{n,n-2}}(X)| = 2(n+1) - 4 = 2n-2$.

Case 3. $d(x,y) > 2$

Now $|N_{Q_{n,k}}(X)| = 2(n+1)$.

The proof is complete.

From the proof of lemma 7, the number of neighbors of any two vertices is at least $2n-2$.

Lemma 8 For $n \geq 3$, let $x, y, z \in V(Q_{n,k})$ with $d(x,y) = d(x,z) = d(y,z) = 2$, then

(i) When $k = n-2$, there are exact four vertices, say s, u, v, w, such that $\{s, u, v, w\} = N_{Q_{n,k}}(x) \bigcap N_{Q_{n,k}}(y) \bigcap N_{Q_{n,k}}(z)$.

(ii) When $k \neq n-2$, there are exact four vertices, say s, u, v, w, such that $\{s, u, v, w\} \subset N_{Q_{n,k}}(\{x, y, z\})$ and each of $\{s, u, v, w\}$ is adjacent to at least two vertices of $\{x, y, z\}$ and there is exact one of $\{s, u, v, w\}$ is a common neighbor of $\{x, y, z\}$.

Proof: Because $Q_{n,k}$ is vertex symmetrically, without loss of generality, we could use some vertices to verify the results

(i) $k = n-2$, assume that

$$\begin{cases} x = 00\cdots0 \\ y = 00\cdots011 \ \ (\text{ or } y = (x^{n-1})^n \ ) \\ z = 00\cdots0110 \ \ (\text{ or } z = (x^{n-2})^{n-1} = (y^{n-2})^n \ ) \end{cases}$$

Then $d(x,y) = d(x,z) = d(y,z) = 2$, and

$$\begin{cases} s = 00\cdots0111 \ (\text{or } s = \overline{x} = y^{n-2} = z^n\ ) \\ u = 00\cdots0100 \ (\text{ or } u = x^{n-2} = \overline{y} = z^{n-1}\ ) \\ v = 00\cdots010 \ (\text{ or } v = x^{n-1} = y^n = z^{n-2}\ ) \\ w = 00\cdots001 \ (\text{ or } w = x^n = y^{n-1} = \overline{z}\ ) \end{cases}$$

Therefore $\{s,u,v,w\} = N_{Q_{n,k}}(x) \bigcap N_{Q_{n,k}}(y) \bigcap N_{Q_{n,k}}(z)$.

(ii) $k \neq n-2$, assume that

$$\begin{cases} x = 00\cdots0 \\ y = 0\cdots010\cdots010\cdots0 \ (\text{ or } y = (x^i)^j\ ) \\ z = 0\cdots010\cdots010\cdots0 \ (\text{ or } z = (x^j)^k = (y^i)^k\ ) \end{cases}$$

Then $d(x,y) = d(x,z) = d(y,z) = 2$, and

$$\begin{cases} s = 0\cdots010\cdots0 \ (\text{ or } s = x^i\ ) \\ u = 0\cdots010\cdots0 \ (\text{ or } u = x^j\ ) \\ v = 0\cdots010\cdots0 \ (\text{ or } v = x^k\ ) \\ w = 0\cdots010\cdots010\cdots010\cdots0 \ (\text{ or } w = ((x^i)^j)^k\ ) \end{cases}$$

Hence

$$\begin{cases} s \in N_{Q_{n,k}}(x) \bigcap N_{Q_{n,k}}(y) \\ u \in N_{Q_{n,k}}(x) \bigcap N_{Q_{n,k}}(y) \bigcap N_{Q_{n,k}}(z) \\ v \in N_{Q_{n,k}}(x) \bigcap N_{Q_{n,k}}(z) \\ w \in N_{Q_{n,k}}(y) \bigcap N_{Q_{n,k}}(z) \end{cases}$$

The proof is complete.

Lemma 9 Let $Q_{n,k}$ be an enhanced hypercube with $n \geq 5, 2 \leq k \leq n-1$, and X be a subset of vertices with $|X| = 3$. Then

(i) When $k \neq n-2, n-1$, $|N_{Q_{n,k}}(X)| \geq 3n-2$.

(ii) When $k = n-2, n-1$ $|N_{Q_{n,k}}(X)| \geq 3n-5$.

Proof: Suppose $X = \{x,y,z\} \subset V(Q_{n,k})$. Because $Q_{n,k}$ is vertex symmetry, the proof will be given according to the distance between x, y and z.

Case 1. $d(x,y) = d(x,z) = d(y,z) = 1$

Since $\{x,y,z\}$ forms a triangle, lemma 4 and lemma 5 leads to $k = n-1$. By lemma 6 (v), $x,y,z \in Q_{n,n-1}$, $\{x,y,z\} = \{x, x^{n-1}, \overline{x}\}$ or $\{x,y,z\} = \{x, x^n, \overline{x}\}$. Therefore $|N_{Q_{n,k}}(X)| = 3(n+1) - 2 - 2 - 2 - 2 = 3n-5$.

Case 2. $d(x,y) = d(x,z) = 1, d(y,z) = 2$

Case 2.1 $k \neq n-2$

By lemma 6, $|N_{Q_{n,k}}(X)| = 3(n+1) - 2 - 2 - 1 = 3n-2$.

Case 2.2 $k = n-2$

If y,z have exact two common neighbors, then $|N_{Q_{n,n-2}}(X)| = 3(n+1)-2-2-1 = 3n-2$.

If y,z have exact four common neighbors, by lemma 6 (iv), $|N_{Q_{n,n-2}}(X)| = 3(n+1) - 4 - 1 - 1 = 3n-3$.

Case 3. $d(x,y)=1, d(x,z)=d(y,z)=2$

There exists a 5-cycle passing through x, y, z, $Q_{n,k}$ contains odd cycle, so $k = n-3$ or $k = n-1$.

Case 3.1 $k = n-3$

By lemma 6 (ii), say $\overline{x} = N_{Q_{n,n-3}}(x) \bigcap N_{Q_{n,n-3}}(z)$, Without loss of generality, let

$$\begin{cases} x = 00\cdots 0 \\ y = 00\cdots 01\ (= x^n) \\ z = 00\cdots 0111 \end{cases}$$

Then $\overline{x} = 00\cdots 01111 (= z^{n-3})$ is the only common neighbor of x and z. $z^{n-1} = 00\cdots 0101 (= y^{n-2})$ and $z^{n-2} = 00\cdots 011 = (y^{n-1})$ are exact two common neighbors of y and z. Therefore $|N_{Q_{n,n-3}}(X)| = 3(n+1)-2-2-2 = 3n-2$.

Case 3.2 $k = n-1$

By lemma 6 (v), x, y are on a triangle, certainly, there is a 5-cycle containing x, y and z. Without loss of generality, set

$$\begin{cases} x = 00\cdots 0 \\ y = 00\cdots 01\ (= x^n) \\ z = 10\cdots 011 \end{cases}$$

Then $x^{n-1} = 0\cdots 010\ (= \overline{y})$ and $\overline{x} = 0\cdots 11\ (= y^{n-1})$ are the exact two common neighbors of x and y. $x^1 = 10\cdots 0\ (= \overline{z})$ and $\overline{x} = 0\cdots 11\ (= z^1)$ are the exact two common neighbors of x and z. $y^1 = 10\cdots 01\ (= z^{n-1})$ and $y^{n-1} = 0\cdots 11\ (= z^1 = \overline{x})$ are the exact two common neighbors of y and z. That is $\overline{x}$ is a common neighbor of x, y and z. Therefore $|N_{Q_{n,n-1}}(X)| = 3(n+1)-1-3-2 = 3n-3$.

Case 4. $d(x,y)=1, d(x,z)=2, d(y,z)>2$

Case 4.1 $k = n-1$

If xy is on a triangle, by lemma 6, $|N_{Q_{n,n-1}}(X)| = 3(n+1)-2-2-1-1 = 3n-3$. Otherwise, $|N_{Q_{n,n-1}}(X)| = 3(n+1)-2 = 3n+1$.

Case 4.2 $k = n-2$,

By lemma 6 (iii) (iv), $|N_{Q_{n,n-2}}(X)| \geq 3(n+1)-2-4 = 3n-3$.

Case 4.3 $k \neq n-2, n-1$ then $|N_{Q_{n,k}}(X)| \geq 3(n+1)-2-2 = 3n-1$.

Case 5. $d(x,y)=1, d(x,z)>2, d(y,z)>2$

By lemma 6, $|N_{Q_{n,k}}(X)| \geq 3(n+1)-2 = 3n+1$.

Case 6. $d(x,y)=d(x,z)=d(y,z)=2$

Case 6.1 $k = n-2$, by lemma 6 (iv), there are three vertices $\{x, y, z\}$ such that $|N_{Q_{n,k}}(X)| = 3(n+1)-4-4 = 3n-5$.

Case 6.2 $k \neq n-2$, by lemma 8, $|N_{Q_{n,k}}(X)| = 3(n+1)-3-2 = 3n-2$.

Case 7. $d(x,y)=d(x,z)=2, d(y,z)>2$

Case 7.1 $k = n-2$, $|N_{Q_{n,n-2}}(X)| \geq 3(n+1)-4 = 3n-1$.

Case 7.2 $k \neq n-2$, $|N_{Q_{n,k}}(X)| \geq 3(n+1)-2-2 = 3n-1$.

Case 8. $d(x,y)=2, d(x,z)>2, d(y,z)>2$

With lemma 6, $|N_{Q_{n,k}}(X)| \geq 3(n+1) - 4 = 3n - 1$.

Case 9. $d(x,y) > 2, d(x,z) > 2, d(y,z) > 2$

With lemma 6, no common neighbors between any two vertices among $\{x, y, z\}$. Then $|N_{Q_{n,k}}(X)| = 3(n+1) = 3n + 3$.

The proof is finished.

From the proof of lemma 9, the number of neighbors of any three vertices is at least $3n - 5$.

Lemma 10 Let $Q_{n,k}$ be an enhanced hypercube with $n \geq 6, k \neq n-2, n-1$, and $F \subseteq V(Q_{n,k})$ with $|F| \leq 2n - 1$, then there exists a connected component W in $Q_{n,k} - F$ such that $|V(W)| \geq 2^n - |F| - 1$.

Proof: Suppose $F \subseteq V(Q_{n,k})$ with $|F| \leq 2n - 1$. Then with lemma 1, $|F| \leq 2n-1 < \theta_{Q_n}(3)-1 = 3n-6$ for $n \geq 6$, and with lemma 2, Q_n has a large component W such that $|W| \geq 2^n - |F| - 2$. Since Q_n is a spanning subgraph of $Q_{n,k}$, then W is a component of $Q_{n,k}$. Therefore $Q_{n,k} - F$ induces at most three components, and $|V(Q_{n,k}) - F - V(W)| \leq 2$. BY lemma 7, $|V(Q_{n,k}) - F - V(W)| = 2$ is not true, because the neighbors of the two vertices are contained in F, which means $Q_{n,k} - F$ has a large component W, such that $|V(W)| \geq 2^n - |F| - 1$. The proof is complete.

Theorem 11 Let $Q_{n,k}$ be an enhanced hypercube with $n \geq 6, k \neq n-2, n-1$, and $V' \subseteq V(Q_{n,k})$ with $|V'| \leq n - 1$, then each pair of vertices x and y in $Q_{n,k} - V'$ are connected by $\min\{d_{Q_{n,k}-V'}(x), d_{Q_{n,k}-V'}(y)\}$ internal disjoint paths.

Proof: Suppose x and y are vertices in $Q_{n,k} - V'$. Without loss of generality, assume that $d_{Q_{n,k}-V'}(x) \leq d_{Q_{n,k}-V'}(y)$, thus $\min\{d_{Q_{n,k}-V'}(x), d_{Q_{n,k}-V'}(y)\} = d_{Q_{n,k}-V'}(x)$. It should be proved that x and y are connected even if $d_{Q_{n,k}-V'}(x) - 1$ vertices are deleted in $Q_{n,k} - V'$.

By contradiction, suppose that x and y is not connected when deleting a vertex set U with $|U| \leq d_{Q_{n,k}-V'}(x) - 1$. Hence $|U| \leq n$ because of $d_{Q_{n,k}-V'}(x) \leq d_{Q_{n,k}}(x) \leq n+1$. Therefore $|V' \cup U| \leq 2n - 1$. Let $F = V' \cup U$, then by lemma 10, there exists a connected component $C \subset Q_{n,k} - F$ such that $|V(C)| \geq 2^n - |F| - 1$. This implies that $Q_{n,k} - F$ is either connected or has exact two components, one of which is a single vertex. If $Q_{n,k} - F$ is connected, then it contradicts to the fact that x and y is not connected. Otherwise, $Q_{n,k} - F$ has exact two components, one of which is a single vertex, say z. Because x and y is not connected, so $z \in \{x, y\}$, say $z = x$. That is, $N_{Q_{n,k}-V'}(x) \subseteq U$. But $|U| \leq d_{Q_{n,k}-V'}(x) - 1$, a contradiction. Therefore x and y are connected even if $d_{Q_{n,k}-V'}(x) - 1$ vertices are deleted in $Q_{n,k} - V'$. This completes the proof.

Consequently, the following corollary is true because FQ_n is a special case of $Q_{n,k}$ when $k = 1$.

Corollary 12 Let FQ_n be a folded hypercube with $n \geq 6$, and $V' \subseteq V(FQ_n)$ with $|V'| \leq n - 1$, then each pair of vertices x and y in $FQ_n - V'$ are connected by $\min\{d_{FQ_n-V'}(x), d_{FQ_n-V'}(y)\}$ internal disjoint paths.

Next, it focus on the conditional fault tolerance, that is, when fault occurs, each vertex is adjacent to at leats two faut-free vertices. Under the conditional fault-tolerance assumption, it will be proved that for any two vertices $x, y \in Q_{n,k} -$

V' ($n \geq 8, k \neq n-2, n-1$) with $|V'| \leq 2n-3$ are connected by $\min\{d_{Q_{n,k}-V'}(x), d_{Q_{n,k}-V'}(y)\}$ internal disjoint paths. In order to obtain the result, the following lemma is needed.

Lemma 13 Let $Q_{n,k}$ be an enhanced hypercube with $n \geq 8$, $k \neq n-2, n-1$, and $F \subsetneq V(Q_{n,k})$ with $|F| \leq 3n-3$, then there exists a connected component W in $Q_{n,k} - F$ such that $|V(W)| \geq 2^n - |F| - 2$.

Proof: Suppose $F \subsetneq V(Q_{n,k})$ with $|F| \leq 3n-3$. Then with lemma 1, $|F| \leq 3n-3 < \theta_{Q_n}(4) - 1 = 4n - 10$ for $n \geq 8$, and with lemma 2, Q_n has a large component W such that $|W| \geq 2^n - |F| - 3$. Because Q_n is a spanning subgraph of $Q_{n,k}$, then W is a component of $Q_{n,k}$. Therefore $Q_{n,k} - F$ has some components with at most three vertices except for component W, $|V(Q_{n,k}) - F - V(W)| \leq 3$. By lemma 9, $|V(Q_{n,k}) - F - V(W)| = 3$ is not true because the neighbors of the three vertices are contained in F, which means $Q_{n,k} - F$ has a large component W, such that $|V(W)| \geq 2^n - |F| - 2$. The proof is complete.

Theorem 14 Let $Q_{n,k}$ be an enhanced hypercube with $n \geq 8$, $k \neq n-2, n-1$, $V' \subsetneq V(Q_{n,k})$ with $|V'| \leq 2n-3$, and each vertex in $V(Q_{n,k}) - V'$ is adjacent to at least two vertices of $V(Q_{n,k}) - V'$, then each pair of vertices x and y in $Q_{n,k} - V'$ are connected by $\min\{d_{Q_{n,k}-V'}(x), d_{Q_{n,k}-V'}(y)\}$ internal disjoint paths.

Proof: Suppose x and y are vertices in $Q_{n,k} - V'$. Without loss of generality, let $d_{Q_{n,k}-V'}(x) \leq d_{Q_{n,k}-V'}(y)$, hence $\min\{d_{Q_{n,k}-V'}(x), d_{Q_{n,k}-V'}(y)\} = d_{Q_{n,k}-V'}(x)$. It is needed to prove that x and y are connected even if $d_{Q_{n,k}-V'}(x) - 1$ vertices are deleted in $Q_{n,k} - V'$.

By contradiction. suppose that x and y is disjoint by any path when a vertex set W is deleted, where $|W| \leq d_{Q_{n,k}-V'}(x) - 1$. Hence $|W| \leq n$ because of $d_{Q_{n,k}-V'}(x) \leq d_{Q_{n,k}}(x) \leq n+1$. Therefore $|V' \cup W| \leq 3n-3$. Let $F = V' \cup W$, then by lemma 13, there exists a connected component $C \subset Q_{n,k} - F$ such that $|V(C)| \geq 2^n - |F| - 2$. This implies that except for the component $C \in Q_{n,k} - F$, there are at most two vertices. There are three cases needed to be considered.

Case 1. $|V(C)| = 2^n - |F|$.

In this case, $Q_{n,k} - F$ is connected, contradict to the assumption that x and y is not connected.

Case 2. $|V(C)| = 2^n - |F| - 1$.

There is only one vertex except for the component C in $Q_{n,k} - V'$. x and y can not be the only vertex because of $|W| \leq d_{Q_{n,k}-V'}(x) - 1 \leq d_{Q_{n,k}-V'}(y) - 1$. It is a contradiction.

Case 3. $|V(C)| = 2^n - |F| - 2$.

Let $u, v \in V(Q_{n,k} - F)$ and $\{u, v\} \cap V(C) = \emptyset$. Two cases are needed to discuss.

Case 3.1. $x \in C$

If $y \in V(C)$, then x and y is connected, it is a contradiction.

If $y \in \{u, v\}$, when $|W| = d_{Q_{n,k}-V'}(x) - 1 = d_{Q_{n,k}-V'}(y) - 1$, $u = y$ and $y \in N(v)$, then y is not adjacent to any vertex of C, but if $W \subsetneq N_{Q_{n,k}-V'}(y)$, then v is not adjacent to any vertex of W because of lemma 5, otherwise there exits a triangle. By the assumption of $d_{Q_{n,k}-V'}(v) \geq 2$, v is adjacent to some vertex of C, hence

y is joined to some vertex of C through v. If W is not contained in the neighbor set of y in $Q_{n,k} - V'$, then y is joined to some vertex of C by lemma 5. When $|W| \leq d_{Q_{n,k}-V'}(y) - 2$, y is joined to a vertex of C, then x and y is connected, it is a contradiction.

Case 3.2. $x \in \{u, v\}$, say $x = u$.

Case 3.2.1. $u \notin N(v)$. Then $N_{Q_{n,k}-V'}(x) \subseteq W$. $|W| \leq d_{Q_{n,k}-V'}(x) - 1$ and $|N_{Q_{n,k}-V'}(x)| = d_{Q_{n,k}-V'}(x) > |W|$, it is a contradiction.

Case 3.2.2. $u \in N(v)$. The assumption leads to $N_{Q_{n,k}-V'}(x) - v \subseteq W$. According to the conditional fault-tolerance, any vertex of $Q_{n,k}-V'$ has at least two neighbors. Since $uv \in Q_{n,k} - V'$ and lemma 5, they have no any common neighbors, therefore $|W| \geq d_{Q_{n,k}-V'}(x) - 1 + 1$, it is a contradiction.

Consequently, x and y is connected by some path although $d_{Q_{n,k}-V'}(x) - 1$ vertices are deleted. This completes the proof.

Corollary 15 There are $\min\{d_{FQ_{n,k}-V'}(x), d_{FQ_{n,k}-V'}(y)\}$ internal disjoint paths joining vertices x and y in $FQ_{n,k} - V'$ with $n \geq 8$, $k \neq n-2, n-1$, and $|V'| \leq 2n-3$ if each vertex in $FQ_{n,k} - V'$ is adjacent to at least two vertices of $FQ_{n,k} - V'$.

4 Conclusion

This paper has performed the structural analysis of $Q_{n,k}$ and identified a number of good features, which then has proved that any two vertices x and y in $Q_{n,k} - V'$ are connected by $\min\{d_{Q_{n,k}-V'}(x), d_{Q_{n,k}-V'}(y)\}$ internal disjoint paths when $n \geq 8$, $k \neq n-2, n-1$, $V' \subseteq V(Q_{n,k})$ with $|V'| \leq n-1$, which generalize the results of folded hypercube FQ_n. We also proved that there exists $\min\{d_{Q_{n,k}-V'}(x), d_{Q_{n,k}-V'}(y)\}$ internally-disjoint paths between two any distinct vertices x and y in conditional fault-tolerance $Q_{n,k} - V'$ with $n \geq 8$, $k \neq n-2, n-1$, $V' \subseteq V(Q_{n,k})$, $|V'| \leq 2n-3$, under the condition that each vertex of $Q_{n,k} - V'$ ia adjacent to at least two vertices of $Q_{n,k} - V'$, that is, for any $u \in Q_{n,k} - V', d_{Q_{n,k}-V'}(u) \geq 2$. These good properties imply that the reliability and fault-tolerance of $Q_{n,k}$ is more strong than hypercube Q_n. There exists more message transmitting routs in fault $Q_{n,k}$ than Q_n. Thus we think that interconnection networks modelled by enhanced hypercube are extremely robust.

Acknowledgment

This research is supported by the National Science Foundation of China under the projects of grant number 11371162 and grant number 11771172.

The authors would also like to sincerely thank the anonymous reviewers and the editor for their insightful and valuable comments.

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