Prospects For Good Embeddings Of Pairs Of Partial Orthogonal Latin Squares And Of Partial Kirkman Triple Systems

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Abstract. It is known that a pair of mutually orthogonal latin squares (MOLS) of order n can be embedded in a pair of MOLS of order t if $t \geq 3n$. Here we discuss the prospects of extending this result to the case when the smaller pair is only a pair of mutually orthogonal *partial* latin squares (MOPLS). We obtain some conditions, analogous to those of Ryser for embedding partial latin squares in complete latin squares, which we show are necessary for the embedding of MOPLS. We discuss also some implications if these conditions are in fact also sufficient.

We also discuss the analogous problem for pairs of partial Kirkman triple systems PKTS.

1. Introduction

For a number of elementary partial combinatorial structures, such as partial latin squares, partial idempotent latin squares, or partial Steiner triple systems, it is either known or conjectured that if the partial structure has order r, then it can be embedded in a complete structure of order n whenever, roughly speaking, $n \geq 2r$,

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this bound being best possible (each specific problem may have its own individual slight variation on this bound). For a selection of such embedding results see [1],[2],[4], [5],[6],[7], and [11]. A number of these results were conjectured many years ago, and have only 'fallen' fairly recently. A common feature of the solution of several of these problems is that the embeddings have been arrived at by adding a point (or a row) at a time, and this has involved a detailed study of the intermediate stages. The prototypical result here is Ryser's theorem, which states that an $r \times s$ latin rectangle R on n symbols can be embedded in an $n \times n$ latin square if and only if each symbol occurs at least $r + s - n$ times in R.

There is another set of embedding problems where it seems natural to speculate that any partial structure of order r can be embedded in a complete structure of order n whenever $n \geq 3r$, this bound being best possible (again for each specific problem this bound may be varied very slightly). Such problems have seemed to be quite out of reach. In this note we consider two of the main problems of this type. The first is the problem of showing that it is possible to embed any pair of partial orthogonal latin squares of order r in a pair of orthogonal latin squares of order n whenever $n \geq 3r$. The second is the problem of showing that it is possible to embed a partial Kirkman triple system of order r in a Kirkman triple system of order n whenever $n \geq 3r$ and $n \equiv 3 \pmod 6$; we also consider the related problem for resolvable triple systems of index λ, when λ is even. We are interested to see if it is possible to arrive at the embedding by adding a row (or a column) at a time in the first case, or by adding a point at a time in the second case. We recognize that most people who are familiar with orthogonal latin squares or Kirkman triple systems will at this point be smiling sceptically, but politely, behind their hands. But nothing ventured, nothing gained. Here we present sets of necessary conditions for each problem. We hope that these conditions will eventually prove to be sufficient.

2. Pairs of Partial Orthogonal Latin Rectangles

A pair (A, B) of *partial orthogonal latin rectangles* of size $r \times s$ is a pair of $r \times s$ matrices $A = (A_{ij}), B = (B_{ij})$ such that the ordered pairs (A_{ij}, B_{ij}) $(i = 1, \ldots, r; j = 1, \ldots, s)$ are all distinct. If $r = s = n$ and each of A and B is on the same set of n symbols, then (A, B) is a pair of *orthogonal latin squares*.

Let (A, B) be a pair of $r \times s$ partial orthogonal latin squares. Suppose that the pair (A, B) either is, or is to be, embedded in a pair (C, D) of orthogonal latin squares of order n. We may suppose that the symbols of C and D are the integers $1, \ldots, n$. For each $i \in \{1, \ldots, n\}$, let $A(i)$ and $B(i)$ be the number of occurrences of i in A and B respectively.

Let X and Y be subsets of $\{1, \ldots, n\}$. Let $\sigma(X, Y)$ be the set of ordered pairs (x, y) with $x \in X$ and $y \in Y$ and such that there are i and j with $A(i, j) = x$ and $B(i, j) = y$. For $1 \leq i \leq r$, let $R_A^{(i)}(X)$ be the set of elements of X which do not occur in row i of A, and let $R_B^{(i)}(Y)$ be the set of elements of Y which do not occur in row i of B. Similarly, for $1 \leq j \leq s$, let $S_A^{(j)}(X)$ be the set of elements of X which do not occur in column j of A, and let $S_B^{(j)}(Y)$ be the set of elements of Y which do not occur in column j of B. If $X = \{1, \ldots, n\}$ shorten the notation $R_A^{(i)}(X)$ to $R_A^{(i)}$, and define $R_B^{(i)}$, $S_A^{(j)}$ and $S_B^{(j)}$ similarly; thus $R_A^{(i)}$ is simply the set of all elements which do not occur in row i of A, and so $|R_A^{(i)}| = n - s$.

Finally, for $1 \leq i \leq r$, let

$$\phi^{(i)}(X,Y) = \max\{|R_A^{(i)}(X)| + |R_B^{(i)}(Y)| - (n-s), 0\}$$

and, for $1 \leq j \leq s$, let

$$\psi^{(j)}(X,Y) = \max\{|S_A^{(j)}(X)| + |S_B^{(j)}(Y)| - (n-r), 0\}.$$

Theorem 1. *Let (A, B) be a pair of $r \times s$ partial orthogonal latin squares. If (A, B) can be completed to a pair (C, D) of orthogonal latin squares of order n on symbols $1, \ldots, n$, then, for each pair $X \subset \{1, \ldots, n\}$, $Y \subset \{1, \ldots, n\}$,*

$$|\sigma(X,Y)| + \sum_{i=1}^{r} \phi^{(i)}(X,Y) + \sum_{j=1}^{s} \psi^{(j)}(X,Y) \leq |X||Y|. \tag{1}$$

Proof: We may suppose that A and B occupy the first r rows and s columns of C and D respectively. The number of ordered pairs (x, y) with $x \in X$ and $y \in Y$ is $|X|\,|Y|$. For each $i \in \{1, \ldots, r\}$, the number of ordered pairs (x, y) of (A, B) with $x \in X$ and $y \in Y$ which occur in cells $(i, s+1), \ldots, (i, n)$ is at least $\max\{|R_A^{(i)}(X)| + |R_B^{(i)}(Y)| - (n-s), 0\} = \phi^{(i)}(X,Y)$. Similarly, for each $j \in \{1, \ldots, s\}$, the number of ordered pairs (x, y) with $x \in X$ and $y \in Y$ is at least $\max\{|S_A^{(j)}(X)| + |S_B^{(j)}(Y)| - (n-r), 0\} = \psi^{(j)}(X,Y)$. Thus the number of ordered pairs (x, y) with $x \in X$ and $y \in Y$ which occur in the first r rows of (C, D) and the first s columns of (C, D) is at least

$$|\sigma(X,Y)| + \sum_{i=1}^{r} \phi^{(i)}(X,Y) + \sum_{j=1}^{s} \psi^{(j)}(X,Y).$$

The desired inequality (1) now follows.

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One rather obvious necessary condition for (A, B) to be completable to (C, D) is as follows. Consider a particular element $x \in \{1, \ldots, n\}$, and suppose without loss of generality that x lies in cells $(1, 1), (2, 2), \ldots, (t, t)$ of A, and does not occur elsewhere in A, so that $t = |A(x)|$. Then the family $(R_B^{(t+1)}, \ldots, R_B^{(r)}, S_B^{(t+1)}, \ldots, S_B^{(s)})$ of sets must have a transversal containing no element of $\{y\colon (x, y) \in$

$\sigma(x, \{1, \ldots, n\})\}$. We show now that condition (I) does imply the existence of such a transversal.

Let $X = \{x\}, N = \{1, \ldots, n\}$ and $N_X = \{y: (x, y) \in \sigma(x, N)\}$. Choose sets $W_A \subset \{t+1, \ldots, r\}$ and $W_B \subset \{t+1, \ldots, s\}$. We show the existence of such a transversal by demonstrating that if (I) holds then Hall's condition also holds. This means that we show that

$$|W_A| + |W_B| \leq |\left(\left(\bigcup_{i \in W_A} R_B^{(i)}\right) \cup \left(\bigcup_{j \in W_B} S_B^{(j)}\right)\right) \backslash N_X|. \tag{2}$$

Let

$$Y = \left(\bigcup_{i = \in W_A} R_B^{(i)}\right) \cup \left(\bigcup_{j \in W_B} S_B^{(j)}\right).$$

For $i \in \{1, \ldots, r\}$,

$$\phi^{(i)}(x, Y) = \max\{|R_A^{(i)}(x)| + |R_B^{(i)}(Y)| - (n - s), 0\}$$

has the value 1 if $t + 1 \leq i \leq r$ (so that x does not lie in row i of A) and $|R_B^{(i)}(Y)| = n - s$, and it has the value 0 otherwise. In particular, in view of the definition of Y, it takes the value 1 if $i \in W_A$. Therefore

$$\sum_{i=1}^{r} \phi^{(i)}(x, Y) \geq |W_A|.$$

Similarly

$$\sum_{j=1}^{s} \psi^{(j)}(x, Y) \geq |W_B|.$$

From the inequality (1) it now follows that

$$|\sigma(x, Y)| + |W_A| + |W_B| \leq |Y|.$$

Consequently

$$|W_A| + |W_B| \leq |Y| - |\sigma(x, Y)|.$$

But

$$|Y| - |\sigma(x, Y)| = |Y \backslash \sigma(x, Y)| = |Y \backslash N_X|.$$

Therefore

$$|W_A| + |W_B| \leq |Y \backslash N_X|$$

which is the inequality (2) rewritten. Therefore the transversal does exist.

Some special cases of Theorem 1 are of particular interest. Suppose that B is a latin square, so that $r = s$, and suppose that the symbols of B are the integers $1, \ldots, r$. Let $x \in \{1, \ldots, n\}$ and suppose, as before, without loss of generality, that x occupies cells (1,1), (2,2),..., (t,t) of A, so that $t = A(x)$. We know by the argument we have just gone through that the inequality (1) implies that the family $(R_B^{(t+1)}, \ldots, R_B^{(r)}, S_B^{(t+1)}, \ldots, S_B^{(r)})$ has a transversal corresponding to x. Since B is a latin square on $\{1, \ldots, r\}$, the $2(r-t)$ symbols in this transversal do not lie in the set $\{1, \ldots, r\}$. There are $t + 2(r-t) = 2r - t$ ordered pairs with first element x accounted for so far. Of these only t have the second element in the set $\{1, \ldots, r\}$. Therefore the remaining $n - (2r - t)$ ordered pairs with first element x must include the remaining $r - t$ ordered pairs with first element x and second element in the set $\{1, \ldots, r\}$. Therefore $n - (2r - t) \geq r - t$ so that $2t \geq 3r - n$. Since $t = A(x)$ and since x was chosen arbitrarily, it follows that

$$A(x) \geq \frac{1}{2}(3r - n) \qquad (\forall x \in \{1, \ldots, n\}). \tag{3}$$

If A were also a latin square and $n > r$ then for some symbols $i, A(i) = 0$. Condition (3) then reduces to

$$n \geq 3r. \tag{4}$$

It has been shown by Heinrich and Zhu [3], following upon the work of several other authors, that this is a sufficient condition for the embedding of (A, B) into (C, D). Thus in this case at least, condition (1) is sufficient for the existence of an embedding.

Another special case of interest is when the symbols of C each occur $\lfloor \frac{r^2}{n} \rfloor$ or $\lceil \frac{r^2}{n} \rceil$ times in A. Thus $A(i) = \lfloor \frac{r^2}{n} \rfloor$ or $\lceil \frac{r^2}{n} \rceil$ for each i. Condition (3) then becomes $\lfloor \frac{r^2}{n} \rfloor \geq \frac{1}{2}(3r - n)$. This implies that $\frac{r^2}{n} \geq \frac{1}{2}(3r - n)$. This is equivalent to $(n - 2r)(n - r) \geq 0$. Since $n > r$ it follows that

$$n \geq 2r \tag{5}$$

This suggests some exciting speculations. If (1) were sufficient in this case also, it would follow that if $A(i) = \frac{r^2}{2r} = \frac{r}{2} (\forall i)$ then (A, B) could be embedded in (C, D), where C and D have order $2r$. If it is true that to any $r \times r$ latin square B there exists a partial $r \times r$ latin square A on $2r$ symbols such that A and B are orthogonal, and if also (1) is sufficient in this case, then it would follow that any $r \times r$ latin square can be embedded in a latin square of order $2r$ which has an orthogonal mate.

3. Partial Kirkman triple systems

An *edge-colouring* of a graph H is a map $\phi: E(H) \to C$, where $E(H)$ is the set of edges of H and C is a set of colours. A *partial Kirkman triple system* on r

points is an edge-colouring of K_r in which each colour class consists of (vertex) disjoint K_2's and K_3's (i.e. (vertex) disoint triangles and edges). If $r \equiv 3 \pmod 6$ and each colour class consists of $\frac{1}{3}r$ K_3's and no K_2's, then the edge-coloured K_r is a *Kirkman triple system.*

A *proper edge-colouring* of a graph H is an edge-colouring in which no vertex of G is incident with more than one edge of each colour. H is called *h-edge-colourable* if it has a proper edge-colouring with h colours.

Define

$$\mu = \begin{cases} \lfloor \frac{1}{3}r\lfloor\frac{1}{2}(r-1)\rfloor\rfloor & \text{for } r \not\equiv 5 \pmod 6, \\ \lfloor \frac{1}{3}r\lfloor\frac{1}{2}(r-1)\rfloor\rfloor - 1 & \text{for } r \equiv 5 \pmod 6. \end{cases}$$

It was shown by Schönheim [12] that the number of triples in a partial Kirkman triple system on r elements is at most $\mu(r)$.

Given a partial Kirkman triple system of order r, let G be the *missing-edge graph*, i.e., the graph whose vertex set is the set of points of the partial Kirkman triple system, and whose edge set is the set of all edges which are not in triples.

In the case when a fixed integer n is given, and the partial Kirkman triple system of order r either is, or is to be, embedded in a Kirkman triple system of order n, let G^o be the regular graph of degree $n-r$ obtained from G by adding $\frac{1}{2}(n-r-d_G(r))$ loops at each vertex $r \in V(G)$ (here a loop counts two towards the degree), and it is not hard to see that if the partial system of order r is embedded in a complete system of order n then $\frac{1}{2}(n-r-d_G(r))$ is even.

We are now in a position to state our main result on partial Kirkman triple systems.

Theorem 2. *Let* $n = 6t+3$ *and let* K_r *be edge-coloured with* $3t+1$ *colours* $c_1, \ldots, c_{3t+1}$. *Let the* i*-th colour class be* C_i, *and let* C_i *consist of* t_i K_3*'s and* e_i K_2*'s, the* K_3*'s and* K_2*'s all being mutually vertex disjoint. If the edge-colouring of* K_r *can be extended to an edge-colouring of* K_n *in which each colour class is a 2-factor consisting of disjoint* K_3*'s then*

(i) $6t_i + 3e_i \geq 3r - n$,

(ii) $0 \leq \binom{n-r}{2} + \binom{r}{2} - \frac{1}{2}(n-r)n - 3\sum_{i=1}^{3t+1} t_i \leq 3\mu(n-r)$,

(iii) G *is* $(n-r)$*-edge-colourable,*

(iv) G^o *has no components of even order with two loops.*

Speaking loosely, we can say that the fact that the edge-coloured K_n is a Steiner triple system implies Conditions (ii), (iii) and (iv), and the fact that the Steiner triple system is resolvable, i.e., is a Kirkman triple system, implies Condition (i).

Let $N(v)$ denote the number of triangles in the edge-coloured K_r which contain v. We remark that the further condition

(v) $N(v) \geq \frac{1}{2}(2r-n-1) \qquad (\forall v \in V(K_n))$,

which the observant reader might suspect has been overlooked, is an easy consequence of Condition (iii).

Proof: Conditions (ii), (iii), and (iv) follow whenever the edge-coloured K_n is a Steiner triple system; they are proved in [5]. It remains to prove Condition (i).

For each i there are $r - 3t_i - 2e_i$ vertices in the K_r which are in triangles with an edge disoint from the K_r (i.e., the edge has no vertex in common with the K_r) These $r - 3t_i - 2e_i$ vertices in the K_r are therefore in triangles with $2(r - 3t_i - 2e_i)$ vertices not in the K_r. There are a further e_i vertices not in the K_r corresponding to the independent edges of C_i in the K_r. There are therefore at least $2(r - 3t_i - 2e_i) + e_i = 2r - 6t_i - 3e_i$ vertices not in the K_r. But there are $n - r$ vertices not in the K_r. Therefore

$$2r - 6t_i - 3e_i \leq n - r,$$

so that

$$3r - n \leq 6t_i + 3e_i.$$

This proves Theorem 2.

A special case of interest is when $r \equiv 1$ or 3 (mod 6), the edge-coloured K_r is a Steiner triple system, and each colour class consists of $\left\lfloor \frac{r(r-1)}{3(n-1)} \right\rfloor$ or $\left\lceil \frac{r(r-1)}{3(n-1)} \right\rceil$ triangles. Then Condition (i) becomes

$$6 \left\lfloor \frac{r(r-1)}{3(n-1)} \right\rfloor \geq (3r - n).$$

This implies that $2r(r-1)/(n-1) \geq (3r - n)$, which is equivalent to $(n - 2r - 1)(n - r) \geq 0$. Since $n > r$ it follows that

$$n \geq 2r + 1. \tag{6}$$

Under these conditions, Conditions (ii), (iii) and (iv) are all satisfied.

This suggests some further interesting speculations. If (i), (ii), (iii), and (iv) are sufficient in this case, then it follows that if $r = 6p+1$, $n = 12p+3$ and each colour class consists of p triples, then the Steiner triple system of order r can be extended to a Kirkman triple system of order n in which, for each $i \in \{1, 2, \ldots, 6p + 1\}$, the i-th colour class C_i becomes part of the i-th parallel class of the Kirkman triple system.

There is a very natural way of associating with a Steiner triple system A of order $6p + 1$ contained in a Kirkman triple system B of order $12p + 3$ a pair (C, D) of orthogonal 1-factorizations of order $6p + 2$; such pairs of orthogonal 1-factorizations are well-known to be equivalent to Room squares. Thus if the conditions of Theorem 2 are sufficient, and if a proof of Theorem 2 is devised which constructs the Kirkman triple system by adding a point at a time, then we would obtain incidentally a method for 'growing' Room squares a row and column at a time.

The pair (C, D) arises as follows. Let the vertices of the Steiner triple system be $v_1, \ldots, v_{6p+1}$; the colour classes (or parallel classes) of the Kirkman triple system are $C_1, \ldots, C_{6p+1}$. Let the points of the Kirkman triple system other than $v_1, \ldots, v_{6p+1}$ be $w_1, \ldots, w_{6p+2}$. Let $W = \{w_1, \ldots, w_{6p+2}\}$. Notice that no three points of W form a triangle of the Kirkman triple system. One of the two 1-factorizations of the K_{6p+2} on $w_1, \ldots, w_{6p+2}$ has as its j-th 1-factor the set of those edges in the K_{6p+2} which form a triangle in the Kirkman triple system with the vertex v_j. The second of the two 1-factorizations has as the i-th 1-factor G_i the edges of the K_{6p+2} which are in a triangle of the i-th colour class C_i. It is easy to see that at most one edge of F_j is in G_i, i.e., the 1-factorizations are orthogonal. The Room square is indexed by $W \times W$, and the (i,j)-th element is 0 if G_i and F_j do not have a vertex in common, and the set $\{w_k, w_\ell\}$ if G_i; and F_j have the edge $W_k W_\ell$ in common. We remark that this construction goes back to Kirkman [9], and is discussed by Mullin and Vanstone in [10].

Finally we consider the analogous problem of embedding partial resolvable triple systems of index λ when λ is even.

The necessary conditions in this case are simpler, and it seems likely to be a good deal easier to show that these necessary conditions are sufficient (if that is in fact the case). At the time of writing, the problem of embedding partial Steiner triple systems has not been completely solved, whereas the problem of embedding partial triple systems of index λ has been solved when $4 \mid \lambda$ (see [7]).

Let λK_n be the graph on n vertices in which each two vertices are joined by λ edges. A *partial resolvable triple system* of index λ is an edge-colouring of λK_r in which each colour class consists of (vertex) disjoint K_3's and K_2's. If $r \equiv 0 \pmod 3$ and each colour class consists of $\frac{1}{3}r$ K_3's and no K_2's, then the edge-coloured λK_r is a *resolvable triple system* of index λ.

Given a partial resolvable triple system of index λ and order r, let G be the graph whose vertex set is the set of points of the partial resolvable triple system, and whose edge set is the set of all edges which are not in triples.

When a fixed integer n is given, and the partial resolvable triple system of order r is, or is to be, embedded in a resolvable triple system of order n, let G^o be the regular graph of degree $\lambda(n-r)$ obtained from G by adding $\frac{1}{2}\{\lambda(n-r) - d_G(r)\}$ loops at each vertex $v \in V(G)$.

Theorem 3. *Let $n = 6t$ or $6t+3$. Let λ be even. Let $r \leq n$ and let λK_r be edge-coloured with $\frac{1}{2}\lambda(n-1)$ colours. Let the i-th colour class be C_i and let C_i consist of t_i K_3's and e_i K_2's, the K_3's and K_2's all being mutually (vertex) disjoint. If the edge-colouring of λK_r can be extended to an edge-colouring of λK_n in which each colour class is a 2-factor consisting of disjoint K_3's (so the edge-colouring gives a resolvable triple system of index λ), then*

(i) $6t_i + 3e_i \geq 3r - n \qquad (\forall i)$,
(ii) $3\sum_i t_i \leq \lambda\{\binom{n-r}{2} + \binom{r}{2} - \frac{1}{2}r(n-r)\}$,

(iii) $\Delta(G) \leq \lambda(n-r)$, *and*
(iv) G° *contains no component with exactly one loop; and if* $n - r = 2$, G° *contains no component with an odd number of loops.*

Let $N(v)$ denote the number of triangles in the edge-coloured λK_n which contain v. We remark that condition (iii) is equivalent to the condition

(v) $N(v) \geq \frac{1}{2}\lambda(2r - n - 1) \qquad (\forall v)$.

Proof: Conditions (ii), (iii) and (iv) follow whenever the edge-coloured λK_n is a triple system of index λ; they are proved in [7]. The proof of Condition (i) is the same as in the proof of Theorem 2, and follows because the triple system is resolvable. This proves Theorem 3.

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