On neutrality of regular graphs

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Abstract. In this note, we give a characterization of regular graphs which are neutral.

In this paper all graphs are finite, simple and undirected. Given a graph $G = (V, E)$, a binary labeling f on G is defined as follow: each vertex of the graph is assigned 0 or 1. An edge joining two vertices having the same vertex label is assigned 0, and an edge joining two vertices having opposite vertex labels is assigned 1. For such a labeling, let $v(i)$ and $e(i)$ denote respectively the numbers of vertices and edges with label i, where $i = 0, 1$.

Cahit [2] called a graph G *cordial* if there exists a binary labeling f such that $|v(0) - v(1)| \leq 1$ and $|e(0) - e(1)| < 1$.

Cahit's consideration of cordial graphs is motivated by the study of graceful graphs. He regards cordial graphs as a weaker version of graceful graphs and harmonious graphs, although there are cordial graphs which are not graceful. While it is not known whether all trees are graceful, Cahit [2] gives the affirmative answer that they are all cordial. In general, it is difficult to decide when a regular graph G is cordial. For other results of cordial graphs, the reader can refer to [1,3,4,5,6].

Recently, the first author of this paper, K.W. Lih and Y.N. Yeh [7] consider another labeling problem. Consider a binary labeling g on the vertices of G, such that each vertex is assigned 0 or 1. A partial function $g^{+}: E \rightarrow \{0, 1\}$ can be defined as follows:

$$g^{+}(\{a,b\}) = \begin{cases} 1 & if\, g(a)=g(b)=0 \\ 0 & if\, g(a)=g(b)=1, \text{ where } \{a,b\} \varepsilon E. \end{cases}$$

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A graph is called *neutral* if there exists a binary labeling g and its induced labeling g^+, such that the following inequality holds

$$|(e(0) + v(0)) - (e(1) + v(1))| \leq 1.$$

The labeling g is called the neutral labeling. A neutral graph is called *strongly neutral* if we have $e(0) + v(0) = e(1) + v(1)$.

We prove the following result.

THEOREM 1. *An r-regular (p, q)-graph G is neutral if and only if*
" (1)" p is even or
" (2)" p is odd and $r = 2$ or 4.
If p is even, then G is strongly neutral.

Proof: For simplicity, we name the vertices of G by $1, 2, \ldots, p$.

Label the first k vertices by 0 and the remaining vertices by 1. Here k can be $0, 1, 2, \ldots, \lfloor p/2 \rfloor$. (Figure 1)

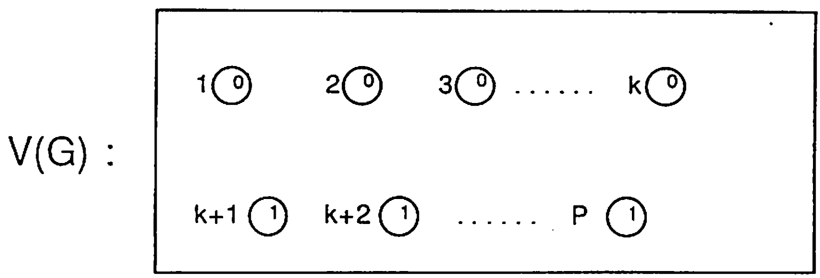

Figure 1

Define $f_i(0)$ (respectively $f_i(1)$ to be the number of edges connecting vertex i to vertices labeled 0 (respectively 1). Then we see that

$$v(1) = p - k$$

$$v(0) = k$$

$$e(0) = \left(\sum_{i=k+1}^{p} f_i(1)\right)/2$$

$$e(1) = \left(\sum_{i=1}^{k} f_i(0)\right)/2$$

If we count the number of edges of the graphs by counting the three types of edges: (1) the end vertices of the edges with label 0, (2) the end vertices of the edges with undefined labels and (3) the end vertices of the edges with label 1, then we have

$$pr/2 = (\sum_{i=1}^{k} f_i(0))/2 + (\sum_{i=k+1}^{p} f_i(0)) + (\sum_{i=k+1}^{p} f_i(1))/2$$

Let $s = v(1) + e(1) - v(0) - e(0)$, then

$$s = p - k + (\sum_{i=1}^{k} f_i(0))/2 - k - (\sum_{i=k+1}^{p} f_i(1)/2)$$

$$= p - 2k + [pr/2 - \sum_{i=k+1}^{p} f_i(0) - (\sum_{i=k+1}^{p} f_i(1)/2)] - (\sum_{i=k+1}^{p} f_i(1)/2)$$

$$= p - 2k + pr/2 - \sum_{i=k+1}^{p} f_i(0) + \sum_{i=k+1}^{p} f_i(1)$$

$= p - 2k + pr/2 - (p - k)r$, (since all vertices have the same degree r)

$$= p - 2k + pr/2 - pr + kr$$

$$= 2k(r/2 - 1) - p(r/2 - 1)$$

$$= (2k - p)(r/2 - 1)$$

If p is even, then G is strongly neutral. For we can let $k = p/2$.
If p is odd, then $2k - p \neq 0$. We consider the following cases:

Case 1. $r = 2$.
Then G is 2-regular which is strongly neutral.

Case 2. $r > 2$.
Then $|r/2 - 1| \neq 0$. Thus $s \neq 0$ and G is not strongly neutral.
For $|(p - 2k)(r/2 - 1)| = 1$, we must have $|p - 2k| = 1$ and $|r/2 - 1| = 1$.
Thus $k = (p - 1)/2$ and $r = 4$.

Corollary 2. *Every 2-regular graph (= a disjoint union of cycles) is strongly neutral.*

Remark. *It is not true that every regular graph of even order is cordial. Indeed, the generalized Petersen graph $P(n, k)$ where $n \geq 5$ and $1 \leq k < n$, has vertex set $V = \{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\}$ and edge set $E = \{\{a_i, a_{i+1}\}: i = 0, 1, \ldots, n-1\} \cup \{\{a_i, b_i\}: i = 0, 1, \ldots, n-1\} \cup \{\{b_i, b_{i+k}\}: i = 0, 1, \ldots, n-1\}$, where all subscripts are taken modulo n is not cordial for $n \neq 2 \pmod 4$ [4].*

We consider a class of regular graphs which are called *circulants* in [2], and which we refer to as step graphs and denoted by $S(n; a_1, a_2, \ldots, a_k)$, where

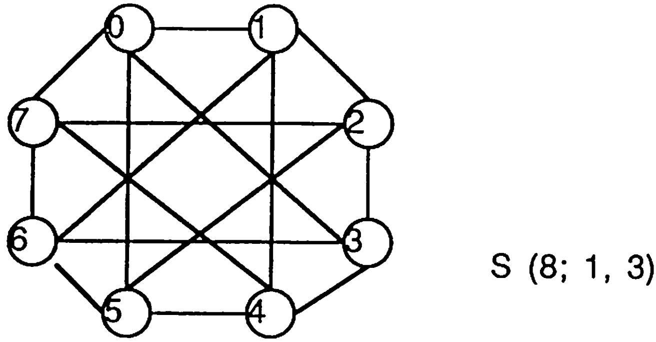

Figure 2. The step graph $S(8; 1, 3)$

$a_1, a_2, \ldots, a_k$ are some integers, and $1 \leq a_1 \leq a_2 \leq \cdots \leq \lfloor n/2 \rfloor$. We represent its vertex set by $\{0, 1, \ldots, n-1\}$. The edge set of $S(n; a_1, a_2, \ldots, a_k)$ is given by $\{\{u, v\}: u, v \text{ in } V(G), v - u = a_j \pmod{n} \text{ for } j = 1, 2, \ldots, k\}$. For example, see Figure 2.

We have the following results which are easy consequences of Theorem 1.

Corollary 3. *The step graph $S(n; a_1, a_2, a_3, \ldots, a_k)$ is neutral for all even n. If n is odd then it is neutral if and only if $k \leq 2$.*

Corollary 4. *Hypercube K_2^n is strongly neutral for all $n \geq 2$.*

Corollary 5. *A regular complete k-partite graph $K(n, n, \ldots n)$ is strongly neutral if either n is even or k is even.*

References

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