Lyndon Graphs are not Hamiltonian for n even

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Abstract. Lyndon graphs are connected subgraphs of the n-cube which arise in the combinatorics of words. It is shown that these graphs are not Hamiltonian when n is even.

Let $\Sigma = \{0, 1\}$ and let n be a positive integer. Then Σ^* is the set of all words over Σ and Σ^n is the set of words over Σ of length n. Σ^n can be made into a graph in the standard way yielding the n-cube.

It is well known that Σ^n is Hamiltonian. It is the purpose of this note to partially answer a question raised in [1] as to whether or not a certain subgraph of Σ^n is Hamiltonian. This subgraph arises in the combinatorics of words: a good reference for this theory is [4].

We first need to make a few definitions.

Definition 1. *Two words $x, y \in \Sigma^n$ are said to be conjugate if there exist words $u, v \in \Sigma^*$ such that $x = uv$ and $y = vu$.*

This notion of conjugacy defines an equivalence relation on Σ^n. In fact x is conjugate to y if and only if y can be obtained from x by a cyclic permutation of the letters in x, from which the result follows easily.

Definition 2. *A word $x \in \Sigma^*$ is said to be primitive if it is not the power of another word.*

If we consider the words in Σ^n as binary expansions we have a natural correspondence between Σ^n and the set $\{0, 1, 2, \ldots, 2^n - 1\}$. The total order relation on this set of integers can then be placed in Σ^n yielding what is known as the lexicographic order on Σ^n.

Definition 3. *A word $x \in \Sigma^n$ is called a Lyndon word if it is primitive and it is minimal in its conjugate class.*

The set of Lyndon words of length n is denoted by Λ_n. Figure 1 shows these graphs for the first few values of n. We have labelled the vertices with the base 10 representations of the corresponding Lyndon words.

It was shown in [1] that Λ_n is connected for all values of $n > 1$. Since Σ^n is bipartite it also follows that Λ_n is bipartite. It is well known that a necessary condition for a bipartite graph to be Hamiltonian is that the sizes of the two vertex sets in the vertex partition be the same. Moreover, in order for the graph to be semi-Hamiltonian it is necessary that the sizes of the two vertex sets differ by at most one. We will make use of these conditions to answer our question in the negative when n is even.

If $x \in \Sigma^n$ we let $w(x)$ be the number of 1's in the word x.

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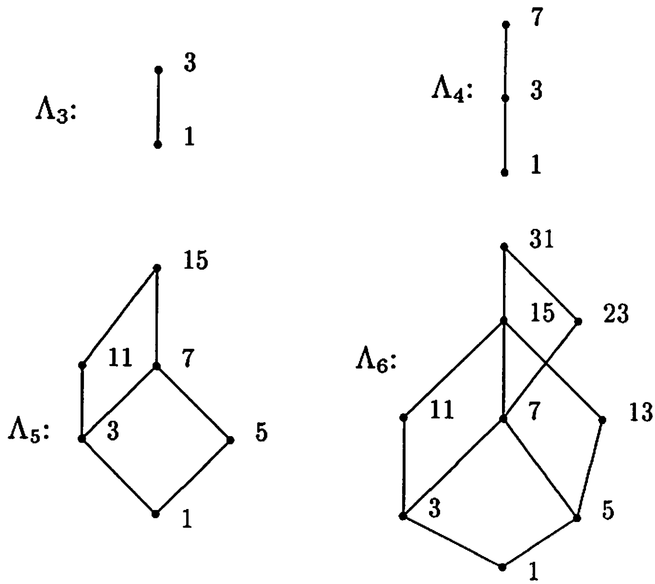

Figure 1

Lemma 1. *Let* $\lambda_{n,k} := |\{x \in \Lambda^n : w(x) = k\}|$. *Then*

$$\lambda_{n,k} = \frac{1}{n} \sum_{d|(k,n)} \mu(d) \binom{n/d}{k/d}$$

where μ *is the Mobius function [3].*

Proof: We first prove that

$$\binom{n}{k} = \sum_{d|(k,n)} \frac{n}{d} \lambda_{n/d, k/d}.$$

Let $x \in \Sigma^n$ satisfy $w(x) = k$.
Then $w = x^d$ for some x which is primitive ([4, p. 7]) with $d \mid (k, n)$.

Clearly x is conjugate to an element in $\Lambda_{n/d}$ with weight k/d. There are n/d such possibilities.

We now obtain the required result by Mobius inversion [3, p. 236]. ∎

Corollary 1. $\lambda_{n,k} = \lambda_{n,n-k}$ *for* $1 \leq k \leq n-1$.

Proof: This is a simple consequence of the symmetry of the binomial coefficient and the divisors. ∎

Corollary 2 [1]. $\lambda_{n,2} = \left\lceil \frac{n-2}{2} \right\rceil$.

Proof: If n is odd,

$$\lambda_{n,2} = \frac{1}{n}\left\{\binom{n}{2}\right\} = \frac{n-1}{2} = \left\lceil \frac{n-2}{2} \right\rceil.$$

If n is even,

$$\lambda_{n,2} = \frac{1}{n}\left\{\binom{n/2}{1}\right\} = \frac{1}{n}\left\{\frac{n(n-1)}{2} - \frac{n}{2}\right\} = \frac{n-2}{2} = \left\lceil \frac{n-2}{2} \right\rceil.$$

∎

Lemma 2. *If* $n \geq 2$,

$$\sum_{k=1}^{n-1} \lambda_{n,k} t^k = \frac{1}{n}\sum_{d|n} \mu(d)(1+t^d)^{n/d}. \tag{1}$$

Proof: From Lemma 1,

$$\sum_{k=1}^{n-1} \lambda_{n,k} t^k = \frac{1}{n}\sum_{k=1}^{n-1}\left\{\sum_{d|(k,n)} \mu(d)\binom{n/d}{k/d} t^k\right\}.$$

Now $\sum_{d|n} \mu(d) = 0$ for $n > 1$ so that

$$\sum_{k=1}^{n-1} \lambda_{n,k} t^k = \frac{1}{n}\sum_{k=1}^{n}\sum_{d|(k,n)} \mu(d)\binom{n/d}{k/d} t^k = \frac{1}{n}\sum_{d|n}\mu(d)\left\{\sum_{\substack{k=sd\\ s\geq 1}} \binom{n/d}{k/d} t^k\right\}$$

$$= \frac{1}{n}\sum_{d|n}\mu(d)\left\{(1+t^d)^{n/d} - 1\right\} = \frac{1}{n}\sum_{d|n}\mu(d)(1+t^d)^{n/d},$$

as required. ∎

Corollary 1[1]. $|\Lambda_n| = \frac{1}{n}\sum_{d|n}\mu(d)2^{n/d}$.

Proof: Immediate: just put $t = 1$ in equation (1). ∎

Corollary 2. *Let* $e_n = |\{x \in \Lambda^n : w(x)$ *is even*$\}|$ *and* $o_n = |\{x \in \Lambda^n : w(x)$ *is odd*$\}|$. *Then*

$$e_n - o_n = \frac{1}{n}\sum_{\substack{d \text{ even}\\ d|n}} \mu(d)2^{n/d}.$$

Proof: Put $t = -1$ in equation (1). ∎

We are now in a position to state and prove our result.

Theorem 1. *If n is an even integer ≥ 4 the Λ_n is not Hamiltonian. If n is an even integer ≥ 8 then Λ_n is not semi-Hamiltonian.*

Proof: From Figure 1, it is clear that Λ_4 and Λ_6 are semi-Hamiltonian, and that Λ_4 is not Hamiltonian. We easily compute that $o_6 - e_6 = 1$, $o_8 - e_8 = 2$ and $o_{10} - e_{10} = 3$. By the remarks made prior to Lemma 1, these facts show that Λ_n is not Hamiltonian for $n = 6, 8, 10$. So we can suppose $n \geq 12$.

Set $n = 2^t m$ with $t \geq 1$ and m odd. If $d \mid (k, n)$ then $d = 2^i \ell$ with $1 \leq i \leq t$ and ℓ odd. But $\mu(d) = 0$ if $i \geq 2$, so we obtain (using Corollary 2 of Lemma 2)

$$\frac{1}{n} \sum_{\substack{d \text{ even} \\ d|n}} \mu(d) 2^{n/d} = -\frac{1}{n} \sum_{\ell|m} \mu(\ell) 2^{n/2\ell} = -\frac{1}{n} \left\{ 2^{n/2} + \sum_{\substack{\ell|m \\ \ell \geq 3}} \mu(\ell) 2^{n/2\ell} \right\}.$$

Now

$$\left| \sum_{\substack{\ell|m \\ \ell \geq 3}} \mu(\ell) 2^{n/2\ell} \right| \leq \sum_{\substack{\ell=3 \\ \ell \text{ odd}}}^{m} 2^{n/2\ell} < \frac{m}{2} 2^{n/6} \leq \frac{n}{4} 2^{n/6},$$

since $m \leq n/2$. Hence $e_n - o_n \leq -\frac{1}{n}(2^{n/2} - \frac{n}{4} 2^{n/6}) \leq -2$, provided that $2^{n/2} - \frac{n}{4} 2^{n/6} \geq 2n$, which is true for $n \geq 10$ and the result is proved. ∎

The case of n being odd is not amenable to this attack, since it is easy to see that $e_n - o_n$ equals 0 in this case. We have managed to prove that Λ_7 is Hamiltonian, but the general case remains open.

Addendum.

After completing this paper I found out that Goldwasser had independently proved that Λ_n was not Hamiltonian, for n even, using auxiliary results of Cummings and Mays. His proof is included in [2].

References

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4. M. Lothaire, *Combinatorics on words*, in "Encyclopedia of Mathematics and its Applications", Vol 17, 1983.