On super-simple $2-(v,4,\lambda)$ designs

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Abstract. It is proven that for all $v \equiv 1 \bmod 3$, $v \geq 7$ there is a $2-(v,4,2)$ design whose blocks have pairwise at most two elements in common. Moreover, for $v \equiv 1,4 \bmod 12$ we have shown that these designs can be generated by two copies of $2-(v,4,1)$ designs.

1. Introduction.

A $t-(v,k,\lambda)$ design is a pair (V,B), where V is a v-set and B is a collection of (not necessarily distinct) k-subsets of V—called blocks—such that any t-subset of V is contained in exactly λ blocks. It is well known (see Hanani [2]) that $2-(v,4,\lambda)$ designs exist if and only if the usual necessary conditions, namely

$$\begin{aligned} \lambda(v-1) &\equiv 0 \bmod 3, \\ \lambda v(v-1) &\equiv 0 \bmod 12, \end{aligned} \tag{1}$$

are satisfied. The constructions of Hanani yield in several cases designs with repeated blocks. This raises the question of existence of $2-(v,4,\lambda)$ designs without repeated blocks, so called *simple* $2-(v,4,\lambda)$ designs. It has been shown that such designs exist for all admissible v (see [5]). Since a simple $2-(v,4,\lambda)$ design is balanced according to the 2-subsets of V only, the 3-subsets of V belonging to the blocks can have a large range of distinct distributions. This is illustrated in the complete enumeration of simple $2-(8,4,3)$ and $2-(8,4,6)$ designs, see Gronau and Reimer [1].

In this paper we go a step further and study *super-simple* $2-(v,4,\lambda)$ designs, that is, designs in which the intersection of any two blocks has at most 2 elements. Note that a $2-(v,3,\lambda)$ design is simple iff it is super-simple. It is easy to see that every $3-(v,4,1)$ design is a super-simple $2-(v,4,\frac{v-2}{2})$ design.

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Since it is known (see Hanani [3]) that $3-(v,4,1)$ designs exist if and only if $v \equiv 2$ or $4 \mod 6$, we obtain series of super-simple $2-(v,4,\lambda)$ designs, but unfortunately, with increasing λ's.

Let $B[4^*,\lambda]$ denote the set of orders v of super-simple $2-(v,4,\lambda)$ designs. Note that $1 \in B[4^*,\lambda]$. Of special interest are super-simple $2-(v,4,\lambda)$ designs which are the union of λ copies of $2-(v,4,1)$ designs. Let $B[4^*,\lambda \star 1]$ denote the set of those orders v. Obviously, $B[4^*,\lambda \star 1] \subseteq B[4^*,\lambda]$. Analogously, we call transversal and group divisible designs super-simple, if their blocks have no 3-subset in common.

Transversal designs $TD_\lambda(m,n)$, $TD(m,n) = TD_1(m,n)$, pairwise balanced designs, and group divisible designs are defined as in Hanani [4].

2. Main results.

We start with the following important

Lemma 2.1. *For any super-simple* $2-(v,k,\lambda)$ *design we have*

$$v \geq 2 + (k-2)\lambda.$$

Proof: Fix a 2-subset P of V and let $P_1, P_2, \ldots, P_\lambda$ denote the blocks containing P. Since the design is super-simple, the sets $P_1 - P, P_2 - P, \ldots, P_\lambda - P$ are mutually disjoint subsets of $V - P$, that is,

$$\lambda(k-2) = \sum_{i=1}^{\lambda} |P_i - P| = \left| \bigcup_{i=1}^{\lambda} (P_i - P) \right| \leq |V - P| = v - 2.$$

∎

Particularly for $k = 4$ and applying the necessary conditions (1) we obtain

Lemma 2.2. *For any super-simple* $2-(v,4,\lambda)$ *design we have*

$$v \geq \begin{cases} 2\lambda + 2 & \text{if } \lambda \not\equiv 2 \mod 3, \\ 2\lambda + 3 & \text{if } \lambda \equiv 2 \mod 3. \end{cases}$$

Note that the existence of $3-(2\lambda+2,4,1)$ designs (see above) implies $2\lambda + 2 \in B[4^*,\lambda]$.

Let B be a block of a $2-(v,k,\lambda)$ design, D. Then the *intersection numbers* n_i of D with respect to B are defined to be the number of blocks of D which meet B in exactly i elements.

By counting the total number of blocks and the occurrences of the elements and pairs of elements in the block B, the following equations are obtained:

$\sum_{i=0}^{k} n_i = b$ (where $b = \lambda(v-1)/k(k-1)$ is the total number of blocks);

$\sum_{i=0}^{k} i n_i = rk$ (where $r = \lambda(v-1)/(k-1)$ is the frequency of occurrence of each element in the blocks);

and

$$\sum_{i=0}^{k} \binom{i}{2} k = \lambda \binom{k}{2}.$$

By imposing the condition that $n_i = 0$ for $i \geq 3$, the intersection numbers are determined uniquely in case of super-simple $2-(v,4,\lambda)$ designs, as is noted below.

Lemma 2.3. *The intersection numbers $n_0, n_1, \ldots, n_4$ corresponding to a block are given by*

$$n_4 = 1,\ n_3 = 0,\ n_2 = 6(\lambda - 1),\ n_1 = \frac{4}{3}[\lambda v - 10\lambda + 6],$$
$$n_0 = \frac{1}{12}[\lambda v^2 - 17\lambda v + 88\lambda - 36].$$

The necessary conditions and Lemma 2.2 yield

$$B[4^*,2] \subseteq \{v: v \equiv 1 \bmod 3,\ v \neq 4\},$$
$$B[4^*,2 \star 1] \subseteq \{v: v \equiv 1 \text{ or } 4 \bmod 12,\ v \neq 4\}.$$

The main results of this paper are:

Theorem A. $B[4^*,2] = \{v: v \equiv 1 \bmod 3,\ v \neq 4\}$.

Theorem B. $B[4^*,2 \star 1] = \{v: v \equiv 1\ or 4 \bmod 12, v \neq 4\}$.

We will prove these theorems in Section 5, and Section 4, respectively.

3. Recursive constructions.

For recursive constructions the following is very useful.

Theorem 3.1. *Let $m \geq 3$, $m \neq 6$. A super-simple $TD_\lambda(4,m)$ exists iff $\lambda \in \{1,2,\ldots,m\}$. For any of these λ's such a transversal design can be constructed as a union of λ $TD(4,m)$ designs.*

Proof: Using the idea of the proof of Lemma 2.1 it follows that $\lambda \leq m$. If $m \geq 3$, $m \neq 6$ it is well known that there is a transversal design T (see [4]) with $\lambda = 1$, 4 groups of size m each and blocks of size 4. Let $G_i = \{(i,j): j = 0,1,\ldots,m-1\}, i = 0,1,2,3$, be the four groups. Let π be the cyclic permutation $(0\ 1\ 2\ \ldots\ (m-1))$. We apply π to T by saying that π acts exactly on the 3rd and 4th group. So, $T, \pi T, \pi^2 T, \ldots, \pi^{m-1} T$ are mutually super-simple. Indeed, assume the contrary, that is assume that $\pi^\alpha T$ and $\pi^\beta T$ $(\alpha < \beta)$ have two blocks S_α and S_β, respectively, with $|S_\alpha \cap S_\beta| \geq 3$. Then two elements of $S_\alpha \cap S_\beta$

belong to the groups 1 and 2 or to the groups 3 and 4. Without loss of generality, we assume the first case, that is,

$$S_\alpha = ((0,a),(1,b),(2,c),(3,d))$$
$$S_\beta = ((0,a),(1,b),(2,e),(3,f))$$

Since $\beta > \alpha$ our construction yields $c \neq e$ and $d \neq f$, that is,

$$|S_\alpha \cap S_\beta| = 2$$

contradicting our assumption. ∎

Corollary 3.1.1. *Let* $m \geq 3, m \neq 6$.

1) *If* $m \in B[4^*, \lambda \star 1]$, *then* $4m \in B[4^*, \lambda \star 1]$.
2) *If* $m + 1 \in B[4^*, \lambda \star 1]$, *then* $4m + 1 \in B[4^*, \lambda \star 1]$.
3) *If* $m \in B[4^*, \lambda]$, *then* $4m \in B[4^*, \lambda]$.
4) *If* $m + 1 \in B[4^*, \lambda]$, *then* $4m + 1 \in B[4^*, \lambda]$.

Proof: The proof is immediate for 1) and 3). For 2) and 4), the result is obtained by adjoining a new point, say x, to each of the groups. ∎

Theorem 3.2. *Let* $m \geq 4$, $m \neq 6, m \neq 10$ *and* $0 \leq n \leq m$ *be integers. Then there exists a super-simple group divisible design with* $\lambda = 2$ *with block size* 4*,* 4 *groups of size* $3m$ *and one group of size* $3n$*. Such a design can be constructed as the union of two group divisible designs with the same block and group sizes, but with* $\lambda = 1$.

Proof: Start with a transversal design $TD(5, m)$ and $\lambda = 1$. It is well known that these designs exist if $m \geq 4, m \neq 6$, $m \neq 10$. Assign to $m - n$ points of the last group weight 0, to all other points weight 3. Apply the fundamental construction of Wilson [7] and use as the ingredient designs

(i) for the blocks of size 4 just a super-simple $TD_2(4,3)$ of Theorem 3, which can be constructed as the union of two $TD(4,3)$ and
(ii) for the blocks of size 5 the following two group divisible designs T_1 and T_2 with $\lambda = 1$, block size 4 and 5 groups of size 3 each:

$$T_1 = \{((0,0),(1,1),(2,1),(3,0)) \bmod (5,3)\}$$
$$T_2 = \{((0,0),(1,2),(2,2),(3,0)) \bmod (5,3)\}.$$

It is easy to check that these two designs have the desired properties. ∎

Corollary 3.2.2. *Let* $m \geq 4, m \neq 6, m \neq 10$ *and* $0 \leq n \leq m$.

1) *If* $3m + 1 \in B[4^*, 2 \star 1]$ *and* $3n + 1 \in B[4^*, 2 \star 1]$, *then* $12m + 3n + 1 \in B[4^*, 2 \star 1]$
2) *If* $3m + 1 \in B[4^*, 2]$ *and* $3n + 1 \in B[4^*, 2]$, *then* $12m + 3n + 1 \in B[4^*, 2]$.

4. Proof of Theorem B.

In this section, we examine $B[4^*, 2 \star 1]$.

Lemma 4.1. $\{13, 16, 25, 28, 37, 40\} \subset B[4^*, 2 \star 1]$.

Proof: For these values, we display two $1-(v, 4, 2)$ designs T_1 and T_2, which have the property that each block of T_1 meets any block of T_2 in at most two points.

v Designs

13 $T_1 = \{(0,1,3,0) \bmod 13\}$,
$T_2 = \{(0,1,5,11) \bmod 13\}$

16 $T_1 = \{(1,2,3,4),(1,5,10,14),(1,6,12,13),(1,7,9,15),(1,8,11,16),$
$(5,6,7,8),(2,6,11,15),(2,5,9,16),(2,8,12,14),(2,7,10,13),$
$(9,10,11,12),(3,7,12,16),(3,8,10,15),(3,5,11,13),(3,6,9,14),$
$(13,14,15,16),(4,8,9,13),(4,7,11,14),(4,6,10,16),(4,5,12,15)\}$.
$T_2 = T_1(0\ 4\ 12\ 14\ 3\ 13\ 8\ 6\ 9)(1\ 5\ 2\ 10\ 7\ 11)$

25 $X = GF(25, x^2 = 2x + 2)$
$T_1 = \{0, x^0, x^8, x^{16}), (0, x^2, x^{10}, x^{18}) \bmod 25\}$
$T_2 = \{(0, -x^0, -x^8, -x^{16}), (0, -x^2, -x^{10}, -x^{18}) \bmod 25\}$

28 $X = Z(3,2) \times GF(9, x^2 = 2x + 1) \cup \{\infty\}$
$T_1 = \{((0,x^0),(0,x^4),(0,x^2),(0,x^6)),$
$((0,x^1),(0,x^5),(1,x^3),(1,x^7)) \bmod (3,9)\}$
$\cup\{((0,0),(1,0),(2,0),\infty) \bmod (-,9)\}$
$T_2 = T_1(0\ 5\ 11\ 14\ 16\ 19\ 4\ 24\ 18\ 12\ 8\ 22\ 10\ 3\ 13\ 7\ 27\ 15\ 9\ 1\ 26\ 17)$
$(2\ 23\ 25)(20\ 21)$,

where these numbers correspond to the elements as follows

elements	$(0,0)$	$(0,1)$	$(0,2)$	$(0,x)$	$(0,x+1)$	$(0,x+2)$	$(0,2x)$	$(0,2x+1)$	$(0,2x+2)$	∞
number	0	1	2	3	4	5	6	7	8	28

the elements $(1,y)$ resp. $(2,y)$ correspond to the
(number of $(0,y)$ +9 resp. (number of $(0,y)$) +18; $y \in GF(9)$.

37 $T_1 = \{(0,1,13,30),(0,2,23,34),(0,4,10,19) \bmod 37\}$
$T_2 = \{0,1,8,25),(0,2,5,16),(0,4,22,31) \bmod 37\}$

40 $T_1 = \{0,10,20,30)$(one-quarter orbit)$,(0,1,26,32),(0,7,19,36),$
$(0,3,16,38) \bmod 40\}$
$T_2 = T_1(0\ 33\ 8\ 1\ 25\ 24\ 17\ 9\ 32)(2\ 19\ 35\ 27\ 11)(3\ 26\ 18\ 10\ 34)$
$(4\ 13,\ 28\ 20\ 36)(7\ 30)(12\ 37)(14\ 31\ 39\ 22\ 23\ 15)$

The check of the super-simple property was done by a computer program. ∎

Lemma 4.2. *If* $v \equiv 1$ *or* 4 (mod 12) *and* $49 \leq v \leq 205$, *then* $v \in B[4^*, 2 \star 1]$.

Proof: For $v \in \{52, 64, 100, 148\}$ we use Corollary 3.1.1, part 1), that is, $v = 4w$ where $w \in B[4^*, 2 \star 1]$.

For $v \in \{49, 61, 73, 76, 97, 109, 112, 121, 124, 133, 136, 145, 157, 160, 169, 172, 181, 184, 193, 196, 205\}$ we use Corollary 3.2.2 part 1) according to the following table.

v	m	n	v	m	n
49	4	0	145	12	0
61	5	0	157	12	4
73	5	4	160	12	5
76	5	5	169	12	8
97	8	0	172	12	9
109	8	4	181	12	12
112	8	5	184	13	9
121	8	8	193	13	12
124	9	5	196	13	13
133	9	8	205	16	4
136	9	9			

This leaves the cases of $v = 85$ and $v = 88$: For the case of $v = 85$, we use a pairwise balanced design on 22 points which contains one block of size 7 and all other blocks of size 4, which we will denote by PBD$[\{4, 7^*\}, 22]$. (Such a design is easily constructed by adjoining 7 "new" points to a resolvable $2-(15,3,1)$ design.) Delete a point which occurs on the block of size 7 to obtain a group divisible design, say D, with 5 groups of size 3, and one group of size 6, that is, group type $3^5 6^1$, and all blocks of size 4. Form two group divisible designs D_1 and D_2 of group type $12^5 24^1$ by inflating each point by a factor of 4 (in accordance with Wilson's Fundamental Construction [7]), as in the proof of Theorem 3.2.

Adjoin a new point w to each group of both D_1 and D_2 to obtain D_1' and D_2'. Now let E_1 and E_2 be a pair of mutually super-simple $2-(25,4,1)$ designs. Replace B_1, the block of size 25 in D_1' by a copy of E_1 on the points of B_1. Since B_1' is also in D_2, we can replace it by a copy of E_2 on the same set of points. Since there is also a pair of mutually super-simple $2-(13,4,1)$ designs, we can similarly "break up" the blocks of size 13 in D_1' and D_2', to obtain a pair of mutually super-simple $2-(85,4,1)$ designs. Therefore, $85 \in B[4^*, 2 \star 1]$.

For $v = 88$, we proceed as follows. An examination of the $2-(28,4,1)$ design exhibited in [6, page 7(v)] shows that this design admits two resolution classes which meet precisely in a block. Let $R_1: B_1, B_2, \ldots, B_7$ and $R_2: B_1, B_2^*, \ldots, B_7^*$

be these resolution classes. Adjoin a new point $[\infty]$ to every block of R_2. Then $B_1 \cup \{\infty\}, B_2, B_3, \ldots, B_7$ can be viewed as groups of a group divisible design, say D, with group type $5^1 4^6$ and blocks of size 4 and 5. Form two group divisible designs D_1 and D_2 of group type $15^1 12^6$ by inflating each point by a factor of 3 (in accordance with Wilson's Fundamental Construction), as in the proof of Theorem 3.2. Adjoin a new point to the groups in each of D_1 and D_2 and substitute pairs of super-simple designs of sizes 13 and 16 for these blocks as in the previous case. ▮

We now prove the second main theorem.

Theorem B. *If* $v \equiv 1$ *or* 4 mod 12, *and if* $v \neq 4$, *then* $v \in B[4^*, 2 \star 1]$.

Proof: If $v \leq 205$, the result follows from Lemma 4.1 and Lemma 4.2. For $v > 205$, we proceed as follows.

Let $w \equiv 1$ or 4 mod 12, $w \geq 49$, then, by Corollary 3.2.2, part 1, we can construct the desired designs (at least) of the orders belonging to $W(w) = \{4(w-1)+13, 4(w-1)+16, \ldots, 4(w-1)+49\}$, which are just 7 consecutive numbers of the type 1 or 4 mod 12. Since $4(w-1)+49 = 4((w+9)-1)+13$ and the maximal gap between two consecutive numbers of type 1 or 4 mod 12 has length 9, $\bigcup_{w \geq 49} W(w)$ covers all remaining orders. ▮

5. Proof of Theorem A.

For $v \equiv 1$ or 4 mod 12, Theorem B implies Theorem A. Therefore, it is only necessary to prove the result for $v \equiv 7$ or 10 mod 12.

Lemma 5.1. *If* $v \in \{7, 10, 19, 22, 31, 34, 43, 46, 79, 82\}$, *then* $v \in B[4^*, 2]$.

Proof: For these values, we use direct constructions for super-simple designs.

v	Design
7	$\{(0,1,2,4) \bmod 7\}$
10	Every 2-(10,4,2) design is a solution, since the intersection numbers include always $n_3 = 0$, for example, $\{((0,0),(0,1),(0,2),(1,4)),((0,0),(1,0),(1,1),(1,3)),((0,0),(0,2),(1,1),(1,2)) \bmod (-,5)\}$
19	$\{(0,1,2,6),(0,2,8,11),(0,3,7,12) \bmod 19\}$
22	$\{((0,0),(0,3),(0,9),(0,10)),((0,0),(1,0),(1,2),(1,7)),((0,0),(1,0),(1,9),(1,10)),((0,0),(0,2),(1,5),(1,8)),((0,0),(0,3),(1,4),(1,7)),((0,0),(0,4),(1,3),(1,9)),((0,0),(0,5),(1,2),(1,6)) \bmod (-,11)\}$

31 $\{(0,1,2,4),(0,3,8,18),(0,4,13,20),(0,5,12,22),(0,6,12,20) \bmod 31\}$

34 $\{(\infty,(0,0),(0,1),(0,2))\ (\infty,(0,0),(1,1),(2,2))\ ((0,0),(1,0),(2,1),(9,1))\ ((0,0),(1,0),(3,1),(0,1))\ ((0,0),(2,0),(5,1),(6,1))\ ((0,0),(2,0),(7,1),(9,1))\ ((0,0),(4,0),(8,1),(10,1))\ ((0,1),(3,1),(0,2),(6,2))\ ((0,1),(4,1),(1,2),(2,2))\ ((0,1),(5,1),(3,2),(4,2))\ ((0,1),(5,1),(7,2),(10,2))\ ((0,1),(10,1),(4,2),(6,2))\ ((6,0),(10,0),(0,2),(3,2))\ ((0,0),(5,0),(0,2),(4,2))\ ((2,0),(8,0),(0,2),(4,2))\ ((2,0),(10,0),(0,2),(5,2))\ ((1,0),(4,0),(0,2),(9,2)) \bmod (11,-)\}$

43 $\{(0,1,6,36),(0,1,7,17),(0,2,14,34),(0,2,16,33),(0,3,8,27),(0,3,18,22),(0,4,23,32) \bmod 43\}$

46 Delete a point from a $2-(16,4,1)$ design to obtain a group divisible design with five groups of size 3. Use Wilson's Fundamental Construction [7] to inflate by a factor of 3 using a super-simple $TD_2(4,3)$ to obtain a super-simple GDD with five groups of size 9. To each group G adjoin a new point ∞, and replace $G \cup \{\infty\}$ with the blocks of a super-simple 2-(10,4,2)*design.Theresultistherequiredsuper* $-$ *simple*2$-(46,4,2)$ design.

79 $\{(0,1,23,55),(0,1,24,61),(0,2,31,46),(0,2,43,48),(0,3,7,69),(0,3,25,72),(0,4,13,62),(0,5,16,43),(0,6,14,59),(0,6,50,65),(0,8,26,45),(0,9,21,49),(0,12,28,39) \bmod 79\}$

82 Proceed as follows. Delete one point from a $2-(28,4,1)$ design to obtain a group divisible design with nine groups of size 3, and blocks of size 4. Use Wilson's Fundamental Construction [7] to inflate the above group divisible design by a factor of 3 using a super-simple $TD_2(4,3)$ to obtain a super-simple GDD with nine groups of size nine. Adjoin a new point ∞ to each group G, and replace $G \cup \{\infty\}$ with the blocks of super-simple $2-(10,4,2)$ design. The result is the required super-simple $2-(82,4,2)$ design. ∎

Lemma 5.2. *If* $v \in \{55,58,67,70,91,94,103,106,115,118,127,130,139\}$, *then* $v \in B[4^*,2]$.

Proof: Employ Corollary 3.2.2 part 2) in accordance with the following table

v	m	n	v	m	n
55	4	2	106	8	3
58	4	3	115	8	6
67	5	2	118	8	7
70	5	3	127	9	6
91	7	2	130	9	7
94	7	3	139	11	2
103	7	6			

This completes the lemma. ∎

We now prove the first main theorem.

Theorem A. *If* $v \equiv 1 \bmod 3$, *and* $v \neq 4$, *then* $v \in B[4^*, 2]$.

Proof: If $v \leq 139$, the result follows from Lemma 5.1 and Lemma 5.2. For $v > 139$, we proceed as follows. Let $w \equiv 1 \bmod 3$, $w \geq 34$. Then by Corollary 3.2.2, part 2), we can construct the desired designs of the orders belonging to $W(w) = \{4(w-1)+7, 4(w-1)+10, \ldots, 4(w-1)+19\}$, which are just 5 consecutive numbers of type 1 mod 3. Since $4(w-1)+19 = 4((w+3)-1)+7$ and the gap between two consecutive numbers of type 1 mod 3 has length 3, $\bigcup_{w \geq 34} W_{(w)}$ covers all of the remaining orders. ∎

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