Tolerance and Weak Tolerance Relations

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1. Introduction

It was an attempt to incorporate the imperfection of human sight into a mathematical model of visual perception that induced E.C. Zeeman to define a tolerance [190]. He defined it as a reflexive and symmetric binary relation, which generalizes an equivalence by abandoning the transitivity condition. The concept of tolerance relation is so simple that it has been rediscovered many times, each time under a different name: compatibility relation, similarity relation ... Therefore we can expect that many results on tolerance are hidden under different names and contained in papers on difficult-to-predict subjects. Certainly we can expect some overlap between tolerance theory and graph theory as a graph can be considered to be a slight modification of a tolerance. For instance all results about cliques and maximal cliques in graphs can be considered as a part of tolerance theory.

In spite of its apparent generality, the various aspects of a tolerance provide a surprisingly extensive field of investigation. We are inclined to accept the opinion that a tolerance is not so much a generalization of an equivalence as an equivalence is a trivialization of a tolerance.

Although this paper is intended to study only some particular properties of tolerance, it contains a fairly extensive and self-contained exposition of tolerance theory. The previous expositions were written as a background for particular problems (automata theory [3, 4], nontrivial topology on discrete structures [173], finite tolerance structures [188], linguistics [169], generalization of congruence [28, 157]) which are far from our interests. Only a few of them (mostly algebraic) were published in readily available, strictly mathematical literature. Moreover although we use the conceptual base of these expositions, ours differs essentially from them in that it is built on the elements of dependence system theory.

We will begin with some basic notions and notation of binary relation algebras. This will make the statements of our exposition much simpler. We will not use the more advanced theory of relation algebras. Most of the proofs are straightforward, and will be omitted.

Section 3 provides the preliminaries of dependence systems to the extent necessary for a description of tolerance relations. Only the rudiments of the theory are given, again without proofs.

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The fourth section contains a description of a tolerance relation and a discussion of preclasses, classes and bases of a tolerance. We correct a major error in one of the earlier expositions.

In Section 5 we consider a weakening of a tolerance relation in which reflexivity is replaced by a more general condition. Then in the sixth section we compare the notions of weak tolerance and abstract orthogonality relation. They turn out to be complementary relations.

Section 7 contains various constructions of tolerance relations and results characterizing when a given tolerance may arise via a particular construction.

Finally we provide a bibliography of tolerance relations, probably the most comprehensive of all available, but still far from complete.

2. Basic facts about binary relation algebras

Definition 2.1: Let $\mathcal{R}(S)$ be the set of all binary relations on a set S, i.e. the power set of the Cartesian square of the set S. Then $\mathcal{R}(S)$ can be equipped with an algebraic structure: $\langle \mathcal{R}(S), \vee, \wedge, 0, 1, {}^c, \circ, E, {}^* \rangle$ of type $\langle 2,2,0,0,1,2,0,1 \rangle$ (the numbers in the brackets give the "arities" of the algebra operations). The first five operations define the Boolean algebra of set operations on the subsets of $S \times S$, while the last three define an involutorial monoid structure, where the binary operation is the composition of relations, the nullary operation E is the equality relation on S, the unity for composition, and the unary operation is the converse operation for relations, playing the role of an involution. For all relations $R, T \in \mathcal{R}(S)$:

(i) $R \vee T = \{\langle a,b \rangle : \langle a,b \rangle \in R$ or $\langle a,b \rangle \in T\}$,
(ii) $R \wedge T = \{\langle a,b \rangle : \langle a,b \rangle \in R$ and $\langle a,b \rangle \in T\}$,
(iii) $R^c = \{\langle a,b \rangle : \langle a,b \rangle \notin R\}$,
(iv) $1 = \{\langle a,b \rangle : a \in S$ and $b \in S\}$,
(v) $0 = \emptyset$,
(vi) $RT = R \circ T = \{\langle a,b \rangle : \exists c \in S : \langle a,c \rangle \in R$ and $\langle c,b \rangle \in T\}$,
(vii) $E = \{\langle a,b \rangle : a = b\}$,
(viii) $R^* = \{\langle a,b \rangle : \langle b,a \rangle \in R\}$.

As a Boolean algebra, $\mathcal{R}(S)$ is partially ordered by inclusion i.e. $R \leq T$ *iff* $\forall a,b \in S$, $\langle a,b \rangle \in R \Rightarrow \langle a,b \rangle \in T$. We will use the symbol $\mathcal{R}(S)$ both for the set of all binary relations and for the relation algebra defined on this set if no confusion is likely.

Proposition 2.2: The reduct of algebra $\mathcal{R}(S)$ to the last three operations is an ordered monoid with respect to the ordering of relations by inclusion. Moreover it is a residuated lattice-monoid: $\forall R, T \in \mathcal{R}(S)$, $\exists V, W \in \mathcal{R}(S)$ such that

(i) $\forall U \in \mathcal{R}(S), TU \leq R$ *iff* $U \leq V$, and
(ii) $\forall U \in \mathcal{R}(S), UT \leq R$ *iff* $U \leq W$.

Indeed V and W are given by:

$$V = R.\,{}^{\cdot}T = \{\langle a,b\rangle : \forall x \in S,\ \langle x,a\rangle \in T \Rightarrow \langle x,b\rangle \in R\} = (T^{*}R^{c})^{c},$$
$$W = R^{\cdot}.T = \{\langle a,b\rangle : \forall x \in S,\ \langle b,x\rangle \in T \Rightarrow \langle a,x\rangle \in R\} = (R^{c}T^{*})^{c}.$$

Proposition 2.3: The converse operation in $\mathcal{R}(S)$ has the following properties:

(i) $R^{**} = R$;
(ii) $(RT)^{*} = T^{*}R^{*}$;
(iii) $(R \vee T)^{*} = R^{*} \vee T^{*}$;
(iv) $(R \wedge T)^{*} = R^{*} \wedge T^{*}$;
(v) $R^{c*} = R^{*c}$;
(vi) $E = E^{*}$.

Proposition 2.4: The standard properties of relations can be expressed in terms of relation algebras in the following way: $\forall R \in \mathcal{R}(S)$,

(i) R is symmetric *iff* $R^{*} = R$,
(ii) R is transitive *iff* $R^{2} \leq R$,
(iii) R is antisymmetric *iff* $R \wedge R^{*} \leq E$,
(iv) R is reflexive *iff* $E \leq R$,
(v) R is a quasiorder *iff* $E \leq R$ and $R^{2} \leq R$,
(vi) R is a partial order *iff* $R \wedge R^{*} = E$ and $R^{2} \leq R$,
(vii) R is a linear partial order *iff* $R \wedge R^{*} = E$ and $R^{2} \leq R$ and $R^{c2} \leq E^{c}$,
(viii) R is an equivalence *iff* $E \leq R^{2} = R^{*} = R$,
(ix) R is a function *iff* $R^{*}R \leq E$ and $E \leq RR^{*}$,
(x) R is a surjective function *iff* $R^{*}R \doteq E$ and $E \leq RR^{*}$,
(xi) R is an injective function *iff* $R^{*}R \leq E$ and $E = RR^{*}$,
(xii) R is a bijective function *iff* $R^{*}R = RR^{*} = E$,
(xiii) R is a tolerance relation *iff* $E \leq R = R^{*}$.

The following proposition gives examples of properties which characterize the operations in a relation algebra $\mathcal{R}(S)$, but in some cases the statements remain true in much more general structures.

Proposition 2.5: $\forall R, T, U \in \mathcal{R}(S)$:

(i) $R \leq RR^{*}R$,
(ii) $(R^{\cdot}.T)(T^{\cdot}.U) \leq R^{\cdot}.U$,
(iii) $(R.\,{}^{\cdot}T)(U.\,{}^{\cdot}R) \leq U.\,{}^{\cdot}T$,
(iv) $(R^{\cdot}.T)^{*} = R^{*}.\,{}^{\cdot}T^{*}$,
(v) $(R.\,{}^{\cdot}T)^{*} = R^{*\cdot}.T^{*}$,
(vi) $E \leq R$ *iff* $R^{\cdot}.R \leq R$ *iff* $R.\,{}^{\cdot}R \leq R$,
(vii) $R \leq T$ *iff* $E \leq T^{\cdot}.R$ *iff* $E \leq T.\,{}^{\cdot}R$,
(viii) $R \leq T \Rightarrow (R^{\cdot}.T)^{2} \leq R^{\cdot}.T$,
(ix) $R \leq T \Rightarrow (R.\,{}^{\cdot}T)^{2} \leq R.\,{}^{\cdot}T$,
(x) $(R^{\cdot}.R)^{\cdot}.(R^{\cdot}.R) = R^{\cdot}.R$,

Definition 2.6: Let $R \in \mathcal{R}(S)$. Define: $\forall a \in S, R(a) = \{x \in S : \langle a, x\rangle \in R\}$. Thus $\langle a, b\rangle \in R$, aRb, $b \in R(a)$, $a \in R^*(b)$ are equivalent.

Proposition 2.7: The operations on relations in $\mathcal{R}(S)$ can be equivalently defined in the following way: $\forall a, b \in S$:

(i) $a(R \wedge T)b$ *iff* $b \in R(a) \cap T(a)$,
(ii) $a(R \vee T)b$ *iff* $b \in R(a) \cup T(a)$,
(iii) aR^cb *iff* $b \in [R(a)]^c$,
(iv) $aRTb$ *iff* $R(a) \cap T^*(b) \neq \emptyset$,
(v) $a(R^{\cdot}.T)b$ *iff* $T(b) \subseteq R(a)$,
(vi) $a(R.^{\cdot}T)b$ *iff* $T^*(a) \subseteq R^*(b)$.

Definition 2.8: The *left* (resp. *right*) *domain* of relation $R \in \mathcal{R}(S)$ is the set: $D(R) = \{x \in S : R(x) \neq \emptyset\}$ (resp. $D^*(R) = \{x \in S : R^*(x) \neq \emptyset\}$).

Definition 2.9: A *relational structure* $\langle S, R\rangle$ is a set S together with a binary relation R on S. When R is a tolerance relation, $\langle S, R\rangle$ is called a *tolerance space.*

Definition 2.10: Given two relational structures $\langle S_1, R_1\rangle$ and $\langle S_2, R_2,\rangle$ a *relational structure homomorphism* is a map $\phi : S_1 \rightarrow S_2$ such that $\forall x, y \in S_1$, $xR_1y \Rightarrow \phi(x)R_2\phi(y)$. If in addition, ϕ is a bijection and ϕ^{-1} is also a relational structure homomorphism, it is called a *relational structure isomorphism.* If R_1 and R_2 are tolerances, ϕ is called a *tolerance homomorphism* (resp. *isomorphism*).

Definition 2.11 [R9]: Given a relational structure $\langle S, R\rangle$, there is an *induced equivalence relation* N on S defined by: $\forall x, y \in S$, xNy *iff* $R(x) = R(y)$ and $R^*(x) = R^*(y)$. For each $x \in S$, let $[x]$ denote the equivalence class of x; let S/N denote the set of equivalence classes. The *quotient relational structure* $\langle S/N, \bar{R}\rangle$ is defined by: $\forall [x], [y] \in S/N$, $[x]\bar{R}[y]$ *iff* xRy. Note that $\bar{R}$ is well defined. When R is a tolerance relation on S, the quotient relational structure will be called the *reduced tolerance space.* If $\langle S_1, R_1\rangle$ and $\langle S_2, R_2\rangle$ are relational structures and $\phi : S_1 \rightarrow S_2$ is a relational structure homomorphism, the *relational structure induced by* ϕ is $\langle S_1, R_\phi,\rangle$ where $\forall x, y \in S_1$, $xR_\phi y$ *iff* $\phi(x)R_2\phi(y)$. Clearly, $R_1 \leq R_\phi$. Moreover, if R_2 is a tolerance relation (resp. equivalence relation), then R_ϕ is also a tolerance relation (resp. equivalence relation); in this case, ϕ is a tolerance homomorphism of $\langle S_1, R_\phi\rangle \rightarrow \langle S_2, R_2.\rangle$

Proposition 2.12: Let $\langle S_1, R_1\rangle$ and $\langle S_2, R_2\rangle$ be relational structures and $\phi : S_1 \rightarrow S_2$ a surjective relational structure homomorphism. Let $\langle S_1, R_\phi\rangle$ be the relational structure induced by ϕ and N_ϕ the equivalence relation induced by R. Define $\bar{\phi} : \langle S_1/N_\phi, \bar{R}_\phi\rangle \rightarrow \langle S_2/N_2, \bar{R}_2\rangle$ by $\bar{\phi}([x]_{N_\phi}) = [\phi(x)]_{N_2}$. Then $\bar{\phi}$ is a well-defined relational isomorphism.

Proof: First we verify that $\bar{\phi}$ is well-defined. Suppose $xN_\phi y$. Then $R_\phi(x) = R_\phi(y)$ and $R^*_\phi(x) = R^*_\phi(y)$. Since ϕ is surjective, this implies $R_2(\phi(x)) = R_2(\phi(y))$ and $R_2^*(\phi(x)) = R_2^*(\phi(y))$, i.e., $\phi(x)N_2\phi(y)$.

Next, suppose $[x]_{N_\phi} \tilde{R}_\phi [y]_{N_\phi}$. Then $x R_\phi y$, hence $\phi(x) R_2 \phi(y)$ and $[\phi(x)]_{N_2} \tilde{R}_2 [\phi(y)]_{N_2}$. Thus $\tilde{\phi}$ is a relational structure homomorphism. Trivially, since ϕ is surjective, $\tilde{\phi}$ is also. If $\tilde{\phi}([x]_{N_\phi}) = \tilde{\phi}([y]_{N_\phi})$, then $[\phi(x)]_{N_2} = [\phi(y)]_{N_2}$, i.e., $R_2(\phi(x)) = R_2(\phi(y))$ and $R_2^*(\phi(x)) = R_2^*(\phi(y))$. This implies $R_\phi(x) = R_\phi(y)$ and $R_\phi^*(x) = R_\phi^*(y)$. Hence $\tilde{\phi}$ is bijective. Trivially, $\tilde{\phi}^{-1}$ is also a relational structure homomorphism.

3. Basic concepts of dependence system theory

There are some similarities in the formal structures of different mathematical theories such as topology, combinatorics, logical consequence theory, general algebra. In all these theories we find, as one of the fundamental notions, some closure operation, which in each case differs in its secondary properties and which sometimes goes by a different name. But no matter what is its name: closure, matroid, consequence operator, generating operator of an algebra, a *closure operator* can be defined as a mapping f of the power set of some set S into itself satisfying the three fundamental conditions: for each $A, B \subseteq S$:

(i) $A \subseteq f(A)$ (f is extensive),
(ii) $A \subseteq B \Rightarrow f(A) \subseteq f(B)$ (f is monotone),
(iii) $f[f(A)] = f(A)$ (f is transitive).

Any structure defined by the first and second conditions alone is called a *dependence system*. In that case, f is called an *operator*.

It is a well-known fact that in topology there is a bijective correspondence between transitive closure operators on a given set S and Moore families (families of subsets of S with S as identity element and closed under arbitrary intersections). The Moore family associated with a given closure operator f is just the set of f-closed subsets of S : $\mathbf{f}\text{-}\mathbf{Cl} = \{A \subseteq S : A = f(A)\}$. Conversely, given a Moore family, for each $A \subseteq S$, $f(A)$ is the intersection of all members of this family containing A as a subset.

In what follows, $|A|$ denotes the cardinality of set A, $|A| < \omega$ means that A is of a finite cardinality, $A^c = S \backslash A$ where S is the universe of the discourse set.

Let f be an operator on a set. We define some distinctive classes of subsets of S:

(i) $A \in \mathbf{f}\text{-}\mathbf{Ind} \subseteq 2^S$ *iff* $\forall x \in A, x \notin f(A \backslash \{x\})$ and we say: A is f-*independent* or simply *independent* if no confusion is likely.
(ii) $A \in \mathbf{f}\text{-}\mathbf{Gen} \subseteq 2^S$ *iff* $f(A) = S$ and we call A an f-*generating* set or simply a *generating* set.
(iii) $A \in \mathbf{f}\text{-}\mathbf{Cl} \subseteq 2^S$ *iff* $f(A) = A$ and we call A an f-*closed* set or simply a *closed* set.
(iv) $A \in \mathbf{f}\text{-}\mathbf{Base} \subseteq 2^S$ *iff* $A \in \mathbf{f}\text{-}\mathbf{Ind} \cap \mathbf{f}\text{-}\mathbf{Gen}$ and we call A an f-*base* or simply a *base*.

Suppose $M \subseteq S$. Define the *restriction* of an operator f to M by: $\forall A \subseteq M$, $f_M(A) = f(A) \cap M$. Certainly f_M is an operator. Moreover if $f(M) = M$, then $\forall A \subseteq M, f_M(A) = f(A)$. Therefore for closed M : $\mathbf{f_M}$**-Cl** = **f-Cl** $\cap 2^M$, $\mathbf{f_M}$**-Gen** $= \{A \subseteq M : f(A) = M\}$, $\mathbf{f_M}$**-Ind** = **f-Ind** $\cap 2^M$, and $\mathbf{f_M}$**-Base** = $\mathbf{f_M}$**-Ind** $\cap$ $\mathbf{f_M}$**-Gen** $= \{A \subseteq M : \forall x \in A, x \notin f(A \backslash \{x\})$ and $f(A) = M\}$.

Every **f-Base** (i.e. element of **f-Base**) is a maximal f-independent set (i.e. a maximal element of **f-Ind** with respect to inclusion), and also it is a minimal f-generating set (i.e. a minimal element of **f-Gen** with respect to inclusion). However, in general it is not true that a maximal f-independent set is an **f-Base**, or that a minimal f-generating set is a base.

4. Tolerance relations

Throughout this section, if not explicitly stated otherwise, T denotes a tolerance relation, i.e. $E \leq T = T^*$. Originally the reflexivity and symmetry properties of a tolerance relation were abstracted from the simple example of points in the Euclidean plane which are less then ϵ apart ($\epsilon > 0$). This example can be generalized to any metric space $\langle M, d\rangle : \forall x, y \in M, xTy$ if $d(x, y) < \epsilon$. This explains the name tolerance borrowed from the vocabulary of engineering, where it is used to express the inaccuracy of measurement or a negligible difference in size.

Equivalence relations (i.e. relations satisfying $E \leq R^2 = R^* = R$) form the most familiar class of tolerance relations, distinguished by the transitivity property ($R^2 \leq R$). There are some well-known alternative ways to define an equivalence relation R on a set S. One is based on the fact that there is a bijective correspondence between the set of partitions of the set S and the set of all equivalence relations on S. Another correspondence (in this case surjective only) connects equivalence relations on S with functions with domain S and range an arbitrary set X. For such a function $\phi : S \to X$, define xRy *iff* $\phi(x) = \phi(y)$. Analogs for tolerance relations will be explored later in this section.

Definition 4.1: A set $L \subseteq S$ is a *preclass* of tolerance T (T-compatibility class in [176]) if $\forall x, y \in S$, $\{x, y\} \subseteq L \Rightarrow xTy$. The set of all preclasses of T will be written: $\mathcal{L}(T)$.

It is an immediate consequence of the Definition that $\forall x, y \in S, xTy$ *iff* $\exists L \in \mathcal{L}(T), \{x, y\} \subseteq L$.

Definition 4.2.: A maximal preclass of tolerance T is called a *class* of T (clique induced by T in [176]). The set of all classes of T will be written $\mathcal{K}(T)$.

By a straightforward application of Zorn's Lemma, every preclass of a tolerance T is contained in some class of T. Clearly $\forall x, y \in S$, xTy *iff* $\exists K \in \mathcal{K}(T)$, $\{x, y\} \subseteq K$. We also note that the family of all classes (resp. preclasses) forms a (not necessarily disjoint) covering of S. The main difference between equivalence classes for an equivalence relation and tolerance classes for a tolerance relation is that the former form a pairwise disjoint covering of S. Also note that the set $\mathcal{L}(T)$ (resp. $\mathcal{K}(T)$) of all preclasses (resp. classes) of T determines T uniquely.

Proposition 4.3: Let R, T be binary relations on a set S and $T_R = RR^*$. Then:

(i) T_R is a symmetric relation,
(ii) $\forall x, y \in S, xT_Ry$ *iff* $R(x) \cap R(y) \neq \emptyset$,
(iii) T_R is a tolerance *iff* R is everywhere defined ($E \leq RR^*$),
(iv) R is a function ($E \leq RR^*$ and $R^*R \leq E$) $\Rightarrow T_R$ is an equivalence relation, but the reverse implication is not necessarily true,
(v) T is an equivalence relation *iff* $\exists R \in \mathcal{R}(S), T = T_R$ and $E \leq RR^*$ and $R^*R \leq E$ (i.e.R is a function),
(vi) Let $E \leq T$. Then T is an equivalence relation *iff* $T = T_T$.

Proof: i)–iii) and vi) are obvious.

For iv) we have: $R^*R \leq E \Rightarrow T_R^2 = RR^*RR^* \leq RR^* = T_R$. The following shows that the reverse implication is false. Let $S = \{a, b, c, d\}$, $T = E \cup \{\langle a, b\rangle, \langle b, a\rangle, \langle c, d\rangle, \langle d, c\rangle\}$. Then for $R = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, a\rangle, \langle c, c\rangle$ $\langle d, c\rangle, \langle c, d\rangle\}$ we have $E \leq T = T^* \leq T^2, T = T_R = RR^*$, but R is not a function.

($\Leftarrow$) of v) follows from iv).

For ($\Rightarrow$), by the Axiom of Choice we can define a selector function $\phi : \mathcal{K}(T) \rightarrow S$, such that $\phi(K) \in K$. Now define $R = \{\langle x, y\rangle : \exists K \in \mathcal{K}(T), x \in K$ and $y = \phi(K)\}$. Then $T = T_R$ and R is a function.

Remark 4.4: Proposition 4.3 provides us with a rich source of examples of tolerances: any everywhere defined relation R defines a tolerance $T = RR^*$. However, different relations can define the same tolerance. For example, let $S = \{a, b, c\}$, $T = E \cup \{\langle a, c\rangle, \langle c, a\rangle, \langle b, c\rangle, \langle c, b\rangle\}$, $R = \{\langle c, a\rangle, \langle c, b\rangle, \langle a, a\rangle, \langle b, b\rangle\}$, $U = \{\langle a, b\rangle, \langle b, c\rangle, \langle c, b\rangle, \langle c, c\rangle\}$. Then $T = RR^* = UU^*$, but $R \neq U$.

The next natural question is: what properties of a family $\mathcal{L}$ of subsets of S ensure that $\mathcal{L}$ is $\mathcal{L}(T)$ for some tolerance relation T? As noted above, $\mathcal{L}$ must be a covering of S.

Definition 4.5 [176]: Let $\mathcal{A}$ be a family of subsets of S. Then $B \subseteq S$ is *induced by* $\mathcal{A}$ if $\forall x, y \in B \exists A \in \mathcal{A}, \{x, y\} \subseteq A$.

Definition 4.6 [176]: Let $\mathcal{A}$ be a family of subsets of S. We say $\mathcal{A}$ is *complete* if $\forall B \subseteq S$, (B is induced by $\mathcal{A}$) $\Rightarrow$ ($B \in \mathcal{A}$). The *completion of* $\mathcal{A}$ is defined to be $\overline{\mathcal{A}} = \{B \subseteq S : B$ is induced by $\mathcal{A}\}$.

Proposition 4.7: Completion of families of subsets of a set S, regarded as an operation $\mathcal{A} \mapsto \overline{\mathcal{A}}$, is a transitive closure operation on the power set of S; in other words, $\forall \mathcal{A} \subseteq 2^S$:

(i) $\mathcal{A} \subseteq \overline{\mathcal{A}}$,
(ii) $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \overline{\mathcal{A}} \subseteq \overline{\mathcal{B}}$,
(iii) $\overline{\mathcal{A}} = \overline{\overline{\mathcal{A}}}$.

Proof: i) and ii) are obvious.

iii) Suppose $\exists A \in \overline{\overline{\mathcal{A}}}, A \notin \overline{\mathcal{A}}$, i.e. $\exists x, y \in A, \forall B \subseteq S, \{x,y\} \subseteq B \Rightarrow B \notin \mathcal{A}$. But $\forall x, y \in A, \exists C \in \overline{\mathcal{A}}, \{x,y\} \subseteq C$. This implies $\exists B \in \mathcal{A}, \{x,y\} \subseteq B$, contradiction.

The following two definitions follow the concepts of Section 3, when S is replaced by 2^S and $f(\mathcal{A}) = \overline{\mathcal{A}}$.

Definition 4.8: A family $\mathcal{A}$ of subsets of S is *independent* ($\mathcal{A} \in$ **Ind**) if $\forall A \in \mathcal{A}$, $A \notin \overline{\mathcal{A}\backslash\{A\}}$, i.e. $\forall A \in \mathcal{A}, \exists\{x,y\} \subseteq A, \forall B \in \mathcal{A}, \{x,y\} \subseteq B \Rightarrow B = A$.

Definition 4.9: A family $\mathcal{A} \subseteq 2^S$ is *generating* for a complete family $\mathcal{B} = \overline{\mathcal{B}} \subseteq 2^S$ if $\overline{\mathcal{A}} = \mathcal{B}$.

Proposition 4.10: A family $\mathcal{L} \subseteq 2^S$ is the set $\mathcal{L}(T)$ of all preclasses of some tolerance relation T *iff*

(i) $\mathcal{L}$ is a covering, and
(ii) $\mathcal{L} = \overline{\mathcal{L}}$ i.e. $\mathcal{L}$ is complete.

Proof: $\Rightarrow$) It is easily verified that $\mathcal{L}(T)$ is a covering of S. Let $B \subseteq S$ and $\forall x, y \in B, \exists A \in \mathcal{L}(T), \{x,y\} \subseteq A$. Then $\forall x, y \in B, xTy$, i.e. $B \in \mathcal{L}(T)$ and $\mathcal{L}(T)$ is complete.

$\Leftarrow$) Let $\mathcal{L} \subseteq 2^S$ be a complete covering of S. Define a relation T by: $\forall x, y \in S$, xTy if $\exists A \in \mathcal{L}, \{x,y\} \subseteq A$. $\mathcal{L}$ is a covering of S, hence T is reflexive i.e. $E \leq T$. Symmetry is obvious, so T is a tolerance relation. Now we have to show $\mathcal{L} = \mathcal{L}(T)$. By the definition of T we have $\mathcal{L} \subseteq \mathcal{L}(T)$, so we need to show the reverse inclusion only. Suppose $A \in \mathcal{L}(T)$. Then $\forall x, y \in A, xTy$. By the Definition of $T, \exists B \in \mathcal{L}, \{x,y\} \subseteq B$, hence $A \in \overline{\mathcal{L}} = \mathcal{L}$.

Corollary 4.11 (for S countable [176]). *There is a bijective correspondence between the set of all complete coverings of a set S (i.e. all closed families that cover S) and the set of all tolerance relations on S.*

Corollary 4.12. *For every covering $\mathcal{A}$ of S there exists a unique tolerance relation T on S such that:*

(i) $\forall A \subseteq S, A \in \overline{\mathcal{A}}$ *iff* $\forall x, y \in A, xTy$,
(ii) $\forall x, y \in S, xTy$ *iff* $\exists A \in \mathcal{A}, \{x,y\} \subseteq A$.

Remark 4.13: In the following we will write $f(\mathcal{A})$ for the closure $\overline{\mathcal{A}}$. According to the content of Section 3, we can consider a restriction of the closure f to every closed $\mathcal{A} = f(\mathcal{A})$. By Proposition 4.10 there is a unique corresponding tolerance T such that $\mathcal{A} = \mathcal{L}(T)$, therefore the restriction can be written f_T. Certainly $\forall \mathcal{B} \subseteq \mathcal{A}, f_T(\mathcal{B}) = f(\mathcal{B}) \cap \mathcal{A} = f(\mathcal{B})$. If $\mathcal{A} = 2^S$, then $\forall \mathcal{B} \subseteq 2^S, f_T(\mathcal{B}) = f(\mathcal{B})$ and $T = S \times S$, the full relation on S. If $\mathcal{A} = \{\{x\} : x \in S\}$, then $T = E$ and $\forall \mathcal{B} \subseteq \mathcal{A} : f_E(\mathcal{B}) = \mathcal{B}$. To avoid a confusion observe that although

$\forall \mathcal{B} \subseteq \mathcal{A} = f(\mathcal{A})$, $f_T(\mathcal{B}) = f(\mathcal{B})$, the domain of f_T is $\mathcal{L}(T) = \mathcal{A}$, while the domain of f is 2^S.

Consider the families: $\mathbf{f_T}$**-Cl** $=$ **f-Cl** $\cap\, 2^{\mathcal{L}(T)}$, $\mathbf{f_T}$**-Ind** $=$ **f-Ind** $\cap\, 2^{\mathcal{L}(T)}$, $\mathbf{f_T}$**-Gen** $= \{\mathcal{B} \subseteq \mathcal{L}(T) : f(\mathcal{B}) = \mathcal{L}(T)\}$, $\mathbf{f_T}$**-Base** $= \{\mathcal{B} \subseteq \mathcal{L}(T) : \forall B \in \mathcal{B}, B \notin f(\mathcal{B}\backslash\{B\})$ and $f(\mathcal{B}) = \mathcal{L}(T)\}$. For elements of $\mathbf{f_T}$**-Ind** and $\mathbf{f_T}$**-Gen** we can apply simply Definition 4.8 and 4.9 respectively to subsets of $\mathcal{L}(T)$. In the case of the family of bases we can formulate the Definition the following way:

Definition 4.14: By a *base* of tolerance T we mean any element of $\mathbf{f_T}$**-Base**, i.e. any family of preclasses $\mathcal{B}$ which satisfies the conditions:

(i) $\forall x, y \in S, xTy$ *iff* $\exists B \in \mathcal{B}, \{x, y\} \subseteq B$,
(ii) $\forall B \in \mathcal{B}, \exists \{x, y\} \subseteq B, \forall A \in \mathcal{B}, \{x, y\} \subseteq A \Rightarrow A = B$.

Remark 4.15: As was mentioned in Section 3, in a dependence system every base is a maximal, independent subset and also a minimal generating subset. In particular, given a tolerance T, any base $\mathcal{B}$ is simultaneously a maximal independent subset and a minimal generating subset of $\mathcal{L}(T)$. This can explain the exceptional role played by bases. They are minimal coverings defining a given tolerance T. Every tolerance T has a base: e.g. $\{\{x, y\} \subseteq S : xTy$ and $(x \neq y$ if $\exists w, z \in S : xTw$ and yTz and $x \neq w$ and $z \neq y)\}$ is one. However, this base can have cardinality greater than that of $\mathcal{K}(T)$, the set of all classes of T. So we can expect that those bases which consist of classes (not preclasses only) have relatively small cardinalities. Certainly $\mathcal{K}(T)$ is a covering that generates T. When S is finite and $n = |S|$, Sperner's Lemma shows that $|\mathcal{K}(T)| \leq \binom{n}{[\frac{n}{2}]}$.

Definition 4.16: A base $\mathcal{B}$ of tolerance T is a *class base* if $\mathcal{B} \subseteq \mathcal{K}(T)$.

Proposition 4.17: Let T be a tolerance on a set S such that $\mathcal{K}(T)$ is finite. Then T has a class base. Moreover, if $\mathcal{A}$ is any generating family for T, then there exists a class base $\mathcal{B}(T)$ satisfying $|\mathcal{B}(T)| \leq |\mathcal{A}|$.

Proof: In a finite number of steps we can reduce the generating family $\mathcal{K}(T)$ to a class base, in each step removing one class and checking if the remaining family is still generating. Now let $\mathcal{A} \subseteq \mathcal{L}(T)$ be any generating family for T. For each $A \in \mathcal{A}$, by the Axiom of Choice, we may select a fixed class K_A in $\mathcal{K}(T)$ such that $A \subseteq K_A$. Let $\mathcal{K}(\mathcal{A}) = \{K \in \mathcal{K}(T) : \exists A \in \mathcal{A}, K = K_A\}$. Then $\mathcal{K}(\mathcal{A})$ is a finite generating family for T, so removing classes one by one from $\mathcal{K}(\mathcal{A})$, we obtain a class base $\mathcal{B}(T)$ such that $|\mathcal{B}(T)| \leq |\mathcal{K}(\mathcal{A})| = |\mathcal{A}|$.

Remark 4.18: Two (class) bases of a tolerance T can have different cardinalities, as shown by the following modification of an example from [169]): Let $S = \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 36, 54, 81\}$ and $T = U \cup U^* \cup E$, where U is defined by: $\forall x, y \in S : xUy$ *iff* $(y = 2x$ or $y = 3x$ or $y = (3/2)x)$. Then: $\mathcal{K}(T) = \{\{x, y, z\} \subseteq S : x \in S$ and $y = 2x \in S$ and $z = 3x \in S\} \cup \{\{x, y, z\} \subseteq S : x \in S$ and $y = (3/2)x \in S$ and $z = 3x \in S\}$. There are exactly two bases: $\mathcal{B}_1 = \{x,y,z\} \subseteq S : x \in S$ and $y = 2x \in S$ and $z = 3x \in S\}, \mathcal{B}_2 = \{\{x,y,z\} \subseteq S : x \in S$ and $y = 2x \in S$ and $z = 3x \in S$ and $x \neq 6\} \cup \{\{x,y,z\} \subseteq S : x \in \{4,6,12\}$ and $y = (3/2)x$ and $z = 3x\}$. $|\mathcal{B}_1| = 10$ and $|\mathcal{B}_2| = 12$.

Example 4.19: In one of the most extensive expositions [169], it is claimed (without proof) that for every tolerance, even if its set of classes is infinite, there exists a class base (called a base in the original terminology of [169]). We will show that this statement is false. There exists a tolerance (on an infinite set) which does not have a class base. First, let us refer to another rich source of examples of tolerance relations. For any partially ordered set $\langle S, \leq \rangle$, define a comparability relation T by: $\forall x,y \in S$, xTy if $x \leq y$ or $y \leq x$, i.e., $T = (\leq) \cup (\leq)^*$. This relation is certainly reflexive and symmetric. Not every tolerance can be represented this way (the conditions for this can be inferred from those given in [78] for a slightly modified structure); nevertheless, there are many interesting examples. For a comparability relation T, $\mathcal{L}(T)$ is the set of all chains in the poset $\langle S, \leq \rangle$, while $\mathcal{K}(T)$ is the set of all maximal chains.

We shall now describe a tolerance relation on the set of natural numbers N which has no class base. Let $a, b \in \mathrm{N}$, and $a = q_1 q_2 \ldots q_k$ and $b = r_1 r_2 \ldots r_m$ be factorizations of a and b into primes. We will always assume without loss of generality that $q_1 \leq q_2 \leq \ldots \leq q_k$, resp. $r_1 \leq r_2 \leq \ldots \leq r_m$. Define $a \ll b$ if $a = b$ or $\exists n \in \mathrm{N}, \exists p_1, \ldots, p_n$ primes in N, each p_i equal to or greater than q_k (the greatest factor of a) so that $b = ap_1 p_2 \ldots p_n$. For illustration we can describe the ordering by its covering relation: a natural number with the prime factorization $q_1 q_2 \ldots q_k$ is covered by an infinite number of elements of the form $q_1 q_2 \ldots q_k p$, where p is prime and for every i, $q_i \leq p$.

Now let T be the comparability relation for $\ll$, i.e. $T = (\ll) \cup (\ll)^*$. Then $\mathcal{K}(T)$ contains all maximal chains of $\langle \mathrm{N}, \ll . \rangle$ As in every tolerance, $\mathcal{K}(T)$ generates T. Now suppose that $\mathcal{A} \subseteq \mathcal{K}(T)$ is a class base. Pick any $A \in \mathcal{A}$. Then by independence of $\mathcal{A}$ there exists a pair a, b, such that $\{a, b\} \subseteq A$ and $\forall B \in \mathcal{A}$, $\{a, b\} \subseteq B \Rightarrow B = A$. W.l.o.g. assume $a \ll b$ (at least one of the inequalities $a \ll b$ or $b \ll a$ is true as $\mathcal{A}$ is generating for T). Pick two distinct primes q_1, q_2 greater than the largest prime factor of b. Then, using the fact that $\mathcal{A}$ is generating, pick $A_1 \in \mathcal{A}$ containing $\{a, bq_1\}$ and $A_2 \in \mathcal{A}$ containing $\{a, bq_2\}$. Then $\{a, b\} \subseteq A_1 \cap A_2$ and $A_1 \neq A_2$, so one of them is different from A, contradiction.

Definition 4.20: Let T be a tolerance on a set S. We define the *dimension* of T to be $\dim T = \inf\{|\mathcal{B}(T)| : \mathcal{B}(T)$ is a base of $T\}$. Certainly $\dim T \leq |S \times S|$, and for an infinite set T, $\dim T \leq |S|$.

Proposition 4.21: Let $\mathcal{A}$ be a generating family for T and $\dim T < \omega$. Then $|\mathcal{A}| = \dim T \Rightarrow \mathcal{A}$ is a base of T.

Proof: Suppose $\mathcal{A}$ is not a base. Since $\mathcal{A}$ is generating, it is not independent, i.e., $\exists A \in \mathcal{A}, \forall x,y \in A, \exists A_{xy} \in \mathcal{A}\backslash\{A\}, \{x,y\} \subseteq A_{xy}$. But then, $\mathcal{A}\backslash\{A\}$ is a generating family with cardinality less than dim T, contradiction.

Remark 4.22: Note that a generating family of cardinality dim T need not be a class base. Indeed, let $S = \{2,3,6,10,15\}$ and let T be defined by xTy if x and y are NOT relatively prime. Then $\dim T = 3$, but the family $\mathcal{B} = \{\{2,6,10\},\{3,6,15\},\{10,15\}\}$ is not a class base, because $\{10,15\}$ is not a class.

Recall from Proposition 4.10 that a family $\mathcal{A} \subseteq 2^S$ is $\mathcal{L}(T)$ for some tolerance T if $\mathcal{A}$ is a covering of S and it is closed with respect to the closure f. We now characterize a family of tolerance classes.

Proposition 4.23 [188]: Let $\mathcal{A}$ be a covering of a set S. Then $\mathcal{A} \subseteq \mathcal{K}(T)$ for some tolerance T on S *iff* $\forall A \in \mathcal{A}, \forall \mathcal{A}_0 \subseteq \mathcal{A}, A \subseteq \cup\{B \subseteq S : B \in \mathcal{A}_0\} \Rightarrow \cap\{B \subseteq S : B \in \mathcal{A}_0\} \subseteq A$.

Proof: $\Leftarrow$) Let T be the tolerance induced by $\mathcal{A}$. Thus, $f(\mathcal{A}) = \mathcal{L}(T)$. Suppose $A \in \mathcal{A}\backslash\mathcal{K}(T)$. Then $A \in \mathcal{L}(T)$, so $\exists x \in S\backslash A, \forall y \in A, xTy$, i.e.$\exists x \in S\backslash A$, $\forall y \in A, \exists B_y \in \mathcal{A}, \{x,y\} \subseteq B_y$. Therefore $A \subseteq \cup\{B_y \in \mathcal{A} : y \in A\}$ but $x \in \cap\{B_y \in \mathcal{A} : y \in A\}\backslash A$, contradiction. Hence $\mathcal{A} \subseteq \mathcal{K}(T)$.

$\Rightarrow$) Suppose $A \in \mathcal{A}, \mathcal{A}_0 \subseteq \mathcal{A}$ and $A \subseteq \cup\{B \subseteq S : B \in \mathcal{A}_0\}$, but there is some $x \in \cap\{B \subseteq S : B \in \mathcal{A}_0\}\backslash A$. Then $\forall y \in A$, xTy and $x \notin A$, so $A \notin \mathcal{K}(T)$ as A is not a maximal preclass, contradiction.

Proposition 4.24: Let T be a tolerance on S, $\mathcal{A} \subseteq \mathcal{K}(T)$ and $f(\mathcal{A}) = \mathcal{L}(T)$. Then $\mathcal{A} = \mathcal{K}(T)$ *iff* $\forall A \subseteq S, (\forall x,y \in A \text{ with } x \neq y, \exists B \in \mathcal{A}, \{x,y\} \subseteq B) \Rightarrow (\exists B \in \mathcal{A}, A \subseteq B)$.

Proof: $\Rightarrow$) If $\forall x,y \in A$ with $x \neq y, \exists B \in \mathcal{A}, \{x,y\} \subseteq B$, then $A \in \mathcal{L}(T)$. Hence $\exists B \in \mathcal{K}(T) = \mathcal{A}, A \subseteq B$.

$\Leftarrow$) Suppose $\exists K \in \mathcal{K}(T)\backslash\mathcal{A}$. Then $\forall x,y \in K$ with $x \neq y, \exists A \in \mathcal{L}(T)$, $\{x,y\} \subseteq A$; hence, $\exists B \in \mathcal{A}, \{x,y\} \subseteq B$. This implies that $\exists B \in \mathcal{A} \subseteq \mathcal{K}(T)$, $K \subseteq B$, contradiction.

Corollary 4.25 [41]: Let $\mathcal{A}$ be a covering of a set S. Then $\mathcal{A} = \mathcal{K}(T)$ for some tolerance T *iff*

(i) $\forall A \in \mathcal{A}, \forall \mathcal{A}_0 \subseteq \mathcal{A}, (A \subseteq \cup\mathcal{A}_0) \Rightarrow (\cap\mathcal{A}_0 \subseteq A)$, and
(ii) $\forall A \subseteq S, (\forall x,y \in A \text{ with } x \neq y, \exists B \in \mathcal{A}, \{x,y\} \subseteq B) \Rightarrow (\exists B \in \mathcal{A}, A \subseteq B)$.

This gives a bijective correspondence between coverings of S satisfying i) and ii) and tolerances on S.

Example 4.26: Proposition 4.24 is not true if we do not assume $f(\mathcal{A}) = \mathcal{L}(T)$, i.e. if $\mathcal{A}$ does not generate T. For again, let $S = \{2,3,5,6,10,15\}$, and xTy *iff* x and y are not relatively prime and $\mathcal{A} = \{\{5,10,15\},\{3,6,15\}\}$. Then $\mathcal{A} \subseteq \mathcal{K}(T), \mathcal{A} \neq \mathcal{K}(T)$ but the condition from Proposition 4.24 is satisfied. This is possible because $f(\mathcal{A}) \neq \mathcal{L}(T)$.

The following result is due to Gilmore (see [9]). It was originally phrased in the language of hypergraphs. A hypergraph consists of a nonempty set S together with a covering $\mathcal{A}$ of nonempty sets. It should be pointed out that different hypergraphs may give rise to the same tolerance relation.

Proposition 4.27: Let S be a finite set. For any $n \in \mathrm{N}$, let $A_1, \ldots, A_n$ be subsets of S; let $\hat{A}_i$ denote $\bigcap_{i \neq j} A_j$. For a tolerance T induced by a covering $\mathcal{A}$ of S, the following are equivalent:

(i) $\mathcal{K}(T) \subseteq \mathcal{A}$.
(ii) $\exists n \geq 3, \forall A_1, \ldots, A_n \in \mathcal{A}, \exists D \in \mathcal{A}, \hat{A}_1 \cup \ldots \cup \hat{A}_n \subseteq D$.
(iii) $\forall n \geq 3, \forall A_1, \ldots, A_n \in \mathcal{A}, \exists D \in \mathcal{A}, \hat{A}_1 \cup \ldots \cup \hat{A}_n \subseteq D$.

Proof: i) $\Rightarrow$ iii) Let $n \geq 3$ and $A_1, \ldots, A_n \in \mathcal{A}$. Then $\hat{A}_1 \cup \ldots \cup \hat{A}_n$ is a preclass, hence is contained in some class, D.

iii) $\Rightarrow$ ii) Trivially.

ii) $\Rightarrow$ i) We will show that each preclass of T is contained in some member of $\mathcal{A}$. Let C be a preclass. If $|C| \leq 2$, this follows from the Definition of the tolerance induced by $\mathcal{A}$. Assume that $|C| = k \geq 3$ and that every preclass of smaller cardinality is contained in some member of $\mathcal{A}$. Let $x_1, x_2, x_3 \in C$ and $C_i = C \backslash \{x_i\}$. By the induction hypothesis, there exists a set $A_i \in \mathcal{A}$ such that $C_i \subseteq A_i$. By the given condition, there exists $D \in \mathcal{A}$ such that $C = (C_1 \cap C_2) \cup (C_2 \cap C_3) \cup (C_3 \cap C_1) \subseteq (A_1 \cap A_2) \cup (A_2 \cap A_3) \cup (A_3 \cap A_1) \subseteq D$.

Corollary 4.28: Let T be the tolerance induced by $\mathcal{A}$, a covering of a finite set S. Then $\mathcal{A} = \mathcal{K}(T)$ *iff*

(i) $\forall A \in \mathcal{A}, \forall \mathcal{A}_0 \subseteq \mathcal{A}, A \subseteq \cup\{B \subseteq S : B \in \mathcal{A}_0\} \Rightarrow \cap\{B \subseteq S : B \in \mathcal{A}_0\} \subseteq A$.
(ii) $\forall A, B, C \in \mathcal{A}, \exists D \in \mathcal{A}, (A \cap B) \cup (B \cap C) \cup (C \cap A) \subseteq D$.

Now we can ask about the analog for tolerance relations of the definition of equivalence via functions.

Remark 4.29: If we consider binary relations between a set S and some other set M (i.e. certain binary relations in the extended set $S \cup M$), then for every tolerance T on S, there is a set M and a relation $R \subseteq S \times M$ such that $\forall x, y \in S, xTy$ *iff* $R(x) \cap R(y) \neq \emptyset$. Indeed, if $M = \mathcal{K}(T)$ and R is defined by: xRK *iff* $x \in K$, then $\forall x, y \in S, xTy$ *iff* $R(x) \cap R(y) \neq \emptyset$. Recall that every function with domain S defines some equivalence relation on S, and the conditions defining a function distinguish equivalences within the class of tolerances. Indeed, the relation R described above is a function *iff* T is an equivalence relation. However, for every tolerance T on S, we can define a function $\phi : S \to A = \{\mathcal{K} : \mathcal{K} \subseteq \mathcal{K}(T)\}$, where $\forall x \in S, \phi(x) = \{K \in \mathcal{K}(T) : x \in K\}$. Since ϕ is a function defined on S, it defines an equivalence relation: xNy *iff* $\phi(x) = \phi(y)$. Equivalently, xNy *iff* $(\forall K \in \mathcal{K}(T), x \in K$ *iff* $y \in K)$ *iff* $T(x) = T(y)$. Note that this is consistent with Definition 2.11.

Definition 4.30: Given a tolerance relation T on a set S, the equivalence relation $N(T)$ (if no confusion is likely we will write just N), defined by: xNy *iff* $T(x) = T(y)$ is called the *nuclear equivalence* of tolerance T. Its equivalence classes are called *nuclei* of the tolerance T. If all nuclei of a given tolerance T are one-element subsets, we say T is non-nuclear: (Nuclei are called by some authors kernels or cores, but both these terms already have a different established meaning in mathematics). $\mathcal{N}(T)$ denotes the family of all nuclei of T.

Proposition 4.31: T is an equivalence relation *iff* $N = T$.

Proposition 4.32: If $\mathcal{K}(T)$ in Remark 4.29 is replaced by any generating family $\mathcal{A}$ of classes for T, and A denotes the power set of $\mathcal{A}$, then the equivalence relation induced by $\phi : S \to A$, where $\phi(x) = \{A \in \mathcal{A} : x \in A\}$ is identical with $N(T)$. In particular, $\mathcal{A}$ can be any class base of T.

Proposition 4.33: Let T be a tolerance on a set S, $\mathcal{A} \subseteq \mathcal{K}(T)$ be a generating family for T. Then $\forall B \subseteq S$, B is a nucleus of T *iff* $\exists \mathcal{B} \subseteq \mathcal{A}, B = \cap\{M \subseteq S : M \in \mathcal{B} \text{ or } M^c \in \mathcal{A}\backslash\mathcal{B}\}$.

Proof: $\Rightarrow$) Suppose B is a nucleus, i.e. i) $\forall x, y \in B, T(x) = T(y)$ and ii) $\forall z \notin B, \forall x \in B, T(z) \neq T(x)$. Let $\mathcal{B} = \{M \in \mathcal{A} : B \subseteq M\}$. We must show that $B = \cap\{M \subseteq S : M \in \mathcal{B} \text{ or } M^c \in \mathcal{A}\backslash\mathcal{B}\}$. Certainly, $B \subseteq \cap\mathcal{B}$. B is a nucleus, so if $D \in \mathcal{A}$ and $B \cap D \neq \emptyset$, then $B \subseteq D$, hence $B \subseteq \cap\{M \subseteq S : M^c \in \mathcal{A}\backslash\mathcal{B}\}$. Therefore, $B \subseteq \cap\{M \subseteq S : M \in \mathcal{B} \text{ or } M^c \in \mathcal{A}\backslash\mathcal{B}\}$. Now suppose $x \in \cap\{M \subseteq S : M \in \mathcal{B} \text{ or } M^c \in \mathcal{A}\backslash\mathcal{B}\}$. Let $b \in B$. For each $A \in \mathcal{A}$, if $b \in A$, then by definition of a nucleus, $B \subseteq A$. Hence $A \in \mathcal{B}$ and $x \in A$. If $b \notin A$, then $B \cap A = \emptyset$, so $B \subseteq A^c$ and $A = A^{cc} \in \mathcal{A}\backslash\mathcal{B}$; thus $x \in A^c$. Hence, $\forall A \in \mathcal{A}, x \in A$ *iff* $b \in A$, i.e., xNb, so $x \in B$.

$\Leftarrow$) Given that $\mathcal{B} \subseteq \mathcal{A}$ and $B = \cap\{M \subseteq S : M \in \mathcal{B} \text{ or } M^c \in \mathcal{A}\backslash\mathcal{B}\}$, then for each $x, y \in B$ we have xTz *iff* yTz, since x and y belong to exactly the same elements of the generating family $\mathcal{A}$.

Applying Proposition 4.33 to the special cases when $\mathcal{A} = \mathcal{K}(T)$ or $\mathcal{A}$ is some class base $\mathcal{B}(T)$, we get:

Corollary 4.34 [169]: B is a nucleus for tolerance T *iff* $\exists \mathcal{B} \subseteq \mathcal{K}(T), B = \cap\{K \subseteq S : K \in \mathcal{B} \text{ or } K^c \in \mathcal{K}(T)\backslash\mathcal{B}\}$ *iff* $\exists \mathcal{A} \subseteq \mathcal{B}(T), B = \cap\{K \subseteq S : K \in \mathcal{A} \text{ or } K^c \in \mathcal{B}(T)\backslash\mathcal{A}\}$.

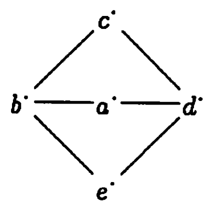
Corollary 4.35: Let $\mathcal{N}(T)$ be the family of all nuclei of a tolerance T and let $\mathcal{B}(T)$ be any class base for T. Then $|\mathcal{N}(T)| \leq |2^{\mathcal{B}(T)}|$.

Corollary 4.36 [169]: If T has a finite class base, then also the family of all tolerance classes of T is finite.

Remark 4.37: In the process of generalizing equivalence to tolerance, the notion of equivalence class splits into two distinct notions: tolerance class and nucleus.

We now explore the relationships between arbitrary tolerance spaces, non-nuclear tolerance spaces and equivalence relations. First, we will show that an arbitrary tolerance space $\langle S,T\rangle$ can be constructed in a natural way from a non-nuclear tolerance space and an equivalence relation on S. By the convention adopted in Def. 4.30, given a tolerance space $\langle S,T\rangle$, the tolerance space, $\langle S/N,\bar{T}\rangle$, reduced by the nuclear equivalence N may be denoted by $\langle \mathcal{N}(T),\bar{T}\rangle$. Let $\langle S,T\rangle$ be a tolerance space and $\langle \mathcal{N}(T),\bar{T}\rangle$ the reduced tolerance space. Then $T_\phi = T$, where $\phi : S \to \mathcal{N}(T)$ is defined by $\phi(x) = N(x) = \{y \in S : xNy\}$. Clearly, ϕ is a tolerance homomorphism from $\langle S,T\rangle$ to a non-nuclear tolerance space. Proposition 2.12 states that a surjective tolerance homomorphism between tolerance spaces induces a tolerance isomorphism between their reduced tolerance spaces. The latter are, of course, non-nuclear.

Example 4.38: In Proposition 4.3, we considered tolerances constructed by composing an everywhere defined relation R ($E \leq RR^*$) with its converse: $T_R = RR^*$. The following example shows that not every tolerance arises this way. Let $S = \{a,b,c,d,e\}$ and let T be the tolerance defined by the following graph (xTy *iff* $x = y$ or there is an edge joining x and y).



Suppose $T = RR^*$. Then xTy *iff* $R(x) \cap R(y) \neq \emptyset$. Therefore: $R(b) \cap R(d) = \emptyset$, also $R(a), R(c), R(e)$ are mutually disjoint, but $R(b)$ intersects $R(a), R(c)$ and $R(e)$; and $R(d)$ intersects $R(a), R(c)$, and $R(e)$. Hence we need at least six different elements: $x_1 \in R(b) \cap R(a)$, $x_2 \in R(b) \cap R(c)$, $x_3 \in R(b) \cap R(e)$, $x_4 \in R(d) \cap R(a)$, $x_5 \in R(d) \cap R(c)$, $x_6 \in R(d) \cap R(e)$. But the set S has only 5 elements, contradiction.

Proposition 4.39: Let T be a tolerance on a set S, $\mathcal{B} \subseteq 2^S$, $f(\mathcal{B}) = \mathcal{L}(T)$ and $|\mathcal{B}| \leq |S|$. Let $\phi : \mathcal{B} \to S$ be any fixed injective function. Define a relation $R \subseteq S \times S$ by: $\forall a \in S, R(a) = \{x \in S : \exists L \in \mathcal{B}, a \in L \text{ and } \phi(L) = x\}$. Then $T = RR^*$. Conversely, if T factors as $T = RR^*$ for some relation R on S, then T has a generating family $\mathcal{B} \subseteq 2^S$ such that $|\mathcal{B}| \leq |S|$.

Proof: $\Rightarrow$) $\forall a,b \in S, R(a) \cap R(b) \neq \emptyset$ *iff* $\exists x \in S, x \in R(a)$ and $x \in R(b)$ *iff* $\exists x \in S$, $[\exists L_1 \in \mathcal{B}, a \in L_1$ and $\phi(L_1) = x]$ and $[\exists L_2 \in \mathcal{B}, b \in L_2$ and $\phi(L_2) = x]$ *iff* $\exists x \in S, \exists L \in \mathcal{B}, \phi(L) = x$ and $\{a,b\} \subseteq L$ (as ϕ is injective i.e. $L_1 = L_2$) *iff* aTb.

$\Leftarrow$) Conversely, consider $\mathcal{B} = \{R^*(a) : a \in S\}$. Certainly $|\mathcal{B}| \leq |S|$. $\forall x,y \in S$, xTy *iff* $R(x) \cap R(y) \neq \emptyset$ *iff* $\exists z \in S$, $z \in R(x) \cap R(y)$ *iff* $\exists z \in S$, $\{x,y\} \subseteq R^*(z)$.

Proposition 4.40 Let $T \subseteq S \times S$ be a tolerance relation on S, $\mathcal{B}(T)$ be a base of T. Then $|\mathcal{B}(T)| \leq |T|$.

Proof: $\mathcal{B}(T)$ is a base of T so it is independent. Hence for every $B \in \mathcal{B}(T)$ we can choose (by Axiom of Choice) a fixed ordered pair $\langle x, y\rangle \in T$ such that $\forall A \in \mathcal{B}(T), \{x, y\} \subseteq A \Rightarrow A = B$. This defines an injective function $\phi : \mathcal{B}(T) \to T$, and therefore $|\mathcal{B}(T)| \leq |T|$.

Corollary 4.41: Let T be a tolerance relation on an infinite set S. Then there exists a relation R on set S, such that $T = RR^*$.

Proof: By Remark 4.15, for every tolerance T there exists a base $\mathcal{B}(T)$ of T. By Proposition 4.40 we have $|\mathcal{B}(T)| \leq |T| \leq |S \times S| \leq |S|$, where the last equality is valid if S is infinite. Now by Proposition 4.39 we get the statement of the corollary.

Remark 4.42: The problem of characterizing factorizable tolerances on a finite set is still open. There are several facts known about it, like: there is no "forbidden subtolerance" characterization. For example the nonfactorizable tolerance T from Example 4.38 can be extended to a factorizable one. Indeed, Let $S_1 = S \cup \{f, g\} = \{a, b, c, d, e, f, g\}$ and $T_1 = T \cup \{\langle f, f\rangle, \langle f, g\rangle, \langle g, f\rangle, \langle g, g\rangle\}$. Now $T_1 = RR^*$, for R defined by $R(a) = \{c, d\}$, $R(b) = \{a, c, e\}$, $R(c) = \{e, f\}$, $R(d) = \{b, d, f\}$, $R(e) = \{a, b\}$, $R(f) = R(g) = \{g\}$.

We also know that such a factorization, if it exists, is in general non-unique. One example was given in Remark 4.4. Also, if $T = RR^*$ and $WW^* = E$ and $U = RW$, then $T = UU^*$. This follows from: $UU^* = (RW)(RW)^* = RWW^*R^* = RR^* = T$. In particular, if $T = RR^*$ and W is a permutation (bijective function $W^*W = WW^* = E$), then $T = (RW)(RW)^*$.

5. Weak tolerance relations

There is a bijective correspondence between partitions (disjoint coverings) of a set S and equivalence relations on S. There is a surjective correspondence between (not necessarily disjoint) coverings of a set S and tolerances on S. What relations correspond to arbitrary families of subsets of a set S?

In Section 4 we considered a closure f. We found a bijective correspondence between f-closed coverings of S and tolerances on S. What are the relations which correspond to arbitrary f-closed families of subsets? The answer to these (and some other) questions will be provided in this section. First we define a generalization of a tolerance relation.

Definition 5.1: A relation $T \in \mathcal{R}(S)$ is a *weak tolerance* relation if:

(i) $T = T^*$ (symmetry),
(ii) $\forall x \in S, xT^cx \Rightarrow \forall y \in S, xT^cy$.

Certainly every tolerance is a weak tolerance as the second condition is satisfied by default.

Most of the results of Section 4 can be obtained for weak tolerance relations with only the difference that a weak tolerance is not necessarily reflexive. Whereas for a tolerance we have a covering of the set by preclasses, for a weak tolerance, the preclasses need not form a covering. Also, if T is a weak tolerance with left and right domain D (T is symmetric so they are equal), then $T|_{D\times D} = T\cap D\times D$ (the restriction of T to set D) is a tolerance relation on D, and $T|_{D^c\times D^c} = T\cap D^c\times D^c = \emptyset$. Therefore elements of D^c do not play an important role in the properties of a weak tolerance T. Almost all definitions for weak tolerance are the same as for tolerances. Also almost all propositions about tolerances which remain true under generalization to weak tolerances have similar or identical proofs. Therefore in most cases the proofs are omitted.

Definition 5.2: i) $L \subseteq S$ is a *preclass* of a weak tolerance T if $\forall x, y \in S$, $\{x, y\} \subseteq L \Rightarrow xTy$. The set of all preclasses of T is denoted by $\mathcal{L}(T)$.

ii) A maximal preclass of a weak tolerance T is called *class* of T. The set of all classes of T is denoted by $\mathcal{K}(T)$.

Proposition 5.3: Let T be a weak tolerance on a set S.

(1) Every preclass of T is contained in some class.
(2) $\forall x, y \in S, xTy$ *iff* $\exists L \in \mathcal{L}(T), \{x, y\} \subseteq L$.
(3) $\forall x, y \in S, xTy$ *iff* $\exists K \in \mathcal{K}(T), \{x, y\} \subseteq K$.
(4) If R is any binary relation on a set S, then RR^* and R^*R are both weak tolerance relations. Moreover $D(RR^*) = D(R) = D^*(RR^*) = D^*(R^*)$, $D(R^*R) = D(R^*) = D^*(R^*R) = D^*(R)$, where D and D^* are the left and right domains of R respectively (Definition 2.8).
(5) A family $\mathcal{A} \subseteq 2^S$ is the set $\mathcal{L}(T)$ of all preclasses of some weak tolerance relation T *iff* $\mathcal{A} = f(\mathcal{A})$, i.e. $\mathcal{A}$ is complete.

Corollary 5.4: There is a bijective correspondence between the set of all complete families of subsets of a set S and the set of all weak tolerance relations on S.

Proposition 5.5: For every family $\mathcal{A}$ of subsets of S, there exists a weak tolerance relation T on S such that:

(i) $\forall x, y \in S, xTy$ *iff* $\exists A \in \mathcal{A}, \{x, y\} \subseteq A$.
(ii) $\forall A \subseteq S, [A \in f(\mathcal{A})$ *iff* $\forall x, y \in A, xTy]$,

Definition 5.6: Let T be a weak tolerance relation on a set S, $\mathcal{B} \subseteq 2^S$. In the same way as for tolerances we define the restriction f_T of an operator f. By a *base* of weak tolerance T we mean any element of $\mathbf{f_T}$**-Base**, i.e., any family $\mathcal{B}$ of preclasses satisfying the conditions:

(i) $\forall x, y \in S, xTy$ *iff* $\exists B \in \mathcal{B}, \{x, y\} \subseteq B$.
(ii) $\forall B \in \mathcal{B}, \exists \{x, y\} \subseteq B, \forall A \in \mathcal{B}, \{x, y\} \subseteq A \Rightarrow A = B$.

As for tolerances, every weak tolerance T on a set S has a base. A *class base* is a base which consists only of weak tolerance classes. If $\mathcal{K}(T)$ is finite, then there exists a class base for T. In particular, if S is finite, then T has a class base. Also all results concerning factorization of tolerances can be extended to weak tolerances without any changes.

6. Orthogonality relations which define syllogistics

There is a close connection between weak tolerances and abstract orthogonality relations (see [R9]). Some orthogonality problems have a much simpler formulation in terms of weak tolerance.

Definition 6.1: Define a *quasi* (sometimes called *abstract*) *orthogonality* as a binary relation $\perp$ on set S satisfying the following conditions:

(i) $\forall x, y \in S, x \perp y \Rightarrow y \perp x$ (symmetry), i.e. $(\perp)^* = (\perp)$,
(ii) $\forall x \in S, x \perp x \Rightarrow [\forall z \in S, x \perp z]$.

If in addition to conditions i), ii) the relation also satisfies the condition:

(iii) $\forall x, y \in S, x = y$ *iff* $\perp(x) = \perp(y)$,

then we call $\perp$ *partial orthogonality*.

The adjectives *quasi* and *partial* used above are not accidental.

Proposition 6.2: Let $\perp$ be a quasi orthogonality on a set S. Then the binary relation $\leq$ on S given by: $\forall x, y \in S, x \leq y$ *iff* $\perp(y) \subseteq \perp(x)$ is a quasi-order on S. If $\perp$ is a partial orthogonality, then $\leq$ is a partial order on S.

Proposition 6.3: Let R be a binary relation on a set S. Then the relation $T = R^{\cdot}.R$ is a quasi-order. Moreover, if $[\forall x, y \in S, R(x) = R(y)$ *iff* $x = y]$, then T is a partial order and conversely.

In view of Proposition 6.3, for any binary relation R on a set S, we adopt the notation: $\leq_R = R^{\cdot}.R$.

Proposition 6.4: If R is a quasi-order, then $\leq_R = R$.

In the following we show that we can also construct a quasi-orthogonality starting from any relation R.

Proposition 6.5: Let R be any binary relation on a set S. Then the relation $R^{\cdot}.R^c$ is a quasi-orthogonality on S.

In light of Proposition 6.5, we will adopt the notation: $R^{\cdot}.R^c = \underset{R}{\perp}$.

Definition 6.6: Let R be a relation on a set S. The relational structure: $\langle S, \leq_R, \underset{R}{\perp}\rangle$ is called the *left syllogistic generated by R on S*, where $\leq_R = R^{\cdot}.R$ and $\underset{R}{\perp} = R^{\cdot}.R^c$.

By Proposition 6.3, the left syllogistic generated by any relation R on S is a quasi-ordered set.

Definition 6.7: The left syllogistic $\langle S, \leq_R, \underset{R}{\perp}\rangle$ generated by $R \in \mathcal{R}(S)$ is *proper* if $\leq_R$ is a partial order. Certainly this is the case when R satisfies: $\forall x, y \in S$: $R(x) = R(y) \Rightarrow x = y$.

Proposition 6.8: $\perp \in \mathcal{R}(S)$ is a quasi-orthogonality on a set S *iff* $\perp^c$ is a weak tolerance on S.

Remark 6.9: For every orthogonality relation $\perp$ on a set S, if there exists a binary relation R on S such that $\perp = \underset{R}{\perp}$, then there exists a syllogistic $\langle S, \leq_R, \underset{R}{\perp} = \perp \rangle$. Is it possible to find such a relation R? This question can be formulated in terms of a weak tolerance relation as: given a weak tolerance relation T on S, can we find a relation R on S such that $T = RR^*$? These questions are equivalent as: $\underset{R}{\perp} = R^{\cdot}.R^c = (R^c R^{*c})^c$, so $(\underset{R}{\perp})^c = R^c(R^c)^* = UU^*$ where $U = R^c$. So $\perp \overset{R}{=} R^{\cdot}.R^c = \underset{R}{\perp}$ *iff* $T = \perp^c = UU^*$ for $U = R^c$. We have already seen (Example 4.38) that even if T is a tolerance on S (not just a weak tolerance) it can happen that for every binary relation R on $S, T \neq RR^*$. However Section 4 essentially shows that a weak tolerance with an infinite (left or right) domain D does factor in this way.

7. Constructions of tolerance spaces

We have already seen several ways to construct a tolerance on a set S. We have considered tolerances defined on pseudometric spaces, tolerances defined by coverings of a set, tolerances defined by composing an everywhere-defined relation with its converse, and tolerances defined via comparability relations. In this section we shall first consider various "intersection tolerances", generalizing an idea used extensively in graph theory.

Definition 7.1: Let $\mathcal{A}$ be a family of subsets of a set S. Then we define the *intersection tolerance* on $\mathcal{A}$, T_{int}, by: $\forall A, B \in \mathcal{A}, AT_{\text{int}}B$ if $A \cap B \neq \emptyset$.

It is a well-known fact that every tolerance is isomorphic to the intersection tolerance of some family of sets. Indeed, let $\langle S, T \rangle$ be a tolerance space. For each $x \in S$, let $\mathcal{M}(x) = \{L \in \mathcal{L}(T) : x \in L\}$ and let $A = \{\mathcal{M}(x) : x \in S\}$. Then $\langle S, T \rangle$ is isomorphic to $\langle A, T_{\text{int}} \rangle$

Several constructions in graph theory can be reformulated in terms of intersection tolerances in subfamilies of $\mathcal{L}(T)$. Some particular cases have been studied by Shreider and Yakubovich [169]. Two interesting cases occur when one considers the set of all two-element subsets of $\mathcal{L}(T)$ or the subfamily $\mathcal{K}(T) \subseteq \mathcal{L}(T)$.

Definition 7.2: Let $\langle S, T \rangle$ be a tolerance space. The *line tolerance* for T is $\langle \mathcal{A}, T_{\text{int}} \rangle$, where $\mathcal{A} = \{A \in \mathcal{L}(T) : |A| = 2\}$. The *class intersection tolerance* for T is $\langle \mathcal{K}(T), T_{\text{int}} \rangle$

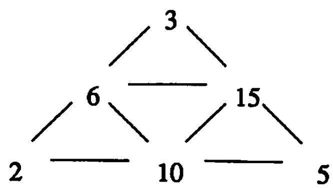
The following is essentially [94, Theorem 8.4]. Note that the set S need not be finite.

Proposition 7.3: The following are equivalent for a tolerance space $\langle S, T \rangle$

(1) $\langle S, T \rangle$ is a line tolerance.
(2) There exists a family $\mathcal{A} \subseteq \mathcal{L}(T)$ such that
    i) each element of S belongs to at most two members of $\mathcal{A}$;
    ii) each two-element preclass in $\mathcal{L}(T)$ is contained in exactly one member of $\mathcal{A}$.

Proposition 7.4 ([157]): Let $\langle S,T\rangle$ be a tolerance space. Then $\langle S,T\rangle$ is isomorphic to the class intersection tolerance for some tolerance space *iff* $\exists \mathcal{A} \in \mathcal{L}(T)$, $\mathcal{A}$ is generating for T and $\forall \mathcal{B} \subseteq \mathcal{A}$, if no two members of $\mathcal{B}$ are disjoint then $\cap\mathcal{B} \neq \emptyset$.

Example 7.5: Not every tolerance is isomorphic to a class intersection tolerance. The following example is due to Hamelinck. Let T be the common prime factor relation on $S = \{2,3,5,6,10,15\}$:



We now turn to a different type of construction of a tolerance. Denote the symmetric difference of two sets A and B by $A \div B = (A\backslash B) \cup (B\backslash A)$.

Definition 7.6: Let S be any set and $\mathcal{M} \subseteq 2^S$ a poset ideal (i.e., if $B \subseteq A \in \mathcal{M}$, then $B \in \mathcal{M}$). Let $\mathcal{A} \subseteq 2^S$. Define the *symmetric difference tolerance* T_{sd} on $\mathcal{A}$ by: $\forall A,B \in \mathcal{A}, AT_{sd}B$ *iff* $A \div B \in \mathcal{M}$.

Proposition 7.7: With notation as above, let $T = T_{sd}$, the symmetric difference. Then T is an equivalence *iff* $\forall A,B \in \mathcal{M}, A \cup B \in \mathcal{M}$.

Proof: $\Leftarrow$) Certainly T is a tolerance. Suppose ATB and BTC, where $A,B,C \subseteq S$. Since $\mathcal{M}$ is an ideal and $(A\backslash C) \cup (C\backslash A) \subseteq (A\backslash B) \cup (B\backslash A) \cup (B\backslash C) \cup (C\backslash B) \in \mathcal{M}$, we have $A \div C \in \mathcal{M}$, so ATC.

$\Rightarrow$) Let $A,B \in \mathcal{M}$. Let $C = (B\backslash A)$. Then $C \in \mathcal{M}$ and $AT\emptyset$ and $\emptyset TC$, so ATC; hence $A \cup B = A \cup C = A \div C \in \mathcal{M}$.

Proposition 7.8: With the notation as above, let $A,B \subseteq S$ and ATB. Then

(i) $\forall C \subseteq S, A \cup C\ T\ B \cup C$.
(ii) $\forall C \subseteq S, A \cap C\ T\ B \cap C$
(iii) $\forall C \subseteq S, (A\backslash C)\ T\ (B\backslash C)$

Proof:

(ii) $[(A \cup C)\backslash(B \cup C)] \cup [(B \cup C)\backslash(A \cup C)] =$
$[(A \cup C) \cap (B^c \cap C^c)] \cup [(B \cup C) \cap (A^c \cap C^c)] =$
$[A \cap B^c \cap C^c] \cup [B \cap A^c \cap C^c] \subseteq A \div B \in \mathcal{M}$.

(ii) $(A \cap C)\backslash(B \cap C) \cup (B \cap C)\backslash(A \cap C) =$
$[A \cap C \cap (B^c \cup C^c)] \cup [B \cap C \cap (A^c \cup C^c)] =$
$[A \cap C \cap B^c] \cup [B \cap C \cap A^c] \subseteq A \div B \in \mathcal{M}$.

(iii) follows immediately from (ii).

Proposition 7.9: Every tolerance is isomorphic to some symmetric difference tolerance.

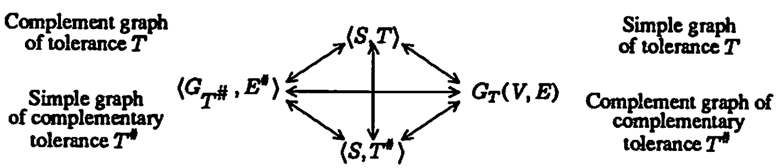
Proof: Let $\langle S,T\rangle$ be a tolerance space. Let $\mathcal{A} = \{\{x\} : x \in S\}$ and $\mathcal{M} = \{\emptyset\} \cup \mathcal{A} \cup \{\{x,y\} : xTy$ and $x \neq y\}$. Then for all $x,y \in S$, xTy *iff* $\{x\} \div \{y\} \in \mathcal{M}$. Thus $x \mapsto \{x\}$ is a tolerance isomorphism of $\langle S,T\rangle$ to $\langle \mathcal{A}, T_{sd\cdot}\rangle$

The following is a generalization of the "common prime factor tolerance" considered in Example 4.26. Let S be a countable set and $\mathcal{A}$ a covering of S such that $\forall x \in S$, $|\{A \in \mathcal{A} : x \in A\}| < \omega$. Let Π denote the set of primes in N. Let $\psi : \mathcal{A} \to \Pi$ be an injection, and for each $A \in \mathcal{A}$, let $\alpha_A : A \to \mathrm{N}$ be an injection. Define $\phi : S \to \mathrm{N}$ by $\phi(x) = \prod\left\{\psi(A)^{\alpha_A(x)} : x \in A\right\}$. Then ϕ is a well-defined injection. Define a tolerance T on S by: $\forall x,y \in S$, xTy *iff* $\phi(x) = \phi(y)$.

In [120], two sets A and B are called *almost disjoint* if $|A \cap B| < \min\{|A|,|B|\}$. The complementary relation, which we call *almost comparable*, may be defined on any subfamily $\mathcal{A} \subseteq 2^S$ by: ATB if $|A \cap B| \geq \min\{|A|,|B|\}$. Trivially, almost comparability is a tolerance relation.

8. Comparison between graph theory and tolerance theory

There is a close relation between graph theory and tolerance space theory. We can even say that these theories deal with the same structure using slightly different conceptual apparatus. However, there are some alternative ways to establish the connection between a tolerance space and the graph related to it. The following diagram shows our choice of the connection between these two structures.



In the diagram, the "simple graph of tolerance T" has $V = S$ and $E = \{\{x,y\} : xTy$ and $x \neq y\}$; the "complement graph of tolerance T" has $V = S$ and edges $E^{\#} = \{\{x,y\} : xT^cy\}$. The "complementary tolerance $T^{\#}$" is defined on S via: $xT^{\#}y$ *iff* xT^cy or $x = y$.

The standard graphic representation of a symmetric relation R on a set S is realized by the assignment of points of a plane to elements of S and joining pairs of those points by lines if the corresponding elements are related by R. Reflexivity of a tolerance relation means that every element should have a loop attached in this representation. This may suggest the choice of the graphs which admit loops. However the majority of the literature in graph theory concerns simple graphs, i.e. those without loops and multiple edges. This justifies our choice of the simple graph $G_T(V, E)$ (we will call it just the graph of T).

The connections illustrated by the diagram above are bijective, hence $\langle S, T\rangle$ and $G_T(V, E)$ can be viewed as cryptomorphic structures.

$T^{\#}$ is not the complement of T^c of T in the relation algebra on S, as T^c is not reflexive. $T^{\#}$ is the complement of T in the lattice of all tolerance relations on a given set S.

Tolerance space theory and graph theory have different terminology. The following dictionary of the concepts which occur in both theories under different names may be helpful to the reader.

The standard terminology of graph theory is given without references, except where the same term has two or more non-equivalent definitions (standard definitions can be found in [94, R2]). In the case of differences in the terminology of papers on topics related to tolerance theory we will provide appropriate references.

We will assume that set S is finite to simplify the comparison with finite graphs.

$$\langle S, T\rangle \longleftrightarrow G_T(V, E)$$

$$S = V$$

$$\{L \in \mathcal{L}(T) : |L| = 2\} = E$$

1.	Family of preclasses of T : $\mathcal{L}(T)$	Family of all complete subgraphs of graph G (Family of cliques of G in [146] as well as in [14, 67, 83, 147, 148, 149, 150])
2.	Family of classes of $T : \mathcal{K}(T)$	Family of cliques of G (maximal cliques in [146])
3.	$T^{\#}$ is an equivalence relation with p classes of cardinalities $m_1, \ldots, m_p$	G is a complete p-chromatic graph $K(m_1, \ldots, m_p)$ (complete p-partite graph in [94])
4.	$\langle S, T^{\#}\rangle$ admits a disjoint covering of S of cardinality p by preclasses	G is p-colorable

5.	The minimal cardinality of a disjoint covering of S by preclasses of $T^{\#}$ is n	The chromatic number $\chi(G)$ is equal to n
6.	As in 5. for the line tolerance T_1 of T instead of T	The line chromatic number $\chi'(G) = \chi(L(G)) = n$

The following concepts will simplify the notation of the next few comparisons. The *set of separated points* in a tolerance space $\langle S,T\rangle$ is $S_0 = \{x \in S : \forall y \in S : xTy \Rightarrow x = y\}$. Let us call $S\backslash S_0$ the *support* of $\langle S,T\rangle$ and $\langle S_S,T_S\rangle = \langle S\backslash S_0, T|_{S\backslash S_0}\rangle$ the *supporting tolerance space* of $\langle S,T\rangle$. Certainly if $\langle S,T\rangle$ does not have separated points, then $\langle S_S,T_S\rangle = \langle S,T\rangle$.

7.	$\mathcal{A}$ is a generating family in $\langle S_S,T_S\rangle$	$\mathcal{A}$ is a covering of edges by complete subgraphs (clique covering in [146])
8.	$\mathcal{A}$ is a generating family in $\langle S_S,T_S\rangle$ such that: $\forall A,B \in \mathcal{A} : \|A \cap B\| \leq 1$	$\mathcal{A}$ is a clique partition [146]

(Observe that this condition is stronger than that defining a base)

9.	$\mathcal{A}$ is a generating family of classes in $\langle S_S,T_S\rangle$ such that $\forall A,B \in \mathcal{A} : \|A \cap B\| \leq 1$	$\mathcal{A}$ is a maximal-clique partition [150]
10.	(By Propositions 4.17, 4.21) dim $\langle S_S,T_S\rangle$ dim $\langle S,T\rangle$	$cc(G)$, the minimum cardinality of a clique covering $cc(G) + \|S_0\|$

One of the main streams of graph theory is connected with $cc(G)$ and its upper bounds for a set V of given cardinality and given specific properties of the graph G. The following items in the bibliography belong to classical graph theory, but are also of great interest for the study of tolerance: [12, 14, 58, 60, 67, 68, 69, 71, 72, 74, 83, 88, 89, 113, 121, 122, 124, 140, 141, 142, 146, 147, 148, 149, 150, 152, 161, 182, 183].

In this paper we studied only one of the dependence systems connected with a tolerance space $\langle S,T\rangle$. Almost all the concepts in the previous sections were defined in terms of the dependence system defined by the closure f_T on $\mathcal{L}(T)$. However, there are some other dependence systems related to a tolerance space. In Section 2 we introduced the image of an element by the relation R, $R(a) = \{x \in S : aRx\}$. We can extend this operation to subsets of S in two possible ways:

i) $R^a(A) = \{x \in S : \forall a \in A,\ aRx\}$

ii) $R^e(A) = \{x \in S : \exists a \in A,\ aRx\}$.

Certainly $R^a(\{a\}) = R(a) = R^e(\{a\})$, and $R^a(A) = \cap\{R(a) : a \in A\}$, $R^e(A) = \cup\{R(a) : a \in A\}$.

For every relation R, the operation $A \mapsto R^{*a}R^a(A)$ is a transitive closure operator [R1, 139]. In this way, every relation R defines a dependence system. Some initial steps in the investigation of the closure T^aT^a, where T is the tolerance determined by a graph $G_T(V, E)$, were taken in [61]. We will not consider this kind of dependence system, but a more comprehensive study will be contained in a separate paper.

Now observe that always $A \subseteq B \Rightarrow R^e(A) \subseteq R^e(B)$. In addition if R is a reflexive relation, then $A \subseteq R^e(A)$. Therefore every reflexive relation defines a dependence system by the (not necessarily transitive) operator $A \mapsto R^e(A)$. Moreover R is a transitive relation *iff* $A \mapsto R^e(A)$ is a transitive closure operator. So while in general if T is a tolerance relation, then $A \mapsto T^e(A)$ is a (not necessarily transitive) operator, it is a transitive closure operator *iff* T is an equivalence relation. Although the name dependence system (introduced much later) and the terminology introduced above did not occur in this context there is a rich literature about the properties of the operator T^e in graph theory. A large part of Berge's book [9] is devoted to it.

Now we return to the dictionary: (dependence system terminology was explained in Section 3)

11.	$A \in$ **T^e-Ind**	A is an independent set of points (stable [9])
12.	$\max\{\lvert A\rvert : A \in$ **T^e-Ind**$\}$	$\beta_0(G)$ [94]$= \alpha(G)$ [9]
13.	$A \subseteq S_2 = \{B \in \mathcal{L}(T) : \lvert B\rvert = 2\}$ and $A \in$ **T$_1^e$-Ind**, where T_1 is the line tolerance of T	A is an independent set of lines
14.	$\max\{\lvert A\rvert : A \in$ **T$_1^e$-Ind**$\}$	$\beta_1(G)$
15.	$A \subseteq S$, $\min\{\lvert A\rvert : A \in$ **T^e-Gen**$\}$	$\alpha_{00}(G)$ [94] $= \sigma_0(G)$ [85] $= \beta(G)$ [9]
16.	$A \subseteq S_2$, $\min\{\lvert A\rvert : A \in$ **T$_1^e$-Gen**$\}$	$\alpha_{11}(G)$ [94] $= \sigma_1(G)$ [85]
17.	$A \subseteq S$ $A \in$ **T^e-Gen**	A is absorbent [9] A is dominant [139]

The dictionary above does not exhaust all parallels between graph theory and tolerance theory. For example, the problem of factorization of tolerance was stated by MHall in the language of combinatorial theory [91]. The graph theoretical result of Mukhopadhyay [125, Thm1] is essentially a corollary to the proof of our Proposition 4.39.

REFERENCES (NOT INCLUDED IN IN THE BIBLIOGRAPHY OF TOLERANCE)

[R1] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloq. Publ. 25, third. ed., Providence, (1967).

[R2] J. W. Essam, M. E. Fisher, *Some basic definitions in graph theory*, Rev. Modern Phys., 42(1970), 272–288.

[R3] C. Greene, D. J. Kleitman, *Proof techniques in the theory of finite sets*, in: Studies in combinatorics, Studies in Math. vol. 17, (G.-C. Rota ed.), MAA, (1978), 22–79.

[R4] G. Kalmbach, *Orthomodular lattices*, Acad. Press, New York, (1983).

[R5] M. D. MacLaren, *Atomic orthocomplemented lattices*, Pacific J. Math., 14 (1964), 597–612.

[R6] M. J. Schroeder, *Description of dependence system by its set of closed subsets*, preprint.

[R7] M. J. Schroeder, *Dependence systems with the operator-image exchange property*, submitted for publication.

[R8] M. J. Schroeder, *Syllogistics defined in relation algebras*, submitted for publication.

[R9] D. Scott, P. Suppes, *Foundational aspects of theories of measurement*, J. Symb. Log., 23(1958), 113–128.

BIBLIOGRAPHY OF TOLERANCE RELATIONS

[1] M. Aigner, *Graphs whose complement and line graph are isomorphic*, J. Comb. Theory, 7(1969), 273–275.

[2] J. Alt, N. Schofield, *Clique analysis of a tolerance relation*, J. Math. Sociology, 6(1978), 155–162.

[3] M. A. Arbib, *Tolerance automata*, Kybernetik (Prague), 3(1967), 223–233.

[4] M. A. Arbib, Automata theory, in Topics in Modern System Theory, R. E. Kal man, P. L. Falb, M. A. Arbib, McGraw-Hill, New York, (1968).

[5] M. Armburst, *A first order approach to correspondence and congruence systems*, Z. Math. Log. Grundl. Math., 19(1973), 215–222. [MR51#12659]

[6] H.-J. Bandelt, *Tolerance relations on lattices*, Bull. Austral. Math. Soc., 23(1981), 367–381. [MR83d:06005]

[7] H.-J. Bandelt, *Tolerance relations as Galois connections*, Acta Sci. Math. (Sze ged), 46(1983), 55–58. (in German) [MR86g:06014]

[8] L. W. Beineke, *Derived graphs and digraphs*, Beiträge zur Graphentheorie, ed. H. Sachs, H. Voss & H. Walter, Teubner, Leipzig, (1968), 17–23.

[9] C. Berge, Graphs and Hypergaphs, North-Holland, Amsterdam, (1973).

[10] F. Bonetti, N.Civolani, *On the product of two tolerances*, Ann. Univ. Ferrara Sec. VII (N. S.), 33(1987), 189–205. [MR89g:08002]

[11] R. Brigham, P. Z. Chinn, R. D. Dutton, *On the clique graph of the line graph*, in: Proc. 15th Southeastern Conf. on Combinatorics, Graph Theory and Computing, (Baton Rouge, 1984), Congr. Numer., 43(1984), 175–184. [MR86j:05118]

[12] N. G. Bruijn, P. Erdös, *On a combinatorial problem*, Indag. Math., 10(1948), 421–423.

[13] L. Cacceta, P. Erdös, E. T. Ordman, N. J. Pullman, *On the difference between clique numbers of a graph*, Ars Combinatoria 19A(1985), 97–106.

[14] L. Cacceta, N. J. Pullman, *On clique covering numbers of cubic graphs*, in: Combinatorial Mathematics X, Proc. Conf. in Adelaide 1982, ed. L. R. A. Casse , Lect. Notes in Math., 1036, Springer, Berlin, (1983), 121–127.

[15] D. deCaen, D. A. Gregory, N. J. Pullman, *Clique covering of complements of paths and cycles*, in Cycles in Graphs, North-Holland Math. Stud. 115, North Holland, Amsterdam, 1985, pp. 257–267. [MR87b:05104]

[16] I. Chajda, *Systems of equations and tolerance relations*, Czech. Math. J., 25(1975), 302–308.

[17] I. Chajda, *A construction of tolerances on modular lattices*, Čas. Pěst. Mat., 101(1976), 195–198.

[18] I. Chajda, *Lattices of compatible relations*, Arch. Math. (Brno), 13(1977), 89–96.

[19] I. Chajda, *Two constructions of compatible relations*, Czech. Math. J., **28**(1978), 439–444.

[20] I. Chajda, *On the tolerance extension property*, Čas. Pěst. Mat., **103**(1978), 327–332. [MR80c:08003]

[21] I. Chajda, *A characterization of distributive lattices by tolerance lattices*, Arch. Math. (Brno), **15**(1979), 203–204.

[22] I. Chajda, *Partitions, coverings and blocks of compatible relations*, Glasnik Mat. (Zagreb), **14**(1979), 21–26.

[23] I. Chajda, *Tolerances and the Chinese remainder theorem*, Math. Slovaca, **30**(1980), 369–371. [MR81m:08006]

[24] I. Chajda, *A Mal'cev characterization of tolerance regularity*, Acta. Sci. Math. (Szeged), **42**(1980), 229–232.

[25] I. Chajda, *Relational classes and their characterizations*, Arch. Math. (Brno), **16**(1980), 199–203.

[26] I. Chajda, *Characterizations of relational blocks*, Alg. Univ., **10**(1980), 65–69.

[27] I. Chajda, *Regularity and permutability of congruences*, Alg. Univ., **10** (1980), 159–162.

[28] I. Chajda, *Recent results and trends in tolerances on algebras and varieties*, in: Finite Algebra and Multiple-Valued Logic, Proc. Colloq. Szeged, 1979, ed. B. Csakany & I. G. Rosenberg, Colloq. Math. Soc. Janos Bolyai, vol. 28, North-Holland, (1981), 69–95. [MR83c:08003]

[29] I. Chajda, *Distributivity and modularity of lattices of tolerance relations*, Alg. Univ., **12**(1981), 247–255. [MR82d:08008]

[30] I. Chajda, *Tolerance Hamiltonian varieties of algebras*, Acta. Sci. Math. (Sze ged), **44**(1982) , 13–16.

[31] I. Chajda, *Tolerance trivial algebras and varieties*, Acta Sci. Math. (Szeged), **46**(1983), 35–40.

[32] I. Chajda, *Relatives of 3-permutability and principle tolerance trivial varieties*, Ann. Univ. Sci. Budapest, Eötwös Sect. Math., **28**(1985), 37–47. [MR87k:08002]

[33] I. Chajda, *Varieties with tolerance and congruence extension property*, Arch. Math. (Brno), **21**(1985), 5–12. [MR87d:08009]

[34] I. Chajda, *Transferable tolerances and weakly tolerance regular lattices*, in: Lectures in Universal Algebra, ed. L. Szabo & Á. Szendrei, Colloq. Math. Soc. Janos Bolyai 43, North-Holland, Amsterdam, (1986), 27–40. [MR87k:08003]

[35] I. Chajda, *Tolerance and congruence conditions; connections and applications*, Mathematica (Cluj), **28**(1986),No. 1, 31–37. [MR88g:08002]

[36] I. Chajda, *Algebras with principal tolerances*, Math. Slovaca, **37**(1987), 169–172. [MR88h:08005]

[37] I. Chajda, *Tolerances in congruence permutable algebras*, Czech. Math. J., **38**(1988), 218–225. [MR89f:08005]

[38] I. Chajda, *Principal tolerances on lattices*, Czech. Math. J., **39**(1989), 176–180. [MR90j:06010]

[39] I. Chajda, J. Dalik, J. Niederle, V. Vesely, B. Zelinka, *How to draw tolerance lattices of finite chains*, Arch. Math. (Brno), **16**(1980), 161–166.

[40] I. Chajda, J. Duda, *Blocks of binary relations*, Ann. Univ. Sci. Sect. Math. (Budapest), **13/14**(1979/80), 3–9.

[41] I. Chajda, J. Niederle, B. Zelinka, *On existence conditions for compatible tolerances*, Czech. Math. J., **26** (1976), 304–311.

[42] I. Chajda, J. Nieminen, *Direct decomposability of tolerances on lattices, semilattices and quasilattices*, Czech. Math. J., **32**(1982), 110–115.

[43] I. Chajda, J. Nieminen, *Atomicity of tolerance lattices*, Czech. Math. J., **30**(1980), 606–609. [MR83c:06006]

[44] I. Chajda, B. Pondelíček, *Algebras with tolerance extension property in 0*, Czech. Math. J., **39**(1989), 142–146. [MR90a:08001]

[45] I. Chajda, B. Zelinka, *Tolerance relations on lattices*, Čas. Pěst. Mat., **99**(1974), 394–399.

[46] I. Chajda, B. Zelinka, *Compatible relations on algebras*, Čas. Pěst. Mat., **100**(1975), 355–360.

[47] I. Chajda, B. Zelinka, *Weakly associative lattices and tolerance relations*, Czech. Math. J., **26**(1976), 259–269.

[48] I. Chajda, B. Zelinka, *Lattices of tolerances*, Čas. Pěst. Mat., **102**(1977), 10–24.

[49] I. Chajda, B. Zelinka, *Permutable tolerances*, Čas. Pěst. Mat., 102(1977), 217–220. [MR56#15509]

[50] I. Chajda, B. Zelinka, *Minimal compatible tolerances on lattices*, Czech. Math. J., 27(1977), 452–459. [MR56#15508]

[51] I. Chajda, B. Zelinka, *Metrics and tolerances*, Arch. Math. (Brno), 14 (1978), 193–200.

[52] I. Chajda, B. Zelinka, *Tolerance relations on direct product of monoids and distributive lattices*, Glasnik Mat. Ser. III, 14(1979), 11–16. [MR81f:08001]

[53] I. Chajda, B. Zelinka, *Tolerances and convexity*, Czech. Math. J., 29(1979), 584–587. [MR81a:06010]

[54] I. Chajda, B. Zelinka, *Every finite chain is the tolerance lattice of some lattice*, Glas. Mat. Ser. III, 20(1985), 3–6. [MR87d:06022]

[55] I. Chajda, B. Zelinka, *A characterization of tolerance-distributive tree semilattices*, Czech. Math. J., 37(1987), 175–180. [MR88c:06009]

[56] I. Chajda, B. Zelinka, *Tolerances and congruences on implication algebras*, Czech. Math. J., 38(1988), 207–217. [MR89k:08001]

[57] V. Chvátal, *Perfect graphs*, in: Surveys in Combinatorics 1987, London Math. Soc. Lect. Notes Series 123, ed. C. Whitehead, Cambridge Univ. Press, Cambridge, (1987), 43–51.

[58] K. Čulik, *The completeness number*, in: Problems presented on special sessions of the symposium, Theory of graphs and its applications, Proc. Symp. Smolenice 1963, ed. M. Fiedler, Acad. Press, New York, (1964), 157–158.

[59] K. Denecke, L. Szabo, *Tolerance-free algebras having majority term functions and admitting no proper subalgebras*, in: Lect. in universal algebra, Szeged 1983, Colloq. Soc. Janos Bolyai, vol. 43, North-Holland, Amsterdam, (1986), 97–107. [MR88a:08001]

[60] G. A. Dirac, *On complete graphs and complete stars contained as subgraphs in graphs*, Math. Scand., 12(1963), 39–46. [MR28#2539]

[61] W. Dörfler, *A complete lattice in graph theory*, in: Combinatorics vol. 1, ed.A. Hajnal, V. T. Sos, Colloq. Math. Soc. Janos Bolyai 17, North-Holland, Amsterdam, (1978), 257–269.

[62] K. Drbohlav, *Remarks on n-semigroups and tolerances*, Proc. 2nd International Symposium on n-ary structures, Varna 1983; Sofia, 1985, pp. 79–82. [MR87a:20002].

[63] K. Drbohlav, *Remarks on tolerance algebras*, Acta Univ. Carolinae-Math. Phys., 22(1981), 11–16. [MR83a:08008]

[64] J. Duda, *Mal'cev conditions for congruence-regular and congruence-permutable varieties*, Comm. Math. Univ. Carolinae, 20 (1979), 795.

[65] J. Duda, *Mal'cev conditions for directly decomposable compatible relations*, Czech. Math. J., 39(1989), 674–680. [MR90i:08006]

[66] A. L. Dulmage, N. S. Mendelsohn, *Coverings of bipartite graphs*, Canad. J. Math., 10(1958), 517–534.

[67] P. Eades, N. J. Pullman, P. J. Robinson, *On clique partition numbers of regular graphs*, in Progress in Graph Theory, Waterloo, Ont. 1982, Acad. Press, Toronto, Ont., (1982), 192–201. [MR86h:05066]

[68] P. Erdös, *On the number of complete subgraphs contained in certain graphs*, Magyar Tad. Akad. Mat. Kutato Int. Közl., 7(1962), 459–464. [MR27#1937]

[69] P. Erdös, *On cliques in graphs*, Israel J. Math., 4(1966), 233–234. [MR34#5700]

[70] P. Erdös, R. Faudree, E. T. Ordman, *Clique partitions and clique coverings*, Discr. Math. 72(1988), 93–101.

[71] P. Erdös, A. W. Goodman, L. Posa, *The representation of graphs by set intersections*, Canad. J. Math., 18(1966), 106–112.

[72] P. Erdös, P. Hajnal, E. C. Milner, *On the complete subgraphs of graphs defined by systems of sets*, Acta Math. Acad. Sci. Hungar., 17(1966), 159–229.

[73] P. Erdös, R. Rado, *Partition relations and transitivity domains of binary relations*, J. London Math. Soc., 42(1967), 624–633.

[74] F. Escalante, B. Toft, *On clique-critical graphs*, J. Combin. Theory B, 17(1974), 170–182. [MR51#5408]

[75] W. L. Fischer, *Tolerance spaces and linguistics*, in: Logic, Language and Probability, ed. R. J. Bogdan, I. Niiniluoto , D. Reidl, Dordrecht, (1973).

[76] P. C. Fishburn, *Intransitive indifference with unequal indifference intervals*, J. Math. Psychology, 7(1970), 144–149.

[77] P. C. Fishburn, Interval orders and interval graphs. A study of partially ordered sets, Wiley, New York, (1985).

[78] P. C. Gilmore, A. J. Hoffman, *A characterization of comparability graphs and of interval graphs*, Canad. J. Math., 16(1964), 539–548.

[79] A. O. Golosov, V. N. Tulovskii, M. Sh. Tsalenko, *Optimization problems in tolerance spaces*, Automatic Documentation and Mathematical Linguistics (Nauchno-Tekhnicheskaya Informatsiya), Ser. 2, 15(No.2)(1981), 70–77.

[80] M. C. Golumbic, C. L. Monma, *A generalization of interval graphs with tolerances*, Congr. Numer., 35(1982), 321–331. [MR85f:05105]

[81] M. C. Golumbic, C. L. Monma, W. T. Trotter, *Tolerance graphs*, Discrete Appl. Math., 9(1984), 157–170. [MR86b:05063]

[82] A. W. Goodman, *Set equations*, Amer. Math. Month., 72(1965), 607–613.

[83] D. A. Gregory, N. J. Pullman, *On a clique covering problem of Orlin*, Discrete Math., 41(1982), 97–99.

[84] J. de Groot, *Graph representations of topological spaces*, in: Topological structures, Proc. Sympos. in honour of J. de Groot, Amsterdam 1973, Math. Centrum, Amsterdam, (1974), 29–37. [MR51#9012]

[85] R. P. Gupta, *Independence and covering numbers of line graphs and total graphs*, in: Proof techniques in graph theory, Proc. Sec. Ann Arbor Graph Theory Conf. 1968, ed. F. Harary, Acad. Press, New York, (1969), 61–62.

[86] V. Ya. Gusakov, S. M. Yakubovich, *Galois connections and binary relation representation theorems*, Informacionnyj analiz, VINITI, Ser. 2, No. 7(1974), 3–6. (in Russian)

[87] V. Ya. Gusakov, S. M. Yakubovich, *Classification algorithms*, Automatic Documentation and Mathematical Linguistics (Nauchno-Tekhnicheskaya Informatsiya), Ser. 2, 10(no.4)(1976), 110–119. [MR45#4923]

[88] R. K. Guy, E. C. Milner, *Covering graphs by complete subgraphs*, in: Theory of graphs, Proc. Colloq. Tihany 1966, Acad. Press, (1968), 151–159.

[89] R. K. Guy, E. C. Milner, *Graphs defined by coverings of a set*, Acta Math. Acad. Sci. Hungar., 19(1968), 7–21.

[90] A. Hajnal, J. Suranyi, *Über die Auflösung von Graphen in vollständige Teilgraphen*, Ann. Univ. Sci. Budapest. Eötwös. Sect. Math., 1(1958), 113–121. [MR21#1944]

[91] M. Hall, *A problem in partitions*, Bull. Amer. Math. Soc., 47(1941), 804–807.

[92] R. C. Hamelink, *A partial characterization of clique graphs*, J. Combin. Theory, 5(1968), 192–197.

[93] F. Harary, *A graph theoretic approach to similarity relations*, Psychometrika, 29(1964), 143–151.

[94] F. Harary, *Graph theory*, Addison Wesley, Reading, Mass., (1969).

[95] F. Harary, *On the intersection number of a graph*, in: Proof techniques in graph theory, Proc. Second Ann Arbor Graph Theory Conf. 1968, ed. F. Harary, Acad. Press, New York, (1969), 71–72.

[96] F. Harary, A. Lempel, *On clique-extremal (p, q)-graphs*, Networks, 4(1974), 371–378. [MR51#5394]

[97] F. Harary, R. Norman, *The dissimilarity characteristic of linear graphs*, Proc. Amer. Math. Soc., 5(1954), 131–135. [MR15, 546]

[98] F. Harary, I. C. Ross, *A procedure for clique detection using the group matrix*, Sociometry, 20(1957), 205–215.

[99] F. Harary, I. C. Ross, *The square of a tree*, Bell System Tech. J., 39(1960), 641–647.

[100] B. Hedman, *Clique graphs of time graphs*, J. Combin. Theory B, 37(1984), 270–278. [MR86b:05064]

[101] I. Ilzinia, *Finding the largest complete subgraph of a graph*, Automat. i Vychisl. Tekh., (1965), No. 11, 43–48. (in Russian)

[102] I. Ilzinia, *Finding the cliques of a graph*, Automat. i Vychisl. Tekh., (1967), No. 2, 7–11. (in Russian)

[103] M. F. Janowitz, *An order theoretic model for cluster analysis*, SIAM J. Appl. Math., 34(1978), 55–72. [MR57#16161]

[104] M. F. Janowitz, *Semiflat L-cluster methods*, Discrete Math., 21(1978), 47–60. [MR81h:06002]

[105] M. F. Janowitz, *Applications of the theory of partially ordered sets to cluster analysis*, in: Universal algebra and applications, Banach Center Publ. 9, PWN, Warsaw, (1982), 305–319.

[106] N. Jardine, R. Sibson, Mathematical Taxonomy, Wiley, New York, (1973).

[107] L. Kalmar, *Semantics, synonymy, and translation*, Report on the Symposium Machine Translation-67, Budapest, (1967).

[108] J. B. Kelly, *Products of zero-one matrices*, Canad. J. Math., **20**(1968), 298–329.

[109] B. P. Kononov, *The difference relation: Duality semantics*, Automatic Documentation and Mathematical Linguistics (Nauchno-Tekhnicheskaya Informatsiya), Ser. 2, **11**(No. 3)(1977), 71–78.

[110] J. Krausz, *Demonstration nouvelle d'un théoréme de Whitney sur les résaux*, Mat. Fiz. Lapok, **50**(1943), 75–85. (in Hungarian)

[111] D. Kurepa, *On reflexive symmetric relations and graphs*, Rozpr. Slov. Akad. Znan. In. Umeth. Razrad. Mat. Fiz. In Tehn. Vede, A**4**(1953), 28. [MR17, 135]

[112] C. G. Lekkerkerker, J. Ch. Boland, *Representation of a finite graph by a set of intervals on the real line*, Fund. Math., **51**(1962), 45–64.

[113] L. Lovász, *On covering of graphs*, in: Theory of graphs, Proc. Colloq. Tihany 1966, Acad. Press, New York, (1968), 231–236.

[114] R. D. Luce, *Connectivity and generalized cliques in sociometric group structure*, Psychometrika, **15**(1950), 169–190. [MR12, 39]

[115] E. Marczewski, *Sur deux propriétes des classes d'ensembles*, Fund. Math., **33**(1945), 303–307.

[116] A. Marczyk, Z. Skupien, *Characterizing line graphs of general graphs*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys., **22**(1974), 235–241.

[117] I. G. Maurer, I. Virág, *Sur les classes de tolérances d'une algébre universelle*, Publ. Math. (Debrecen), **25**(1978), 237–240.

[118] V. V. Menon, *On repeated interchange graphs*, Amer. Math. Month., **73** (1966), 986–989. [MR34#2493]

[119] E. C. Milner, K. Prikry, *Almost disjoint sets*, in: Surveys in Combinatorics 1987, London Math. Soc. Lect. Note Series 123, ed. C. Whitehead, Cambridge Univ. Press, Cambridge, (1987), 155–172.

[120] E. C. Milner and R. Woodrow, *On directed graphs with an independent covering set*, Graph. Combin. 5(1989), 363–369. [MR90m:05061]

[121] J. Moon, *On the number of complete subgraphs of a graph*, Canad. Math. Bull., 8(1965), 831–834. [MR33#1249]

[122] J. Moon, *On independent complete subgraphs in a graph*, Canad. J. Math., 19(1967), 95–102.

[123] J. W. Moon, *A note on approximating symmetric relations by equivalence relations*, SIAM J. Appl. Math., 14(1966), 226–227. [MR34#5691]

[124] J. Moon, L. Moser, *On cliques in graphs*, Israel J. Math., 3(1965), 23–28. [MR32#60]

[125] A. Mukhopadhyay, *The square root of a graph*, J. Combin. Theory, 2(1967), 290–295.

[126] J. Mycielski, *Complete subgraphs and measure*, Proc. Sympos. Rome, (1966), 255–256.

[127] J. Niederle, *Relative bicomplements and tolerance extension property in distributive lattices*, Čas. Pěst. Mat., 103(1978), 250–254.

[128] J. Niederle, *Tolerance extensions in distributive lattices*, Arch. Math. (Brno), 16(1980), 99–104.

[129] J. Niederle, *On atoms in tolerance lattices of distributive lattices*, Čas. Pěst. Mat., 106(1981), 311–315. [MR83a:08005]

[130] J. Niederle, *Conditions for transitive principal tolerances*, Czech. Math. J., 39(1989), 380–381.

[131] J. Nieminen, *Tolerance relations on simple ternary algebras*, Arch. Math. (Brno), 13(1977), 105–109. [MR57#206]

[132] J. Nieminen, *Tolerance relations on join-semilattices*, Glasnik Mat. (Zagreb), 12(1977), 243–246.

[133] J. Nieminen, *On structural properties of normality relations on lattices*, Arch. Math. (Brno), 13(1977), 159–168.

[134] J. Nieminen, *Tolerance relations on complete lattices*, Comment. Math. Univ. Carolinae, 18(1977), 639–645. [MR57#207]

[135] J. Nieminen, *Screens and rounding mappings*, Z. Angew. Math. Mech., 58(1978), 495–511.

[136] J. Nieminen, *Fuzzy mappings and algebraic structures*, Fuzzy Sets and Systems, **1**(1978), 231–235.

[137] J. Nieminen, *Blocks, error algebras and flow sets*, Glasnik Mat. (Zagreb), **14**(1979), 381–385.

[138] R. Z. Norman, M. O. Rabin, *An algorithm for a minimum cover of a graph*, Proc. Amer. Math. Soc., **10**(1959), 315–319.

[139] O. Ore, *Theory of graphs*, Amer. Math. Soc. Colloq. Publ. vol. 38, Amer. Math. Soc., Providence, R. I., (1962).

[140] J. Orlin, *Contentment in graph theory: Covering graphs with cliques*, Indag. Math., **39**(1977), 406–424.

[141] M. D. Plummer, *Some covering concepts in graphs*, J. Combin. Theory, **8**(1970), 91–98.

[142] B. Pondeliček, *On tolerances on periodic semigroups*, Czech. Math. J., **28**(1978), 647–649.

[143] B. Pondeliček, *Tolerance distributive and tolerance Boolean varieties of semigroups*, Czech. Math. J. **36**(1986), 617–622.

[144] B. Pondeliček, *Commutative semigroups whose lattice of tolerances is Boolean*, Czech. Math. J., **38**(1988), 226–230.

[145] B. Pondeliček, *Direct decomposability of tolerances and congruences on semigroups*, Czech. Math. J., **38**(1988), 701–704.

[146] N. J. Pullman, *Clique coverings of graphs – A survey*, in: Combinatorial Mathematics X, Proc. Conf. in Adelaide 1982, ed. L. R. A. Casse, Lect. Notes in Math. 1036, Springer, Berlin, (1983), 72–85.

[147] N. J. Pullman, D. de Caen, *Clique coverings of graphs III: clique coverings of regular graphs*, Congr. Numer., **29**(1980), 795–808.

[148] N. J. Pullman, D. de Caen, *Clique coverings of graphs I: clique partitions of regular graphs*, Utilitas Math., **19**(1981), 177–205. [MR82j:05089a]

[149] N. J. Pullman, A. Donald, *Clique coverings of graphs II: complements of cliques*, Utilitas Math., **19**(1981), 207–213.

[150] N. J. Pullman, H. S. Shank, W. D. Wallis, *Clique coverings of graphs. V. Maximal-clique partitions*, Bull. Austral. Math. Soc., **25**(1982), 337–356. [MR84k:05072b]

[151] D. K. Ray-Chaudhuri, *Characterization of line graphs*, J. Combin. Theory, 3(1967), 201–214.

[152] K. B. Reid, *The number of graphs on N vertices with 3 cliques*, J. London Math. Soc., 8(1974), 94–98. [MR51#2979]

[153] P. L. Renz, *Intersection representations of graphs by arcs*, Pacific J. Math., 34(1970), 501–510. [MR42#5839]

[154] F. S. Roberts, *Indifference*, in: Proof techniques in graph theory, Proc. Second Ann Arbor Graph Theory Conf. 1968, ed. F. Harary, Acad. Press, New York, (1969), 139–146.

[155] F. S. Roberts, *On nontransitive indifference*, J. Math. Psychology, 7(1970), 243–258.

[156] F. S. Roberts, *Tolerance geometry*, Notre Dame J. Formal Logic, 14(1973), 68–76. [MR50#6822]

[157] F. S. Roberts, J. Spencer, *A characterization of clique graphs*, in: Combinatorial structures and their applications, Proc. Calgary Internat. Conf., Calgary 1969, Gordon and Breach, New York, (1970), 367–368.

[158] D. G. Rogers, *Similarity relations on finite ordered sets*, J. Comb. Theory A, 23(1977), 88–98, Erratum 25(1978), 95–96.

[159] A. van Rooij, H. Wilf, *The interchange graph of a finite graph*, Acta Math. Acad. Sci. Hungar., 16(1965), 263–269.

[160] I. G. Rosenberg, D. Schweigert, *Compatible orderings and tolerances of lattices*, in: Finite Algebra and Multiple-Valued Logic, Proc. Colloq. Szeged, 1979, ed. B. Csakany, I. G. Rosenberg, Colloq. Math. Soc. Janos Bolyai, vol. 28, North-Holland, (1981), 119–150.

[161] M. Rosenfeld, *How many graphs have n vertices and k cliques?*, Amer. Math. Month., 78(1971), 49–50.

[162] D. Schweigert, *Compatible relations of modular and orthomodular lattices*, Proc. Amer. Math. Soc., 81(1981), 462–464.

[163] D. Schweigert, *Tolerances and commutators on lattices*, Bull. Austral. Math. Soc., 37(1988), 213–219. [MR89b:08003]

[164] J. Sedlaček, *Some properties of interchange graphs*, in: Theory of graphs and its applications, Proc. Symp. Smolenice 1963, ed. M. Fiedlel, Acad. Press, New York, (1964), 145–150.

[165] L. N. Ševrin, N. D. Filippov, *Partially ordered sets and their comparability graphs*, Siberian Math. J., **11**(1970), 497–509. [MR42#4451]

[166] H.-C. H. Sherman, *Compatible and pseudocompatible tolerances*, Tamkang J. Math., **14**(1976), 1–9.

[167] V. M. Sitnikov, *Semigroups with Boolean algebra of stable tolerances*, Ural Gos. Univ. Mat. Zap. **13**(1983), 146–158.

[168] Yu. A. Shreider, *A mathematical model of classification theory*, Automatic Documentation and Mathematical Linguistics (Nauchno-Tekhnicheskaya Informatsiya), Ser. 2, **2**(No. 4)(1968), 1–15.

[169] Yu. A. Shreider, *Equality, similarity and order*, Nauka, Moscow, (1971) (in Russian) [MR52#10430][Polish transl. WNT, Warszawa, (1975)].

[170] Yu. A. Shreider, *Stratified tolerances (Toward a mathematical theory of classification)*, Automatic Documentation and Mathematical Linguistics (Nauchno-Tekhnicheskaya Informatsiya), Ser. 2, **7** (No. 1)(1973), 47–51. [MR53#2073]

[171] Yu. A. Shreider, *The logic of classification*, Automatic Documentation and Mathematical Linguistics (Nauchno-Tekhnicheskaya Informatsiya), Ser. 2, **7** (No. 2)(1973), 30–36.

[172] Yu. A. Shreyder, *On the question of the structure of scientific collectives*, in: Questions in the theory and practice of informational and information-search systems, Ukr. NIINTI, Kiev, (1968). (in Russian)

[173] Yu. A. Shreyder, *Tolerant spaces*, J. of Cybernetics, **1** (No. 2)(1971), 115–122. [MR45#9879]

[174] R. G. Simonyan, *Tolerance spaces induced by majorant spaces*, Mat. Voprosy Kibernet. Vychisl. Tekhn., **16**(1986), 50–67, 220. (in Russian) [MR88h:08002]

[175] J. Spencer, *Nonconstructive methods in discrete mathematics*, in: Studies in Combinatorics, Studies in Math. vol. 17, (G.-C. Rota ed.), MAA, (1978), 142–178.

[176] A. P. Street, W. D. Wallis, *Combinatorics: A first course*, The Charles Babbage Research Centre, Winnipeg, (1982). [MR56#2831]

[177] L. Szabo, *Tolerance-free infinite algebras having majority term functions and no proper subalgebras*, in: Contributions to general algebra 3, (Vienna 1984), Hölder-Pichler-Tempsky, Vienna, (1985), 355–363.

[178] M. Sh. Tsalenko, *Algebraic description of the composition and structure of information files (tolerant ringoids)*, Automatic Documentation and Mathematical Linguistics (Nauchno-Tekhnicheskaya Informatsiya), Ser. 2, **11** (No. 2) (1977), 102–114.

[179] P. Turan, *On the theory of graphs*, Colloq. Math., 3(1954), 19–30.

[180] C.-H. Tzeng, C.-S. O. Tzeng, *Tolerance spaces and almost periodic functions*, Bull. Inst. Math. Acad. Sinica, 6(1978), 159–173.

[181] G. Wagner, *Common phrases and minimum space text*, *Comm. of the Assoc.* for Computing Machinery, **16**(no.3)(1973), 148–152.

[182] W. D. Wallis, *Asymptotic values of clique partition numbers*, Combinatorica, 2(1982), 99–101. [MR83j:05049]

[183] W. D. Wallis, *Clique partitions of the complement of a one-factor*, in: Proc. of 14th Manitoba Conf. on Numerical Math. and Computing, Winnipeg, Man., 1984, Congr. Numer., **46**(1985), 317–319.

[184] H. Werner, *A Mal'cev condition for admissible relations*, Alg. Univ., 3(1973), 263.

[185] H. Whitney, *Congruent graphs and the connectivity of graphs*, Amer. J. Math., 54(1932), 150–168.

[186] E. S. Wolk, *The comparability graph of a tree*, Proc. Amer. math. Soc., 13(1962), 789–795.

[187] E. S. Wolk, *A note on "The comparability graph of a tree"*, Proc. Amer. Math. Soc., **16**(1965), 17–20.

[188] S. M. Yakubovich, *An axiomatic theory of similarity*, Automatic Documentation and Mathematical Linguistics (Nauchno-Tekhni-cheskaya Informatsiya), Ser. 2, 2(No.t4)(1968), 16–25.

[189] C. T. Zahn, Jr., *Approximating symmetric relations by equivalence relations*, SIAM J. Appl. Math., 12(1964), 840–847.

[190] E. C. Zeeman, *The topology of the brain and visual perception*, in: Topology of 3-manifolds and related topics, Proc. The Univ. of Georgia Institute, 1961, ed. M. K. Fort, Jr., Prentice-Hall, Englewood Cliffs, N. J., (1962), 240–256. [MR25#3796]

[191] E. C. Zeeman, O. Buneman, *Tolerance spaces and the brain*, in: On the Path to Theoretical Biology, Moscow, (1970).

[192] B. Zelinka, *On the number of independent complete subgraphs*, Publ. Math. Debrecen, **13**(1966), 95–97. [MR34#5707]

[193] B. Zelinka, *Tolerance graphs*, Comm. Math. Univ. Carolinae, **9**(1968), 121–131.

[194] B. Zelinka, *Tolerance in algebraic structures*, Czech. Math. J., **20**(1970), 179–183. [MR41#5277]

[195] B. Zelinka, *Tolerance in algebraic structures II*, Czech. Math. J., **25**(1975), 175–178.

[196] B. Zelinka, *Tolerances and congruences on tree algebras*, Czech. Math. J., **25**(1975), 634–637.

[197] B. Zelinka, *Tolerance relations on semilattices*, Comm. Math. Univ. Carolinae, **16**(1975), 333–338.

[198] B. Zelinka, *Tolerance relation on periodic commutative semigroups*, Czech. Math. J., **27**(1977), 167–169.

[199] B. Zelinka, *A remark on systems of maximal cliques of a graph*, Czech. Math. J., **27**(1977), 617–618.

[200] B. Zelinka, *Tolerances on graph algebras*, Czech. Math. J., **37**(1987), 313–317. [MR88c:08001]

[201] B. Zelinka, *Tolerances of polyunary algebras*, Arch. Math. (Brno), **24**(1988), 1–4.