Irregularity Strength of Uniform Hypergraphs

A. Gyárfás[1], M. S. Jacobson[1], L. Kinch, J. Lehel[2], R.H. Schelp[3]

Abstract. A hypergraph is irregular if no two of its vertices have the same degree. It is shown for all $r \geq 3$ and $n \geq r + 3$, that there exist irregular r-uniform hypergraphs of order n. For $r \geq 6$ it is proved that almost all r-uniform hypergraphs are irregular. A linear upper bound is given for the irregularity strength of hypergraphs of order n and fixed rank. Furthermore the irregularity strength of complete and complete equipartite hypergraphs is determined.

Introduction

In this paper the concept of irregularity, introduced in [1] for graphs, is generalised to the study of hypergraphs. A *hypergraph* H is a set of vertices, denoted $V(H)$, and a collection of the elements from the power set of $V(H)$. This collection will be called the edge set and will be denoted $E(H)$. The rank of a hypergraph is the maximum number of vertices in any edge. A hypergraph is *r-uniform* if each edge has exactly r elements while it is *r-regular* if each vertex is contained in exactly r edges. Throughout the paper only hypergraphs without multiple edges will be considered. A hypergraph H' is a *partial hypergraph* of H if $V(H') \subseteq V(H)$, $E(H') \subseteq E(H)$. A hypergraph H is *vertex-distinguishable* if for each pair of vertices there is an edge H which contains precisely one of them. Finally, the *degree* of a vertex $x \varepsilon V(H)$ is the number of edges in which x is contained.

The study of irregularity was started in [1] for integer weighted graphs. The same concepts are extended here to hypergraphs. A hypergraph is *irregular* if all the degrees are distinct. The only irregular graph is the graph with a single vertex; this is not the case for hypergraphs. In section 3 irregular hypergraphs are studied. It is shown by a direct construction that there exist irregular r-uniform hypergraphs with n vertices if and only if $r \geq 3$ and $n \geq r + 3$ (Theorem 3.3). Moreover, using probabilistic techniques, it is shown that almost all r-uniform hypergraphs are irregular if $r \geq 6$ (Theorem 3.4). In [1], the concept of irregularity strength of a graph was presented. This is given in the context of hypergraphs. Consider a hypergraph H with positive integer weights assigned to its edges. This is an irregular weighting if for all $x \varepsilon V(H)$ the degrees (the sum of the weights of the hyperedges containing x) are distinct. The minimum of the largest weights assigned over all such irregular weightings of H will be called the *irregularity strength* of H and will be denoted $s(H)$. If H has no irregular weighting (i.e. H is not vertex distinguishable) then $s(H) = \infty$. Note that if H is an irregular hypergraph then $s(H) = 1$.

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Section 2 of this paper gives some general results on $s(H)$, in particular it includes upper and lower bounds for $s(H)$. In case of connected graphs of order $n \geq 4$, the upper bound $s(G) \leq n-1$ has been proved in [4] (this bound is sharp as the star $K_{1,n-1}$ shows). In the case of r-uniform hypergraphs ($r \geq 3$) it seems to be difficult to get sharp upper bounds. It is shown that $s(H) \leq r \mid V(H) \mid -r + 1$ (Theorem 2.4). On the other hand, it is unknown whether there exists a hypergraph H of order n with $s(H) \geq 2n$. The best known example is based on properties of B_2-sequences investigated in additive number theory, giving a hypergraph H with $s(H) \geq 2n - o(n)$ (Corollary 2.3). The lower bound given in terms of the degree sequence in [4] for graphs remains true for hypergraphs (Proposition 2.1).

In section 4 the irregularity strength of complete and complete equipartite hypergraphs are determined. (Theorem 4.1 and Theorem 4.7)

General results on the irregularity strength.

The first objective of this section is to present upper and lower bounds for the irregularity strength of a hypergraph. The lower bound given in [4] generalizes to hypergraphs and is the content of the first proposition.

Proposition 2.1. *If H has n_i vertices of degree i, then*

$$s(H) \leq \max_{i \leq j} \frac{(\sum_{k=i}^{j} n_k) + i - 1}{j}$$

Proof: For fixed i and j, $i \leq j$, define $S(i,j)$ as the set of vertices with degrees between i and j (inclusive). If there is an irregular weighting with maximum weight t, then the weighted degrees of $S(i,j)$ are in the interval $[i, jt]$. But these weighted degrees are all different. Therefore

$$jt - i + 1 \geq |S(i,j)| = \sum_{k=1}^{j} n_k$$

and

$$t \geq \frac{\sum_{k=1}^{j} n_k + i - 1}{j} \text{ follows.}$$

∎

Note that the lower bound of Proposition 2.1 does not depend on the rank of the hypergraph, so it is not surprising that the lower bound can not be attained if the rank is high. For example, it H is a 2-regular hypergraph, the lower bound of Proposition 2.1 is $\frac{n+1}{2}$ but Corollary 2.3 gives a lower bound of $2n - o(n)$.

To see how the irregularity strength of hypergraphs relates to some problems of additive number theory, it is fruitful to translate its definition to the dual hypergraph.

Assume that $X = \{x_1, x_2, \dots x_m\}$ is a set of variables and $S = \{S_1, S_2, \dots, S_n\}$ is a set of distinct sums involving these variables:

$$S_i = \sum_{j=1}^{m} \varepsilon_{ij}.x_j$$

where $\varepsilon_{ij} = 0$ or 1, for $1 \leq j \leq m$ and $1 \leq i \leq n$. An assignment of positive integers $x_j = a_j, 1 \leq j \leq m$ is called irregular if $S_1, \dots, S_n$ become distinct integers. Consider the problem of determining

$$r^*(X, S) = \min \max_{1 \leq j \leq m} a_j,$$

the minimum taken over all irregular assignments. Let (X, S) be the hypergraph H with $V(H) = X$ and edges $\{x_j : \varepsilon_{ij} = 1\}, 1 \leq i \leq n$. Further let H^* be the dual hypergraph of $H : V(H^*) = S$ and edges $\{S_j : \varepsilon_{ij} = 1\}, 1 \leq j \leq m$. Obviously $s^*(H) = s(H^*)$.

In the case when $H = K_m$, the complete graph of order m, an irregular assignment for $x_1, x_2, \dots, x_m$ is called a B_2'-sequence. In other words, a B_2'-sequence is $a_1 < a_2 < \dots < a_m$ where the sums $a_i + a_j$ are distinct for all $1 \leq i < j \leq m$. The name B_2' was chosen because of the resemblance to B_2*-sequences* introduced in [3]: a sequence of positive integers $a_1, < a_2 < \dots < a_m$ with distinct sums $a_i + a_j$, for $1 \leq i \leq j \leq m$.

It is well known that B_2-sequences satisfy

$$\sqrt{a_m}(1 - \varepsilon) < m \leq \sqrt{a_m} + o(\sqrt{a_m}). \tag{1}$$

The lower bound is due to Chowla and Erdos using a construction of Singer[6] and the upper bound to Erdos and Turan [2], also see [5]. A detailed discussion appears in [3]. The lower bound in[1] is obviously valid for B_2' sequences (since a B_2 sequence is automatically a B_2'-sequence). Erdos pointed out (personal communication) that the upper bound in [1] also holds for B_2' sequences. Therefore [1] holds for any B_2'-sequence which give the following result.

Theorem 2.2. *$s^*(K_m) / \binom{m}{2} \longrightarrow 2$ as m approaches infinity.* ∎

The consequence of Theorem 2.2 is that the dual hypergraph H of K_m (having m edges of cardinality $m - 1$ and $n = \binom{m}{2}$ vertices each of degree two) satisfies $s(H) \geq 2n - o(n)$.

Corollary 2.3. *There exist 2-regular hypergraphs of order n with irregularity strength $2n - o(n)$.* ∎

It is not known how large $s(H)$ can be for 2-regular hypergraphs of order n, or dually how large $s^*(G)$ can be for graph with n edges.

In [4], it is proved that $s(G) \leq |V(G)| - 1$ for connected graphs of order at least four. This bound does not extend to hypergraphs as Corollary 2.3 shows. For hypergraphs of rank r (the maximum cardinality of an edge), the next theorem proves $s(H) \leq r|V(H)| - r + 1$. Most likely, there is no absolute constant c with $s(H) \leq c|V(H)|$.

Theorem 2.4. *If H is a vertex distinguishable hypergraph of rank $r(|s(H)|) < \infty$ then*

$$s(H) \leq r|V(H)| - r + 1.$$

Proof: The proof is induction on $n = |V(H)|$ and $m = |E(H)|$. The cases $n = 1$ and $m = 1$ are trivial. Let $|E(H_1)| \geq 2$ and remove an arbitrary edge $f \varepsilon E(H)$ from H. In case this resulting hypergraph is not vertex distinguishable after the removal of f, then consider a minimal vertex set $Y \subset V(H)\backslash f$ such that H' with $V(H') = V(H)\backslash Y$ and $E(H') = \{e' = e\backslash Y : e \varepsilon E(H)\backslash\{f\}\}$ becomes a vertex distinguishable hypergraph. It should be noted that every hyperedge $e \varepsilon E(H)$ with $e \cap Y \neq \phi$ also satisfies $e \cap f \neq \phi$. In particular, H' has no isolated vertices and no multiple edges. By the inductive hypothesis $s(H') \leq r|V(H')| - r + 1 \leq rn - r + 1$.

An irregular weighting $w(e'), e' \varepsilon E(H')$, of minimum strength can be extended to H in a natural way: $w(e) = w(e')$ for $e \varepsilon (E\backslash\{f\})$ and let $w(f)$ be the smallest positive integer k such that

$$k + d_w(u) \neq d_w(v)$$

for all $u \varepsilon f$ and $v \varepsilon (V(H)\backslash f)$. (Here $(d_w(x)$ denotes the weighted degree of vertex X.)

Since there are at most $r(n-1)$ forbidden values for $k, w(f) \leq ru - r + 1$. Also the irregularity of the extension $w(e), e \varepsilon E(H)$ can be easily verified. ∎

The next three propositions are formulated for use in the last two sections of this paper.

Proposition 2.5. *Let w be an irregular weighting of $H = (V, E)$ and let $\overline{H} = (V, \overline{E})$ be the complement of H with edge set $\overline{E} = \{\overline{e} = V\backslash e : e \varepsilon E\}$. If $\overline{w}$ is defined by $\overline{w}(\overline{e}) = w(e)$, then w is an irregular weighting of H if and only if $\overline{w}$ is an irregular one for $\overline{H}$.*

Proof: For any vertex $v \varepsilon V$ and any edge $e \varepsilon E$, either $v \varepsilon e$ or $v \varepsilon \overline{e}$. Therefore

$$\sum_{e \varepsilon E} w(e) = d_w(v) + d_{\overline{w}}(v).$$

(Here d_w and $d_{\overline{w}}$ denote the weighted degree of vertex v in H and $\overline{H}$ respectively.)

Hence for $u, v \varepsilon V$ $d_w(v) \neq d_w(u)$ iff $d_{\overline{w}}(v) \neq d_{\overline{w}}(u)$. ∎

An immediate corollory is the next proposition.

Proposition 2.6. $s(H) = s(\overline{H})$. ∎

A partial hypergraph H' of H is called *spanning* if $V(H') = V(H)$.

Proposition 2.7. *A regular hypergraph H has irregularity strength $s(H) = 2$ if and only if H contains an irregular spanning hypergraph.*

Proof: Let H be a regular hypergraph with an irregular weighting w such that $s(H) = 2$. Form the spanning hypergraph H' of H by defining $E(H') = \{e | e \varepsilon H$ and $w(e) = 2\}$. Conversely if H' is an irregular spanning hypergraph of the d-regular hypergraph H, then define a weighting w on H as follows: for $e \varepsilon E(H)$ let

$$w(e) = \begin{cases} 2 & \text{if } e \varepsilon E(H') \\ 1 & \text{otherwise} \end{cases}$$

In both cases $d_{H'}(v) = d_H(v) + d$ for all $v \varepsilon V(H) = V(H')$. Hence $s(H) = 2$ if and only if H contains an irregular spanning hypergraph H'. ∎

3. Direct and random constructions of irregular hypergraphs.

In this section the existence of irregular r-uniform hypergraphs of order n is studied. It is clear that for $r = 2$ there are no irregular hypergraphs and for $r \geq 6$ it will be shown that almost all are irregular.

The smallest irregular uniform hypergraph (apart from the one-vertex hypergraph) is shown in Figure 1 in terms of its vertex-edge incidence matrix (it has six vertices, seven edges, it is of rank 3). Its irregularity is shown by the distinct row sums. Starting from this example, an inductive construction will give irregular r-uniform hypergraphs on n vertices for each n and r satisfying $n \geq r + 3, r \geq 3$.

						1
	1				1	
1		1		1		
1	1	1	1			
1			1	1	1	1
	1	1	1	1	1	1

Figure 1. The vertex-edge incidence matrix of an irregular 3-uniform hypergraph

A pair (n, r) will be called irregular if there exists an irregular r-uniform hypergraph of order n. A vertex of degree $\binom{n-1}{r-1}$ is called a *full* vertex in an r-uniform hypergraph on n vertices.

Proposition 3.1. *If (n, r) is irregular then $(n+1, r)$ is also irregular.*

Proof: Let H be an irregular r-uniform hypergraph on n vertices. If H has a full vertex then define H' by adding an isolated vertex to H. If H has no full vertex then H' is defined by adding a new vertex to H and making this new vertex full in H'. It is easy to check that H' is irregular. ∎

Since irregularity is preserved by complementation (Proposition 2.6), the next proposition follows.

Proposition 3.2. *If $(r+3, 3)$ is irregular then $(r+3, r)$ is also irregular.* ∎

Theorem 3.3. *If $n \geq r+3$ and $r \geq 3$ then there exist irregular r-uniform hypergraphs with n vertices.*

Proof: Since $(6, 3)$ is irregular, $(r+3, 3)$ is also irregular by Proposition 3.1. Proposition 3.2 implies that $(r+3, r)$ is irregular and applying Proposition 3.1 again gives that (n, r) is irregular. ∎

Note that if $H(n)$ is the irregular hypergraph constructed inductively for the irregular pair (n, r), then $H(n, r)$ may contain (one) isolated vertex x. It is easy to alter $H(n, r)$ so that the resulting hypergraph is irregular and has no isolated vertex. One way is to add $n-1$ new edges $E_1, E_2, \ldots, E_{n-1}$ to $H(n, r)$ in such a way that $|E_i| = r, x \varepsilon E_i$ for $1 \leq i \leq n-1$ and the sets $E_i - x$ form an $(r-1)$-regular $(r-1)$ uniform hypergraph.

Also observe that Theorem 3.3 characterizes the irregular (n, r) pairs. This is stated formally in Proposition 4.2 which says that hypergraphs with $r \leq n \leq r+2$ are not irregular.

The next objective is to show that for $r \geq 6$, almost all r-uniform hypergraphs are irregular or equivalently, the probability of identical vertex degrees in a random r-uniform hypergraph of order n approaches 0 as $n \longrightarrow \infty$.

Theorem 3.4. *Almost all r-uniform hypergraphs are irregular.*

Proof: The proof proceeds by giving an upper bound on the number of r-uniform hypergraphs on n vertices having at least two vertices of the same degree. It will be shown that this number is "small" in comparison to the total number of r-uniform hypergraphs on n vertices.

Choose x_1 and x_2, two fixed vertices of the hypergraph. The number of r-uniform hypergraphs of order n such that x_1 and x_2 have the same degree is at most

$$\sum_{t=0}^{\binom{n-2}{r-1}} \binom{\binom{n-2}{r-1}}{t}^2 2^{\binom{n-2}{r-2}} 2^{\binom{n-2}{r}}.$$

This follows by considering three types of edges. Those that contain x_1 or x_2 but not both, those that contain both x_1 and x_2, and those edges that contain neither. Since x_1 and x_2 are to have the same degree, only the number of edges containing exactly one of them needs to be the same number, that being $\left(\binom{\binom{n-2}{r-1}}{t}\right)$. All other edges appear arbitrarily, so the sum is taken over all possible t. Since x_1 and x_2 are fixed, but chosen randomly, the total number of r-uniform hypergraphs having at least two vertices of the same degree is at most

$$\binom{n}{2}\sum_{t=0}^{\binom{n-2}{r-1}}\binom{\binom{n-2}{r-1}}{t}^2 2^{\binom{n-2}{r-2}}2^{\binom{n-2}{r}}$$
$$=\binom{n}{2}\binom{2\binom{n-2}{r-1}}{\binom{n-2}{r-1}}2^{\binom{n-2}{r-2}}2^{\binom{n-2}{r}}.$$

Asymptotically, this is equal to

$$\binom{n}{2}\frac{1}{\pi\binom{n-2}{r-1}^{\frac{1}{2}}}2^{\binom{2n-2}{r-1}}2^{\binom{n-2}{r-2}}2^{\binom{n-2}{r}}$$

by Stirling's formula. But,

$$2\binom{n-2}{r-1}+\binom{n-1}{r-2}+\binom{n-2}{r}=\binom{n}{r},$$

so it follows that the number of the r-uniform hypergraphs on n vertices with at least two vertices of the same degree is at most

$$\frac{n(n-1)}{2}\cdot\frac{1}{\pi\binom{n-2}{r-1}^{\frac{1}{2}}}\cdot 2^{\binom{n}{r}}$$

Since $2^{\binom{n}{r}}$ is the total number of r-uniform hypergraph of order n, and for $r \geq 6$,

$$\lim_{n\to\infty}\frac{n(n-1)}{2}\frac{1}{\pi\binom{n-2}{r-1}^{\frac{1}{2}}}=0,$$

the theorem follows. ∎

This result is also true for a special class of r-uniform hypergraphs. A hypergraph is r-partite if the vertices of the hypergraph can be partitioned into r sets $A_1, A_2, \ldots, A_r$ so that the edges contain at most one vertex from A_i, for $i = 1, 2, \ldots, r$. Note that in case of an r-uniform r-partite hypergraph each edge contains exactly one vertex for each A_i, for $i = 1, 2, \ldots, r$. An $\langle r\times m\rangle$ hypergraph is an r-partite, r-uniform hypergraph with m vertices in each vertex class. Using the same argument as given for Theorem 4.1 the following result is obtained.

Theorem 3.5. *For either fixed $m \geq 6$ or fixed $r \geq 6$, almost all $\langle r \times m\rangle$ hypergraphs are irregular.* ∎

4 The Strength of Complete Hypergraphs

In this section the irregularity strength of two particular classes of complete uniform hypergraphs are studied: $K_n^{(r)}$, the complete r-uniform hypergraph of order n, and $K\langle r\times m\rangle$, the complete r-partite r-uniform hypergraph with m vertices in each vertex (partite) class. As mentioned earlier each edge in $K\langle r\times m\rangle$ contains one vertex from each of the r partite classes.

For the most part the hypergraphs in each of the two mentioned classes will have irregularity strength 2. In such cases it follows from Propostion 2.6 that this is equivalent to showing the existence of an irregular r-uniform hypergraph of a given order (and possibly given type). With the results in Section 3 this leads immediately to the determination of the strength of $K_n^{(r)}$.

Theorem 4.1. *For* $r\geq 3$,

$$s(K_n^{(r)})=\begin{cases} n & \text{if } n=r+1\\ 3 & \text{if } n=r+2\\ 2 & \text{if } n\geq r+3\end{cases}$$

Proof: By Proposition 2.5 $s(K_n^{(r)}) = s(\overline{K_n^{(r)}}) = s(K_n^{(n-r)})$. If $n=r+1$, the edges are singletons and thus $s(K_n^{(r)}) = n$. For $n=r+2$, $K_n^{(n-r)})$ is the complete graph K_n and its strength is well known to be 3, see[1]. In case $n\geq r+3$, by Theorem 3.3, there exists an irregular r-uniform hypergraph H' of order n. Since H' is a spanning partial hypergraph of $K_n^{(r)}$ it follows from Propositions 2.6 that $s(K_n^{(r)}s)=2$. ∎

A consequence of Theorem 4.1 and Proposition 2.6 is the following result.

Proposition 4.2. *There exist no irregular r-uniform hypergraphs on n vertices for $r\leq n\leq r+2$.*

The remaining portion of this section is devoted to r-partite hypergraphs.The next two propositions are given without proof, since the proofs either involve small order irregular $\langle r\times m\rangle$—hypergraphs or giving special arguments showing no such irregular hypergraphs exist for certain small orders. The propositions are needed to start later inductive arguments and to determine exceptional cases (cases where $K\langle r\times m\rangle$ is not of strength 2). Only one of the eight irregular hypergraphs needed to verify Proposition 4.4 is included (Figure 2). The constructions of these eight hypergraphs are nontrivial and a vital part of the final result, Theorem 4.7.

Propoition 4.3. *If K is one of the hypergraphs $K\langle 3\times 2\rangle, K\langle 4\times 2\rangle$, or $K\langle 3\times 3\rangle$, then $s(K)=3$.* ∎

Propositon 4.4. *There exist irregular $\langle r\times m\rangle$-hypergraphs for every r,m with $(r,m)\varepsilon\{(3,m)|4\leq m\leq 7\}\cup\{(4,m)|3\leq m\leq 4\}\cup\{(5,2)\}$.* ∎

In order to determine that "most" complete r-partite r-uniform hypergraphs are of strength 2, the existence of appropriate irregular $\langle r \times m \rangle$-hypergraphs needs to be established. This is the remaining major objective of the section.

Theorem 4.5. *There exist irregular* $\langle r \times m \rangle$ *hypergraphs for each* $r \geq 5$ *and* $m = 2, 3$.

x_1	1	1	1	1	1														
y_1						1	1	1	1	1	1	1	1	1	1	1	1	1	1
x_2						1	1	1	1	1	1								
y_2	1	1	1	1	1							1	1	1	1	1	1	1	1
x_3	1					1	1					1	1	1	1				
y_3		1	1	1	1			1	1	1	1					1	1	1	1
x_4		1	1					1	1			1	1			1	1		
y_4	1			1	1	1	1			1	1			1	1			1	1
x_5		1		1		1		1		1		1		1		1		1	
y_5	1		1		1		1		1		1		1		1		1		1

Figure 2. The vertex-edge incidence matrix of an irregular (5×2)-hypergraph

Proof:

Case 1. $m = 2$.

The proof is by induction on r, with the hypergraph of Figure 2 establishing the result for $r = 5$. Thus assume H is an irregular $\langle r \times 2 \rangle$ - hypergraph with vertex classes $\{x_i, y_i\}$, $1 \leq i \leq r$, such that $1 \leq d_H(x_1) < \cdots < d_H(x_r)$ and such that $\{x_1, x_2, \ldots x_r\} \notin E(H)$. Note that when $r = 5$, the hypergraph of Figure 2 satisfies these conditions.

An irregular $\langle (r + l) \times 2 \rangle$-hypergraph H' is constructed from H by adjoining to $V(H)$ the new vertex class $\{x_0, y_0\}$ with $E(H') = \{e \cup \{y_0\} | e \varepsilon E(H)\} \cup \{x_1, x_2, \ldots, x_r, y_0\} \cup \{y_1, y_2, \ldots, y_r, x_0\}$. Note that $d_{H'}(x_0) = 1 < d_{H'}(u) = d_H(u) + 1 < |E(H)| + 1 = d_{H'}(y_0)$ for all $u \varepsilon V(H)$, so that H' is irregular. In fact in $H' 1 \leq d_{H'}(x_0) < d_{H'}(x_1) < \cdots < d_{H'}(x_r)$ and $\{x_0, x_1, \ldots, x_r\} \in E(H')$. This completes the induction and the proof when $m = 2$.

Case 2. $m = 3$.

In the construction of H given in Case 1, careful examination of this irregular $\langle r \times 2\rangle$-hypergraph shows that each of the following hold:

(1) the vertex classes are $\{x_i, y_i\}, i = 1, 2, \ldots, r$, with $1 \leq d_H(x_1) < d_H(x_2) < \cdots < d_H(x_r) < d_H(y_r) < d_H(y_r - 1) < \cdots < d_H(y_1)$,

(2) $\{x_1, x_2, \ldots, x_r\} \notin E(H)$, and

(3) $d_H(x_r) = r + 4$

The existence of these irregular $\langle r \times 2\rangle$-hypergraphs is used to construct the irregular $\langle r \times 3\rangle$-hypergraphs.

To accomplish this first add a new vertex z_i to each of the r classes $\{x_i, y_i\}$ of the irregular $\langle r \times 2\rangle$ - hpergraph H. Letting H' denote the hypergraph under construction define $E(H') = E(H) \cup E_1 \cup E_2$ where

$$E_1 = \{\{x_r, x_{r-1}, \ldots, x_{r-i}, z_{r-i-1}, \ldots, z_1\} \mid i = 0, 1 \ldots, r-1\}$$

and

$$E_2 = E(H^*) - \{\{y_1, y_2, \ldots, y_r\}\}.$$

Here H^* denotes the $K\langle r \times 2\rangle$ hypergraph with vertex classes $\{y_i, z_i\}, 1 \leq i \leq r$.

Only the fact that the $\langle r \times 3\rangle$-hypergraph H' is irregular needs verification. From the definition and conditions (1),(2), and (3) it follows that

(i) H' has no multiple edges,

(ii) $1 < d_{H'}(x_1) < \cdots < d_{H'}(x_r) \leq (r+4) + r$,

(iii) $2^{r-1} - 1 = d_{H'}(z_r) < \cdots < d_{H'}(z_1) \leq (2^{r-1} - 1) + (r - 1)$,

(iv) $(2^{r-1}) - 1 + r + 4 < d_{H'}(y_r) < \cdots < d_{H'}(y_1)$.

Since $2r + 4 < 2^{r-1} - 1$ for $r \geq 5$, H' is irregular, completing the proof. ∎

The hypergraphs constructed in the last proposition are used to construct irregular $\langle r \times m\rangle$-hypergraphs for all $r \geq 5$ and $m \geq 4$ by concatenating vertex classes. For a precise contruction let $r \geq 3$ and $2 \leq m_2 \leq m_1 \leq m_2 + 1$. Further let $H_i = (V_i, E_i)$ be irregular $\langle r \times m_i\rangle$-hypergraphs, $i = 1, 2$. Form the $\langle r \times (m_1 + m_2)\rangle$-hypergraph H by letting $V(H) = V_1 \cup V_2$ and $E(H) = E_1 \cup E_2 \cup E_0$, where E_0 is the set of all r-tuples with one vertex from each vertex class and precisely one vertex from V_2.

In the proof of the lemma that follows it is assumed that either no vertex of H_2 has maximum degree m_2^{r-1}, or each vertex of H_1 has degree at least 2. If in fact H_2 has a vertex of maximum degree, then each of its vertices has degree at least $m_2^{r-2} \geq 2$. In such a case the roles of H_1 and H_2 can simply be interchanged. Thus assume without loss of generality that maximum degree of a vertex in H_2 is strictly less than m_2^{r-1}.

Lemma 4.6. *The $\langle r \times (m_1 + m_2)\rangle$-hypergraph constructed above is irregular.*

Proof: Let $d_1(x), d_2(x)$, and $d(x)$ denote the degree of x in H_1, H_2, and H respectively. As noted above $1 \leq d_1(x) \leq m_1^{r-1}, x \varepsilon V_1$, and $1 \leq d_2(x) \leq m_2^{r-1} - 1, x \varepsilon V_2$.

From the construction give

$$d(v_1) = d_1(v_1) + (r-1)m_1^{r-2}m_2 \qquad \text{and}$$
$$d(v_2 = d_2(v_2) + m_1^{r-1}, \text{for } v_1 \varepsilon V_1, v_2, \varepsilon V_2.$$

The irregularity of H is verified by showing $d(v_2) < d(v_1)$ for all $v_1 \varepsilon V_1$ and $v_2 \varepsilon V_2$.

Observe that

$$d(v_2) \leq m_2^{r-1} + m_1^{r-1} - 1 \qquad \text{and}$$
$$d(v_1) > (r-1)m_1^{r-2}m_2, v_1 \varepsilon V_1, v_2 \varepsilon V_2. \tag{1}$$

If $r = 3$, then (noting $m_2 \leq m_1 \leq m_2 + 1)(1)$ gives $d(v_2) \leq m_2^2 + m_1^2 - 1 \leq 2m_1m_2 < d(v_1)$.

For $r > 3$ and $m_1 = m_2 + 1$,

$$\left(\frac{m_2}{m_1}\right)^{r-2} m_2 + m_1 < m_2 + m_1 = 2m_1 - 1 \leq 3(m_1 - 1) \leq (r-1)m_2,$$

so that by (1)

$$d(v_2) \leq m_2^{r-1} + m_1^{r-1} < (r-1)m_1^{r-2}m_2 < d(v_1).$$

When $m_1 + m_2$, then again

$$d(v_2) \leq 2m_1^{r-1} < (r-1)m_1^{r-1} < d(v_1).$$

Hence $d(v_2) < d(v_1)$ for all $v_1 \varepsilon V_1$ and $v_2 \varepsilon V_2$ completing the proof of the lemma. ∎

Theorem 4.7. *Let $K = K\langle r \times m\rangle$ with $r \geq 3$ and $m \geq 2$. Then*

$$s(K) = \begin{cases} 3 & \text{if } r + m \leq 6 \\ 2 & \text{otherwise.} \end{cases}$$

Proof: Proposition 4.3 gives the result of this theorem when $r + m \leq 6$. When $r + m > 6$, using Propositions 4.4 and 4.5 together with Lemma 4.6, the existance of irregular $\langle r \times m\rangle$-hypergraphs is obtained. Since each of these hypergraphs is a spanning partial hypergraph of $K\langle r \times m\rangle$, the result follows from Proposition 2.6. ∎

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