A construction of addition-multiplication magic squares using orthogonal diagonal latin squares

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Abstract. An addition-multiplication magic square of order n is an $n \times n$ matrix whose entries are n^2 distinct positive integers such that not only the sum but also the product of the entries in each row, column, main diagonal, and back diagonal is a constant. It is shown in this paper that such a square exists for any order mn, where m and n are positive integers and $m, n \notin \{1,2,3,6\}$.

1. Introduction

In [9] and [10], W.W. Horner first investigated construction methods of addition-multiplication magic squares. An *addition-multiplication magic square* of order n, denoted by AMMS(n), is an $n \times n$ matrix whose entries are n^2 distinct positive integers such that not only the sum but also the product of the entries in each row, column, main diagonal, and back diagonal is a constant. The sum is called *magic sum* and the product *magic product*. The main diagonal contains the cells (i,i), $1 \le i \le n$, and the back diagonal contains the cells $(i, n+1-i)$, $1 \le i \le n$.

In [3], J. Denes and A.D. Keedwell asked the following open question (page 489, question 6.3): *For what orders n do addition-multiplication magic squares exist?* Examples for orders 8 and 9 are given in [9] and [10], where the methods are also suitable for other orders such as 16 and some odd orders. In this paper, we shall use orthogonal diagonal latin squares to show the existence of an AMMS(mn) for positive integers m and n, where $m, n \notin \{1,2,3,6\}$.

A *latin square* of order v, based on a v-set S, is a $v \times v$ array such that in each row and column each element in S occurs exactly once. We usually take $S = \{1,2,\dots,v\}$. A latin square is called *diagonal* if for both the main diagonal (cells (i,i), $1 \le i \le v$) and the back diagonal (cells $(i, v+1-i)$, $1 \le i \le v$) each has distinct entries. Two latin squares $A = (a_{ij})$ and $B = (b_{ij})$ of order v, based on S_1 and S_2 respectively, are said to be *orthogonal* if every ordered pair from $S_1 \times S_2$ occurs exactly once among the v^2 pairs (a_{ij}, b_{ij}), $1 \le i \le v$, $1 \le j \le v$. A pair of orthogonal diagonal latin squares of order v is denoted by ODLS(v). Many authors investigated the existence of ODLS(v), see [1–2], [4–8], [11], [13–17]. The existence was finally solved in [1] as follows.

Theorem 1.1. *There exists a pair of orthogonal diagonal latin squares of order n if and only if n is a positive integer and $n \ne 2,3,6$.*

Our construction will also need the use of quasi-magic rectangles.

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Definition1.2: An $m \times n$ matrix is called a *quasi-magic rectangle* if its mn entries are distinct non-negative integers such that the sum of the entries in each row is a constant S_r, called *row magic sum*, and such that the sum of the entries in each column is also a constant S_c, called *column magic sum*. A quasi-magic rectangle is called a *magic rectangle* if its entries are consecutive integers ranging from 0 to $mn - 1$. To avoid triviality we assume that $m + n > 1$.

We shall describe our construction for AMMS using ODLS and quasi-magic rectangles in the next section. For the quasi-magic rectangles needed, we shall discuss the existence of the necessary quasi-magic rectangles in Section 3. However, we have the known existence result for magic rectangles as follows.

Theorem 1.3 ([12]). *An $m \times n$ magic rectangle exists if and only if $m + n \geq 6$, $m \equiv n \pmod 2$ and $m, n \geq 2$.*

As a consequence we shall prove our main result in Section 4, which can be presented in the following.

Main Theorem 1.4. *An addition-multiplication magic square of order mn exists if m and n are positive integers and $m, n \notin \{1, 2, 3, 6\}$.*

2. A construction

Suppose $A = (a_{x,y})$ and $B = (b_{x,y})$ are ODLS(m), based on $0, 1, \ldots, m-1$. Suppose $C = (c_{x,y})$ and $D = (d_{x,y})$ are ODLS(n), based on $0, 1, \ldots, n-1$. For any $x \in \{0, 1, \ldots, mn-1\}$, there is a unique pair (p, q) such that $x = p + qm$, $p \in \{0, 1, \ldots, m-1\}$ and $q \in \{0, 1, \ldots, n-1\}$. We write $(x) = p$, and $\langle x \rangle = q$.

Let G be an $m \times n$ quasi-magic rectangle based on T, having its (x, y) entry $g(x, y)$ and having its rows and columns labelled with $0, 1, \ldots, m-1$ and $0, 1, \ldots, n-1$, respectively. Let $E = (e_{x,y})$ and $F = (f_{x,y})$ be defined such that for $x, y \in \{0, 1, \ldots, mn-1\}$,

$$e_{x,y} = a_{(x),(y)} + mc_{\langle x\rangle,\langle y\rangle}, \tag{1}$$

$$f_{x,y} = g\left(b_{(x),(y)}, d_{\langle x\rangle,\langle y\rangle}\right). \tag{2}$$

Lemma 2.1. *E and F form a pair of orthogonal diagonal latin squares of order mn.*

Proof: It is easy to see that E is a diagonal latin square of order mn, based on $\{0, 1, \ldots, mn-1\}$. For the matrix F, it is well defined on T. For a fixed x, if $f_{x,y} = f_{x,z}$, we have from (2) that

$$\begin{aligned} b_{(x),(y)} &= b_{(x),(z)}, \\ d_{\langle x\rangle,\langle y\rangle} &= d_{\langle x\rangle,\langle y\rangle}. \end{aligned} \tag{3}$$

Since B and D are both latin squares, $(y) = (z)$ and $\langle y\rangle = \langle z\rangle$. So, $y = z$, and each row of F contains distinct entries in T. In a similar way, we can show that each column, the main diagonal, and the back diagonal all have the same property. Thus F is also a diagonal latin square. We next prove orthogonality.

Suppose $(e_{x,y}, f_{x,y}) = (e_{i,j}, f_{i,j})$. We have from (1) and (2) that

$$\begin{aligned} a_{(x),(y)} &= a_{(i),(j)}, \\ b_{(x),(y)} &= b_{(i),(j)}, \\ c_{\langle x\rangle,\langle y\rangle} &= c_{\langle i\rangle,\langle j\rangle}, \\ d_{\langle x\rangle,\langle y\rangle} &= d_{\langle i\rangle,\langle j\rangle}. \end{aligned} \tag{4}$$

The orthogonality of A and B implies $(x) = (i)$ and $(y) = (j)$. The orthogonality of C and D implies $\langle x\rangle = \langle i\rangle$ and $\langle y\rangle = \langle j\rangle$. Therefore, $x = i$ and $y = j$. The proof is complete.

Example 2.2: Let

$$A = \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \end{array} \qquad B = \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \end{array}$$

$$C = \begin{array}{ccccc} 0 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 0 & 2 \\ 3 & 0 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 0 \\ 1 & 3 & 0 & 2 & 4 \end{array} \qquad D = \begin{array}{ccccc} 3 & 1 & 4 & 2 & 0 \\ 2 & 0 & 3 & 1 & 4 \\ 1 & 4 & 2 & 0 & 3 \\ 0 & 3 & 1 & 4 & 2 \\ 4 & 2 & 0 & 3 & 1 \end{array}$$

$$G = \begin{array}{ccccc} 0 & 8 & 18 & 19 & 5 \\ 9 & 14 & 13 & 3 & 11 \\ 16 & 17 & 7 & 6 & 4 \\ 15 & 1 & 2 & 12 & 20 \end{array}$$

We then obtain from Lemma 2.1 ODLS(20) E and F as follows.

$$E = \begin{array}{cccccccccccccccccccc}
0 & 1 & 2 & 3 & 8 & 9 & 10 & 11 & 16 & 17 & 18 & 19 & 4 & 5 & 6 & 7 & 12 & 13 & 14 & 15 \\
2 & 3 & 0 & 1 & 10 & 11 & 8 & 9 & 18 & 19 & 16 & 17 & 6 & 7 & 4 & 5 & 14 & 15 & 12 & 13 \\
3 & 2 & 1 & 0 & 11 & 10 & 9 & 8 & 19 & 18 & 17 & 16 & 7 & 6 & 5 & 4 & 15 & 14 & 13 & 12 \\
1 & 0 & 3 & 2 & 9 & 8 & 11 & 10 & 17 & 16 & 19 & 18 & 5 & 4 & 7 & 6 & 13 & 12 & 15 & 14 \\
16 & 17 & 18 & 19 & 4 & 5 & 6 & 7 & 12 & 13 & 14 & 15 & 0 & 1 & 2 & 3 & 8 & 9 & 10 & 11 \\
18 & 19 & 16 & 17 & 6 & 7 & 4 & 5 & 14 & 15 & 12 & 13 & 2 & 3 & 0 & 1 & 10 & 11 & 8 & 9 \\
19 & 18 & 17 & 16 & 7 & 6 & 5 & 4 & 15 & 14 & 13 & 12 & 3 & 2 & 1 & 0 & 11 & 10 & 9 & 8 \\
17 & 16 & 19 & 18 & 5 & 4 & 7 & 6 & 13 & 12 & 15 & 14 & 1 & 0 & 3 & 2 & 9 & 8 & 11 & 10 \\
12 & 13 & 14 & 15 & 0 & 1 & 2 & 3 & 8 & 9 & 10 & 11 & 16 & 17 & 18 & 19 & 4 & 5 & 6 & 7 \\
14 & 15 & 12 & 13 & 2 & 3 & 0 & 1 & 10 & 11 & 8 & 9 & 18 & 19 & 16 & 17 & 6 & 7 & 4 & 5 \\
15 & 14 & 13 & 12 & 3 & 2 & 1 & 0 & 11 & 10 & 9 & 8 & 19 & 18 & 17 & 16 & 7 & 6 & 5 & 4 \\
13 & 12 & 15 & 14 & 1 & 0 & 3 & 2 & 9 & 8 & 11 & 10 & 17 & 16 & 19 & 18 & 5 & 4 & 7 & 6 \\
8 & 9 & 10 & 11 & 16 & 17 & 18 & 19 & 4 & 5 & 6 & 7 & 12 & 13 & 14 & 15 & 0 & 1 & 2 & 3 \\
10 & 11 & 8 & 9 & 18 & 19 & 16 & 17 & 6 & 7 & 4 & 5 & 14 & 15 & 12 & 13 & 2 & 3 & 0 & 1 \\
11 & 10 & 9 & 8 & 19 & 18 & 17 & 16 & 7 & 6 & 5 & 4 & 15 & 14 & 13 & 12 & 3 & 2 & 1 & 0 \\
9 & 8 & 11 & 10 & 17 & 16 & 19 & 18 & 5 & 4 & 7 & 6 & 13 & 12 & 15 & 14 & 1 & 0 & 3 & 2 \\
4 & 5 & 6 & 7 & 12 & 13 & 14 & 15 & 0 & 1 & 2 & 3 & 8 & 9 & 10 & 11 & 16 & 17 & 18 & 19 \\
6 & 7 & 4 & 5 & 14 & 15 & 12 & 13 & 2 & 3 & 0 & 1 & 10 & 11 & 8 & 9 & 18 & 19 & 16 & 17 \\
7 & 6 & 5 & 4 & 15 & 14 & 13 & 12 & 3 & 2 & 1 & 0 & 11 & 10 & 9 & 8 & 19 & 18 & 17 & 16 \\
5 & 4 & 7 & 6 & 13 & 12 & 15 & 14 & 1 & 0 & 3 & 2 & 9 & 8 & 11 & 10 & 17 & 16 & 19 & 18
\end{array}$$

$$F=\begin{matrix}
19 & 3 & 6 & 12 & 8 & 14 & 17 & 1 & 5 & 11 & 4 & 20 & 18 & 13 & 7 & 2 & 0 & 9 & 16 & 15 \\
12 & 6 & 3 & 19 & 1 & 17 & 14 & 8 & 20 & 4 & 11 & 5 & 2 & 7 & 13 & 18 & 15 & 16 & 9 & 0 \\
3 & 19 & 12 & 6 & 14 & 8 & 1 & 17 & 11 & 5 & 20 & 4 & 13 & 18 & 2 & 7 & 9 & 0 & 15 & 16 \\
6 & 12 & 19 & 3 & 17 & 1 & 8 & 14 & 4 & 20 & 5 & 11 & 7 & 2 & 18 & 13 & 16 & 15 & 0 & 9 \\
18 & 13 & 7 & 2 & 0 & 9 & 16 & 15 & 19 & 3 & 6 & 12 & 8 & 14 & 17 & 1 & 5 & 11 & 4 & 20 \\
2 & 7 & 13 & 18 & 15 & 16 & 9 & 0 & 12 & 6 & 3 & 19 & 1 & 17 & 14 & 8 & 20 & 4 & 11 & 5 \\
13 & 18 & 2 & 7 & 9 & 0 & 15 & 16 & 3 & 19 & 12 & 6 & 14 & 8 & 1 & 17 & 11 & 5 & 20 & 4 \\
7 & 2 & 18 & 13 & 16 & 15 & 0 & 9 & 6 & 12 & 19 & 3 & 17 & 1 & 8 & 14 & 4 & 20 & 5 & 11 \\
8 & 14 & 17 & 1 & 5 & 11 & 4 & 20 & 18 & 13 & 7 & 2 & 0 & 9 & 16 & 15 & 19 & 3 & 6 & 12 \\
1 & 17 & 14 & 8 & 20 & 4 & 11 & 5 & 2 & 7 & 13 & 18 & 15 & 16 & 9 & 0 & 12 & 6 & 3 & 19 \\
14 & 8 & 1 & 17 & 11 & 5 & 20 & 4 & 13 & 18 & 2 & 7 & 9 & 0 & 15 & 16 & 3 & 19 & 12 & 6 \\
17 & 1 & 8 & 14 & 4 & 20 & 5 & 11 & 7 & 2 & 18 & 13 & 16 & 15 & 0 & 9 & 6 & 12 & 19 & 3 \\
0 & 9 & 16 & 15 & 19 & 3 & 6 & 12 & 8 & 14 & 17 & 1 & 5 & 11 & 4 & 20 & 18 & 13 & 7 & 2 \\
15 & 16 & 9 & 0 & 12 & 6 & 3 & 19 & 1 & 17 & 14 & 8 & 20 & 4 & 11 & 5 & 2 & 7 & 13 & 18 \\
9 & 0 & 15 & 16 & 3 & 19 & 12 & 6 & 14 & 8 & 1 & 17 & 11 & 5 & 20 & 4 & 13 & 18 & 2 & 7 \\
16 & 15 & 0 & 9 & 6 & 12 & 19 & 3 & 17 & 1 & 8 & 14 & 4 & 20 & 5 & 11 & 7 & 2 & 18 & 13 \\
5 & 11 & 4 & 20 & 18 & 13 & 7 & 2 & 0 & 9 & 16 & 15 & 19 & 3 & 6 & 12 & 8 & 14 & 17 & 1 \\
20 & 4 & 11 & 5 & 2 & 7 & 13 & 18 & 15 & 16 & 9 & 0 & 12 & 6 & 3 & 19 & 1 & 17 & 14 & 8 \\
11 & 5 & 20 & 4 & 13 & 18 & 2 & 7 & 9 & 0 & 15 & 16 & 3 & 19 & 12 & 6 & 14 & 8 & 1 & 17 \\
4 & 20 & 5 & 11 & 7 & 2 & 18 & 13 & 16 & 15 & 0 & 9 & 6 & 12 & 19 & 3 & 17 & 1 & 8 & 14
\end{matrix}$$

Let $H = (h_{x,y})$ be an $mn \times mn$ matrix such that

$$h_{x,y} = (e_{x,y} + 1)(mnf_{x,y} + 1). \tag{5}$$

Lemma 2.3. *H is an addition-multiplication magic square of order mn.*

Proof: By (5) we know that each entry in H is a positive integer. If $h_{x,y} = h_{i,j}$, then

$$mnf_{x,y}(e_{x,y} + 1) + e_{x,y} = mnf_{i,j}(e_{i,j} + 1) + e_{i,j}. \tag{6}$$

Since $e_{x,y}, e_{i,j} \in \{0, 1, \ldots, mn - 1\}$, we have

$$\begin{aligned} e_{x,y} &= e_{i,j}, \\ f_{x,y} &= f_{i,j}. \end{aligned} \tag{7}$$

The orthogonality of E and F implies that $(x, y) = (i, j)$. Therefore, H contains mn distinct entries.

For a fixed x, we compute the product.

$$\begin{aligned} P &= \prod_{0 \le y \le mn-1} h_{x,y} \\ &= \prod_{0 \le y \le mn-1} (e_{x,y} + 1)(mnf_{x,y} + 1) \\ &= (mn)! \prod_{0 \le i \le m-1, 0 \le j \le n-1} (mng(i,j) + 1) \\ &= (mn)! \prod_{t \in T} (mnt + 1) \end{aligned}$$

In a similar way, we can show that the product of the entries in each column, the main diagonal, and the back diagonal is also P.

Let $S = mn(mn+1)(1+\sum_{t\in T} t)/2$. Let S_r and S_c be the row magic sum and the column magic sum of G, respectively. For a fixed x, we compute the sum

$$\begin{aligned}\sum_{0\le y\le mn-1} h_{x,y} &= \sum_{0\le y\le mn-1} \left(mne_{x,y}f_{x,y} + e_{x,y} + mnf_{x,y} + 1\right) \\ &= mn \sum_{0\le y\le mn-1} e_{x,y}f_{x,y} + mn\frac{mn-1}{2} + mn\sum_{t\in T} t + mn\end{aligned} \tag{8}$$

Using (1) and (2) we have

$$\begin{aligned}\sum_{0\le y\le mn-1} e_{x,y}f_{x,y} &= \sum_{0\le y\le mn-1} \left(a_{(x),(y)} + mc_{(x),(y)}\right) g\left(b_{(x),(y)}, d_{(x),(y)}\right) \\ &= \sum_{0\le i\le m-1}\ \sum_{0\le j\le n-1} \left(a_{(x),i} + mc_{(x),j}\right) g\left(b_{(x),i}, d_{(x),j}\right) \\ &= \sum_{0\le i\le m-1} a_{(x),i}S_r + m\sum_{0\le j\le n-1} c_{(x),j}S_c \\ &= m(m-1)S_r/2 + mn(n-1)S_c/2\end{aligned} \tag{9}$$

Combine (8) and (9). Noticing that $mS_r = nS_c = \sum_{t\in T} t$ we obtain

$$\sum_{0\le y\le mn-1} h_{x,y} = S.$$

Similarly, we can show that the sum of entries in each column, the main diagonal, and the back diagonal is also S.

Therefore, H is an addition-multiplication magic square. The proof is complete.

We conclude this section with the following.

Theorem 2.4. *Suppose there exist ODLS(m), ODLS(n) and an $m \times n$ quasi-magic rectangle, then there exists an addition-multiplication magic square of order mn.*

Example 2.5: An AMMS(20) is constructed in the appendix using (5) and the squares E and F in Example 2.2.

3. Quasi-magic rectangles

In this section we shall establish the existence of quasi-magic rectangles.

Lemma 3.1. *For any positive integer k, there exists a $3 \times (2k)$ quasi-magic rectangle.*

Proof: Let $A = (a_{x,y})$ be a $3 \times (2k)$ matrix where

$$\begin{aligned} a_{x,y} &= 2k + 2y \text{ for } x = 0 \text{ and } y = 0, 1, \ldots, 2k-1, \\ &= 6k + 2y \text{ for } x = 1 \text{ and } y = 0, 1, \ldots, k-1, \\ &= 2y - 2k \text{ for } x = 1 \text{ and } y = k, k+1, \ldots, 2k-1, \\ &= 4k - 3 - 4y \text{ for } x = 2 \text{ and } y = 0, 1, \ldots, k-1, \\ &= 12k - 3 - 4y \text{ for } x = 2 \text{ and } y = k, k+1, \ldots, 2k-1. \end{aligned}$$

It is readily checked that A has the minimum entry 0, the maximum entry $h = 8k - 2$, and all entries distinct. Also, it has the row magic sum $S_r = kh$ and the column magic sum $S_c = 3h/2$. Therefore, A is the required quasi-magic rectangle.

Lemma 3.2. *For any positive integer k, there exists a $5 \times (2k)$ quasi-magic rectangle.*

Proof: It is readily checked that the following $5 \times (2k)$ matrix $A = (a_{x,y})$ is the required one. It has distinct entries between 0 and $h = 12k$, the row magic sum is $S_r = kh$ and the column magic sum is $S_c = 5h/2$, where

$$\begin{aligned} a_{x,y} &= 4k + 2y \text{ for } x = 0 \text{ and } y = 0, 1, \ldots, k-1, \\ &= 4k + 2 + 2y \text{ for } x = 0 \text{ and } y = k, k+1, \ldots, 2k-1, \\ &= 12k - 2y \text{ for } x = 1 \text{ and } y = 0, 1, \ldots, k-1, \\ &= 4k - 2 - 2y \text{ for } x = 1 \text{ and } y = k, k+1, \ldots, 2k-1, \\ &= 8k + 2 + 2y \text{ for } x = 2 \text{ and } y = 0, 1, \ldots, k-1, \\ &= 2y \text{ for } x = 2 \text{ and } y = k, k+1, \ldots, 2k-1, \\ &= 2y + 1 \text{ for } x = 3 \text{ and } y = 0, 1, \ldots, k-1, \\ &= 8k + 1 + 2y \text{ for } x = 3 \text{ and } y = k, k+1, \ldots, 2k-1, \\ &= 6k - 3 - 4y \text{ for } x = 4 \text{ and } y = 0, 1, \ldots, k-1, \\ &= 14k - 1 - 4y \text{ for } x = 4 \text{ and } y = k, k+1, \ldots, 2k-1. \end{aligned}$$

Lemma 3.3. *For any positive integers t and k, there exists a $(2t+1) \times (2k)$ quasi-magic rectangle.*

Proof: First, we construct a $(4t) \times (2k)$ matrix $B = (b_{x,y})$ where

$$
\begin{aligned}
b_{x,y} &= 4ty + x \text{ for } 0 \le y \le k-1 \text{ and } 0 \le x \le t-1 \\
&\qquad \text{or } 3t \le x \le 4t-1, \\
&= 4t(2k-y) - 1 - x + s \text{ for } 0 \le y \le k-1 \text{ and } t \le x \le 3t-1, \\
&= 4t(y+1) - 1 - x + s \text{ for } k \le y \le 2k-1 \text{ and } 0 \le x \le t-1 \\
&\qquad \text{or } 3t \le x \le 4t-1, \\
&= 4t(2k-1-y) + x \text{ for } k \le y \le 2k-1 \text{ and } t \le x \le 3t-1.
\end{aligned}
$$

It is not difficult to verify that B is a quasi-magic rectangle with row magic sum $k(8kt-1+s)$, column magic sum $2t(8kt-1+s)$ and distinct entries in the intervals $[0, 4kt-1]$ and $[4kt+s, 8kt-1+s]$. Here, s is a non-negative integer.

Next, for $d = 3, 5$ we have from Lemmas 3.1 and 3.2 a $d \times (2k)$ quasi-magic rectangle $A = (a_{x,y})$ with distinct entries in $[0, h]$, row magic sum kh and column magic sum $dh/2$.

At last, we construct a $(4t+d) \times (2k)$ matrix $C = (c_{x,y})$ where

$$
\begin{aligned}
c_{x,y} &= a_{x,y} + 4kt \text{ for } 0 \le x \le d-1 \text{ and } 0 \le y \le 2k-1, \\
&= b_{x-d,y} \text{ for } d \le x \le 4t+d-1 \text{ and } 0 \le y \le 2k-1.
\end{aligned}
$$

Take $s = h+1$ in B. It is readily checked that C is a $(4t+d) \times (2k)$ quasi-magic rectangle.

Theorem 3.4. *An $m \times n$ quasi-magic rectangle exists if and only if $m, n \ge 2$ and $m + n \ge 5$.*

Proof: By Definition 1.2, it is easy to check the necessity. For the sufficiency, the conclusion follows from Theorem 1.3 when $m - n \equiv 0 \pmod 2$, and from Lemma 3.3 when $m - n \equiv 1 \pmod 2$.

4. Main result

We are now in a position to prove our main result.

Proof of Main Theorem 1.4: If m and n are positive integers and $m, n \not\equiv \{1, 2, 3, 6\}$, then there are ODLS(m) and ODLS(n) from Theorem 1.1, and also an $m \times n$ quasi-magic rectangle from Theorem 3.4. Therefore, we can apply Theorem 2.4 to obtain an addition-multiplication magic square of order mn.

In order to completely solve the existence question of AMMS(n) posted by J. Denes and A.D. Keedwell, one should further consider the following remaining orders: $n = 24, 27, 54, p, 2p, 3p$, and $6p$ where p is a prime integer. To the authors' knowledge, these cases are still open except the nonexistence of an AMMS(2) and the known AMMS(9).

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Appendix

An addition-multiplication magic square of order 20

381	122	363	964	1449	2810	3751	252	1717	3978	1539	8020	1805	1566	987	328	13	2534	4815	4816
723	484	61	762	231	4092	2529	1610	7619	1620	3757	1818	287	1128	1305	2166	4515	5136	2353	14
244	1143	482	121	3372	1771	210	3069	4420	1919	7218	1377	2088	2527	246	705	2896	15	4214	4173
242	241	1524	183	3410	189	1932	3091	1458	6817	2020	4199	846	205	2888	1827	4494	3913	16	2715
6137	4698	2679	820	5	1086	2247	2408	4953	854	1815	3856	161	562	1023	84	909	2210	891	4812
779	2820	4437	6498	2107	2568	905	6	3615	1936	793	5334	63	1364	281	322	4411	972	1989	1010
5220	6859	738	2397	1448	7	1806	1605	976	5715	3374	1573	1124	483	42	341	2652	1111	4010	729
2538	697	7220	4959	1926	1505	8	1267	1694	3133	6096	915	682	21	644	843	810	3609	1212	2431
2093	3934	5115	336	101	442	243	1604	3249	2610	1551	492	17	3258	6099	6020	1905	366	847	1928
315	5456	3653	2254	1203	324	221	202	451	1692	2349	3610	5719	6420	3077	18	1687	968	305	2286
4496	2415	294	4433	884	303	802	81	3132	3971	410	1269	3620	19	5418	5457	488	2667	1446	605
4774	273	2576	4215	162	401	404	663	1410	369	4332	2871	5778	5117	20	3439	726	1205	3048	427
9	1810	3531	3612	6477	1098	2299	4820	805	1686	2387	168	1313	3094	1215	6416	361	522	423	164
3311	3852	1629	10	4579	2420	1037	6858	147	2728	1405	966	6015	1296	2873	1414	123	564	261	722
2172	11	3010	2889	1220	7239	4338	2057	2248	1127	126	1705	3536	1515	5614	1053	1044	1083	82	141
3210	2709	12	1991	2178	4097	7620	1159	2046	105	1288	1967	1134	5213	1616	3315	282	41	1444	783
505	1326	567	3208	4693	3654	2115	656	1	362	963	1204	3429	610	1331	2892	2737	5058	6479	420
2807	648	1105	606	615	2256	3393	5054	903	1284	181	2	2651	1452	549	3810	399	6820	4777	2898
1768	707	2406	405	4176	5415	574	1833	724	3	602	321	732	4191	2410	1089	5620	3059	378	5797
486	2005	808	1547	1974	533	5776	3915	642	301	4	543	1210	2169	4572	671	6138	357	3220	5339