A Lower Bound for Mixed Ramsey Numbers: Total Chromatic Number Versus Stars

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Abstract. The total chromatic number $\chi_2(G)$ of a graph G is the smallest number of colors which can be assigned to the vertices and edges of G so that adjacent or incident elements are assigned different colors. For a positive integer m and the star graph $K_{1,n}$, the mixed Ramsey number $\chi_2(m, K_{1,n})$ is the least positive integer p such that if G is any graph of order p, either $\chi_2(G) \geq m$ or the complement $\overline{G}$ contains $K_{1,n}$ as a subgraph. In this paper we introduce the concept of total chromatic matrix and use it to show the following lower bound: $\chi_2(m, K_{1,n}) \geq m + n - 2$ for $m \geq 3$ and $n \geq 1$. Combining this lower bound with the known upper bound (Fink) we obtain that $\chi_2(m, K_{1,n}) = m + n - 2$ for m odd and n even, and $m + n - 2 \leq \chi_2(m, K_{1,n}) \leq m + n - 1$ otherwise.

1. Introduction

The graphs discussed in this paper are simple graphs, i.e., finite undirected graphs with neither loops nor multiple edges. The *k-total coloring* of a graph $G = (V, E)$ is a coloring with k colors assigned to the elements of $V \cup E$ such that adjacent or incident elements are assigned different colors. The *total chromatic number* $\chi_2(G)$ of a graph G is the minimum k, for which a k-total coloring of G exists. Let f be a graphical parameter, m a positive integer and H a graph; then the *mixed Ramsey number* $f(m, H)$ is the smallest positive integer p such that for every graph G of order p, either $f(G) \geq m$ or $H \subseteq \overline{G}$. For total chromatic number $\chi_2(G)$ and the star graph $K_{1,n}$, Fink [4] determined the following upper bound for $\chi_2(m, K_{1,n})$ and gave the conjecture below:

Theorem 1. *If $m \geq 3$ and $n \geq 1$, then*

$$\chi_2(m, K_{1,n}) \leq \begin{cases} m+n-2 & \text{if } m \text{ is odd and } n \text{ is even} \\ m+n-1 & \text{otherwise.} \end{cases}$$

Conjecture (Fink). For $m \geq 5$ and $n \geq 2$, then

$$\chi_2(m, K_{1,n}) = \begin{cases} m+n-2 & \text{if } m \text{ is odd and } n \text{ is even} \\ m+n-1 & \text{otherwise.} \end{cases}$$

The conjecture was verified in [4] for some special cases: $m = n$ or $m = n + 3 \equiv 1 \pmod 2$. It was further verifed by Cleves and Jacobson [3] for the small values $m = 5$ and 6. In this paper we shall give further evidence to support the conjecture by showing a lower bound $\chi_2(m, K_{1,n}) \geq m + n - 2$ for $m \geq 3$ and $n \geq 1$, thus verifying the conjecture for the case when m is odd and n is even.

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2. Definitions and notation

Suppose we have a k-total coloring of a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and with colors $x_1, x_2, \ldots, x_k$, which are distinct positive integers. We consider the integer matrix $M = (a_{ij})$ where a_{ii} = color assigned to vertex v_i and a_{ij} = color assigned to edge v_iv_j; $a_{ij} = 0$ when $v_iv_j \notin E(G)$. This leads to the following definitions.

A symmetric matrix $M = (a_{ij})$ of order n is called a *k-total chromatic matrix* if M satisfies:

(i) $a_{ij} \in \{0, x_1, x_2, \ldots, x_k\}$ and $a_{ii} \neq 0$ for $1 \leq i, j \leq n$, where $x_1, x_2, \ldots, x_k$ are distinct positive integers;

(ii) If two entries in any row or two entries in any column are equal then they must be zero;

(iii) If $a_{ii} = a_{jj}$ for $1 \leq i \leq j \leq n$ then $a_{ij} = 0$.

Suppose $M = (a_{ij})$ is a k-total chromatic matrix of order n. If G is a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $a_{ij} = 0$ iff $v_iv_j \notin E(G), 1 \leq i \leq j \leq n$, M is called a *k-total chromatic matrix* of G.

Clearly, a k-total chromatic matrix of G can determine a k-total coloring of G as follows: assign color a_{ii} to vertex v_i and color a_{ij} to edge v_iv_j for $1 \leq i, j \leq n$; therefore $\chi_2(G) \leq k$. It is also easy to see that every k-total chromatic matrix of order n is a k-total chromatic matrix of a graph G of order n which is uniquely determined by the matrix under isomorphism. Thus a k-total chromatic matrix of order n determines both a graph G and a k-total coloring of the graph.

To construct suitable k-total chromatic matrices we shall start with a matrix $L_n = (a_{ij})$, where $i, j, a_{ij} \in \{1, 2, \ldots, n\}$ and $a_{ij} \equiv i + j - 1 \pmod{n}$. Let M be any real matrix. By $M + Z(i,j)$ we mean a matrix obtained from M by changing the (i,j) entry into zero. By $M + A_r(i,j)$ we mean a matrix obtained from M by adding a real number r to the (i,j) entry. In other words, in the sums $M + Z(i,j)$ and $M + A_r(i,j)$, if $M = (m_{st})$, $Z(i,j) = (z_{st})$ and $A_r(i,j) = (a_{st})$, then $Z(i,j)$ and $A_r(i,j)$ have the same number of rows and the same number of columns as M, and

$$z_{st} = \begin{cases} -m_{st} & \text{for } (s,t) = (i,j) \\ 0 & \text{otherwise,} \end{cases}$$

$$a_{st} = \begin{cases} r & \text{for } (s,t) = (i,j) \\ 0 & \text{otherwise.} \end{cases}$$

The notation $M + \sum_{i,j} Z(i,j)$ is self-explanatory and the notation $M + \sum_{i,j} A_r(i,j)$ is similar.

3. Main results

In this section we shall give a lower bound for the mixed Ramsey numbers $\chi_2(m, K_{1,n})$.

Theorem 2. *If $m \geq 3$ and $n \geq 1$, then $\chi_2(m, K_{1,n}) \geq m+n-2$.*

Proof: It has been proved in [3] that $\chi_2(3, K_{1,n}) = n+1$, and

$$\chi_2(4, K_{1,n}) = \begin{cases} n+3 & \text{if } 3\,|(n+2) \\ n+2 & \text{otherwise.} \end{cases}$$

It then follows that $\chi_2(m, K_{1,n}) \geq m+n-2$ for $3 \leq m \leq 4$ and $n \geq 1$. So we assume $m \geq 5$ below.

First, for $m-2 \leq 2k \leq 2m-5$ we shall construct a graph H_{2k} of order $2k$ such that the total chromatic number $\chi_2(H_{2k}) \leq m-1$ and the minimum degree $\delta(H_{2k}) \geq m-3$.

If $2k = m-2$, let H_{2k} be the complete graph K_{m-2}. Evidently, $\chi_2(K_{m-2}) = m-1$ and $\delta(K_{m-2}) = m-3$. If $2k > m-2$, we denote $s = 2k-m+1$. It is easily seen that $0 \leq s \leq k-1$. Let

$$M_{2k} = L_{2k} + \sum_{i,j} Z(i,j),$$

where the summation is taken over the set $\{(i,j)\,|1 \leq i,j \leq 2k, i-j \equiv k \pmod{2k}\} \cup \{(i,j)|1 \leq i,j \leq 2k, i+j \equiv 2p+1 \pmod{2k}, 1 \leq p \leq s\}$. Then M_{2k} is an $(m-1)$-total chromatic matrix of order $2k$ using $m-1 = 2k-s$ colors in the set $\{1,2,\ldots,2k\}\backslash\{2,4,\ldots,2s\}$. This matrix M_{2k} determines a graph H_{2k} of order $2k$ such that the total chromatic number $\chi_2(H_{2k}) \leq m-1$ and the minimum degree $\delta(H_{2k}) \geq (2k-1)-1-s = m-3$.

Next, we shall construct a graph H_{2k+1} of order $2k+1$ for each odd number $2k+1$ in the interval $[m-2, 2m-5]$ such that $\chi_2(H_{2k+1}) \leq m-1$ and $\delta(H_{2k+1}) \geq m-3$.

If $2k+1 \leq m-1$, $H_{2k+1} = K_{2k+1}$ can satisfy the requirement. Otherwise, $2k+1 > m-1$. Let $t = 2k-m+2$, then $1 \leq t \leq k-1$.

When $1 \leq t \leq k-2$, delete the last row and last column from the above defined matrix L_{2k} and denote by L^*_{2k-1} the resultant matrix of order $2k-1$. Add one to each entry from column $(k+1)$ to column $(2k-1)$ in the last row of L^*_{2k-1} and from row $(k+1)$ to row $(2k-2)$ in the last column of L^*_{2k-1}. Here, the result is taken from the set $\{1,2,\ldots,2k\}$ modulo $2k$, and the resultant matrix is denoted by L^{**}_{2k-1}. Let

$$M^*_{2k-1} = L^{**}_{2k-1} + \sum_{i,j} Z(i,j) + \sum_{i,j} A_{2k+1}(i,j) + \sum_{i,j} A_{k-1}(i,j),$$

where the first summation is taken over the set $\{(i,j)\,|1 \leq i,j \leq 2k-1, |i-j| \in \{k, k-1\}\}$, the second over the set $\{(i,j)|1 \leq i,j \leq 2k-1, \{i,j\} \neq \{k, 2k-1\}, |i-j| = k-1\}$ and the third over the set $\{(i,j)\,|\{i,j\} = \{k, 2k-1\}\}$. On the other hand, we construct a $(2k-1)\times 2$ matrix B as follows. Keeping the i-th row we transfer the (i,j) entries in the set $\{(i,j)\,|\,1\le i,j\le 2k-1, i-j=k,1-k\}$ of L^*_{2k-1} into the first column of B. Similarly, move the set $\{(i,j)\,|\,1\le i,j\le 2k-1, j-i=k,1-k\}$ into the second column of B. Denote by B^t the transpose of B. Let

$$M'_{2k+1}=\left(\begin{array}{c|cc} M^*_{2k-1} & \multicolumn{2}{c}{B} \\ \hline B^t & 2k+1 & 0 \\ & 0 & 2k+1 \end{array}\right)$$

It is easily checked that M'_{2k+1} is a $(2k+1)$-total chromatic matrix of order $2k+1$. In M'_{2k+1} we replace all of the elements $2,4,\ldots,2t$ by zeros. Since they cannot appear on the main diagonal and $2k+1-t=m-1$, we obtain an $(m-1)$-total chromatic matrix of order $2k+1$ using colors in $\{1,2,\ldots,2k+1\}\backslash\{2,4,\ldots,2t\}$, say M_{2k+1}. This matrix M_{2k+1} determines a graph H_{2k+1} of order $2k+1$ satisfying both $\chi_2(H_{2k+1})\le m-1$ and $\delta(H_{2k+1})\ge 2k-1-t=m-3$.

It remains to discuss the case when $t=k-1$, that is $2k+1=2m-5$. If m is odd, let

$$M_{2k+1}=\left(\begin{array}{c|c} L_{m-2} & 0 \\ \hline 0 & L^*_{m-3}\end{array}\right)+\sum_{i,j}A_{m-1}(i,j),$$

where L^*_{m-3} is obtained by deleting the last row and last column in L_{m-2}, and the summation is taken over the set $\{(i,j)\,|\,1\le i,j\le 2m-5, i\ne j, i+j=2m-4\}$. If m is even, let

$$M_{2k+1}=\left(\begin{array}{c|c} L'_{m-3} & 0 \\ \hline 0 & N_{m-2}\end{array}\right)+\sum_{i,j}A_{m-1}(i,j)+\sum_{i,j}A_{m/2}(i,j),$$

where L'_{m-3} is obtained by adding one to each entry $\ge m/2$ in L_{m-3}, the first summation is the same as above in the odd case, the second summation is taken over the set $\{(i,j)\,|\,\{i,j\}=\{m-3,m-2\}\}$, and

$$N_{m-2}=\begin{cases} L_{m-2}+\sum_{i,j}Z(i,j) & \text{if } 4\nmid m \\ L_{m-2}+\sum_{i,j}Z(i,j)+A_{\frac{m}{2}-1}(3m/4-1,3m/4-1) & \text{if } 4\,|\,m\end{cases}$$

both the summations here are taken over the same set $\{(i,j)\,|\,1\le i,j\le m-2, |i-j|=m/2-1\}$. Therefore, M_{2k+1} is an $(m-1)$-total chromatic matrix of order $2k+1$, which determines a graph H_{2k+1} of order $2k+1$ satisfying $\chi_2(H_{2k+1})\le m-1$ and $\delta(H_{2k+1})\ge m-3$.

Now we complete the proof. When $m\ge 3$ and $n\ge 1$, the number $m+n-3$ may be expressed as a sum of multiples of integers in the interval $[m-2,2m-5]$. Suppose

$$m+n-3=\sum_{m-2\le h\le 2m-5} c_h\cdot h,$$

where c_h is a non-negative integer. Let

$$H = \sum_{m-2 \leq h \leq 2m-5} c_h H_h,$$

where H_h is constructed above. Then H is a graph of order $m+n-3$. Since for each number h in the interval $[m-2, 2m-5]$ there exists a graph H_h such that $\chi_2(H_h) \leq m-1$ and $\delta(H_h) \geq m-3$, then $\chi_2(H) \leq m-1$ and $\delta(H) \geq m-3$. The maximum degree of the complementary graph $\overline{H}$ is $\Delta(\overline{H}) \leq (m+n-4) - (m-3) = n-1$. $\overline{H}$ cannot contain any $K_{1,n}$ as a subgraph. Thus we obtain the bound $\chi_2(m, K_{1,n}) \geq m+n-2$.

Combining Theorem 1 and Theorem 2 we obtain

Theorem 3. *If $m \geq 3$ is odd and $n \geq 2$ is even, then $\chi_2(m, K_{1,n}) = m+n-2$.*

References

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