Complexity of graph covering problems for graphs of low degree

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Abstract. We consider the following three problems: Given a graph G,

1. What is the smallest number of cliques into which the edges of G can be partitioned?
2. How many cliques are needed to cover the edges of G?
3. Can the edges of G be partioned into maximal cliques of G?

All three problems are known to be NP-complete for general G. We show here that (1) is NP-complete for $\Delta(G) \geq 5$, but can be solved in polynomial time if $\Delta(G) \leq 4$ (the latter has already been proved by Pullman [P]); (2) is NP-complete for $\Delta(G) \geq 6$, and polynomial for $\Delta(G) \leq 5$; and (3) is NP-complete for $\Delta(G) \geq 8$ and polynomial time for $\Delta(G) \leq 7$.

1 Introduction

We consider simple graphs, graphs having no loops or multiple edges.

$$G = (V, E) = (V(G), E(G)) = (\text{vertices of } G, \text{edges of } G)$$

A K_n is a complete graph on n vertices. A *clique* of G is a complete subgraph of G. A K_2 will be called an *edge* (and will be loosely identified with the one edge it contains) and a K_3 will be called a triangle. A family C of cliques of G such that every edge of G belongs to a member of C is called a *clique cover* of G. If the members of C are edge disjoint, then C is called a *clique partition* of G. The *clique covering number* of G, $cc(G)$ is the size of a minimum cardinality clique cover (hereafter *minimum clique cover*) of G, and $cp(G)$, the clique partition number of G, is the size of a minimum cardinality clique partition (*minimum clique partition*) of G. We will be concerned with the following problems.

CC (Clique covering of edges)
Instance: A graph G and natural number k.
Question: Is $cc(G) < k$?

CP (Clique partition of edges)
Instance: Graph G, integer k.
Question: Is $cp(G) < k$?

PMC (Partition of edges into maximal cliques)
Instance: Graph G.
Question: Does G have a clique partition into maximal cliques?

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[1] While this research was being carried out, the author was an NSERC University Research Fellow. The research was also supported by an NSERC Operating Grant.
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These three problems are known to be NP-complete. For CC this was shown by Kou, Stockmeyer, and Wong [KSW78], and by Orlin [Orl77], but a far more elegant proof, which shows all three problems to be NP-complete, was given by Holyer [Hol81]. We are interested here in the following refinements of these problems.

For n a fixed integer, let CC(n) denote CC with the restriction that $\Delta(G) \leq n$. Define CP(n) and PMC(n) similarly. The smaller n is, the easier CC(n), CP(n) or PMC(n) will be. Holyer's proof shows that these problems are still NP-complete for $n = 14$. Our aim is to find the smallest values of n for which the problems are NP-complete.

A problem is *polynomial* if there is an algorithm which solves the problem in time polynomial in the length of the input. We say that a minimization or existence problem is *constructively polynomial* if there is an algorithm which constructs a minimizer or an example if there is one—in the forgoing cases, a minimum cardinality clique partition or clique cover, or a partition into maximal cliques—or states that there is none of there is not.

We assume that a graph G is represented in a natural way as a string of characters, it does not matter exactly how. The length of G as an input means the length of this string. Given an integer k, there are polynomial time algorithms which, on input a graph G of degree not more than k, computes in polynomial time (the polynomial depending on k) a list of all cliques of G, the clique graph of G, a list of all clique components of G, and other objects of a similar nature.

Theorem 1. *1. CP(5) is NP-complete.*
2. CP(4) is constructively polynomial.

Theorem 2. *1. CC(6) is NP-complete.*
2. CC(5) is constructively polynomial.

Theorem 3. *1. PMC(8) is NP complete.*
2. PMC(7) is constructively polynomial.

Pullman [P], has shown that CC(4) and CP(4) are polynomial.

Notation: Vertices of graphs will be denoted by letters $a, b, c, \ldots$. A string of letters standing for vertices indicates a clique with those vertices. For instance, ab is an edge between a and b, $abcd$ is a K_4 with vertices a, b, c, d. A path through the vertices $a_1, \ldots, a_n$ is denoted $(a_1, \ldots, a_n)$. The number of edges of G is $e(G)$, and $v(G)$ is the number of vertices.

2 NP-completeness proofs for CP(5) and CC(6)

We shall prove Theorems 1.1 and 2.1 by reducing the following two problems, which were proved NP-complete by Garey, Johnson, and Stockmeyer [GJS76]. Due to space constraints, we will only sketch the necessary constructions, and leave the details to the reader.

An *independent set* in a graph G is a collection of nonadjacent vertices of G. $is(G)$, the independent set number of G, is the maximum cardinality of an independent set in G. A *vertex cover* of G is a collection C of vertices of G such that every edge of G is incident to a vertex in G. $vc(G)$, the vertex cover number of G is the minimum cardinality of a vertex cover of G. Since the complement of an independent set is a vertex cover and vice versa, we have $vc(G) = v(G) - is(G)$. Now here are the problems.

IS(3) (independent set for graphs of degree 3)
Instance: Graph G with $\Delta(G) \leq 3$, integer k.
Question: Is $is(G) \geq k$?

VC(3) (vertex cover for graphs of degree 3)
Instance: Graph G with $\Delta(G) \leq 3$, integer k.
Question: Is $vc(G) \leq k$?

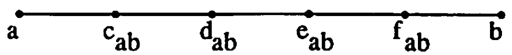


Figure 1: Path replacing the edge ab

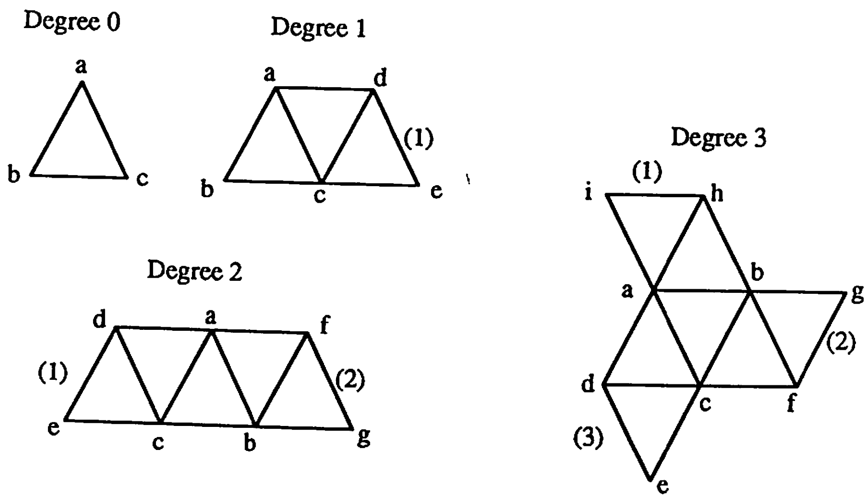


Figure 2: Complexes for reducing IS(3) to CP(5)

Proof of Theorem 1.1: Let (G, k) be an instance of IS(3). Form the graph G' by replacing each edge ab of G by a path $(a, c_{ab}, d_{ab}, e_{ab}, f_{ab}, b)$ as in Fig. 1. Observe that

$$is(G') = is(G) + 2e(G) \tag{1}$$

Now form a graph G'' by replacing each vertex v of G' by one of the complexes in Figure 2, according to the degree of v.

The edges (1), (2), (3) correspond to edges e_1, e_2, e_3 of G which are incident to v. If v and w are joined by an edge e_1, then identify edge (1) in the v complex with edge (1) of the w complex, and so forth. The resulting graph G'' has no vertex with degree more than 5 and no clique larger than a triangle. Further, $t(G'')$, the triangle graph of G'', is isomorphic to the graph G'. Hence by (1),

$$cp(G'') = 3e(G'') - 2(is(G) + 2e(G))$$

Theorem 1.1 follows. ∎

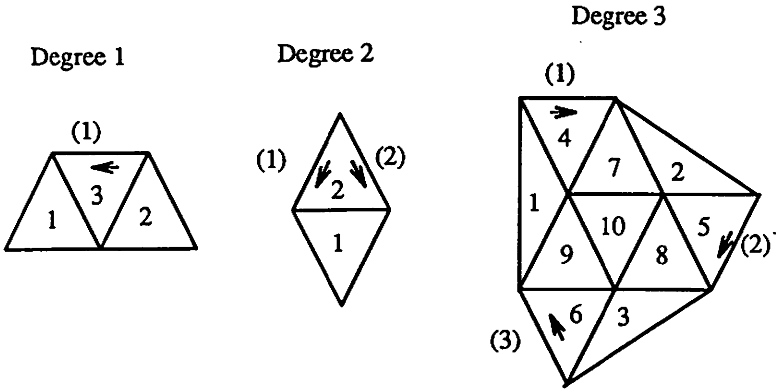


Figure 3: Complexes for reduction of VC(3) to CC(6)

Proof of Theorem 2.1: We reduce VC(3) to CC(6). Observe that if a graph G' has no cliques larger than a triangle, then $cc(G')$ is just $vc(t(G'))$ plus the number of edges of G' which are not contained in a triangle. Let (G, k) be an instance of VC(3). Assume that G has no isolated vertices, since such a vertex can be omitted from any vertex cover. Form a graph G' by replacing each vertex v of G by one of the complexes $C(v)$ in Figure 3, according to the degree of v. As in the construction for CP(5), (i), (ii), (iii) are edges of G incident to v. If v and w are joined by an edge e_1 in G, then the (1) edge of $C(v)$ is identified with the (1) edge of $C(w)$, and so forth. To ensure $\Delta(G') \leq 6$, this must be done so that the arrow on (1) in $C(v)$ has the opposite orientation to the arrow on (1) in $C(w)$. Then the resulting graph G' has degree at most 6. Let $v_i(G)$ indicate the number of vertices of G which have degree i. We have

$$cc(G') = vc(G) + 2v_1(G) + v_2(G) + 6v_3(G).$$

The result follows. ∎

3 NP-completeness of PMC(8)

We shall prove that PMC(8) is NP-complete by reducing to it the One-in-Three Satisfiability problem (1-in-3 SAT) without negated propositional variables ([LO4] in [GJ79]). 1-in-3 SAT was shown by Shaefer [Sha78] to be NP-complete.

The problem 1-in-3 SAT (without negated variables) is as follows. A clause C is a set of at most three propositional variables. A valuation V is a set of propositional variables. A set of clauses S is satisfied by a valuation V iff V contains exactly one member of each clause in S. Now here is the problem.

1-in-3 SAT.

Input. A finite set S of clauses.

Question. Is there a valuation which satisfies every clause in S?

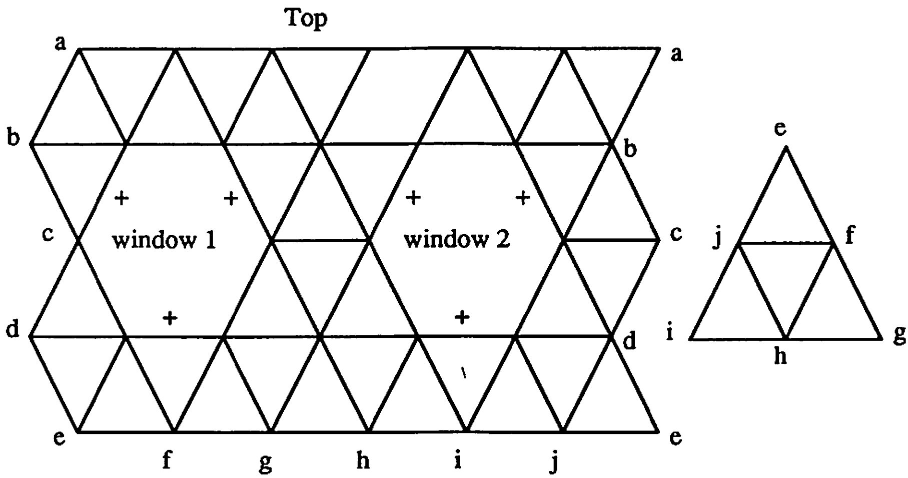


Figure 4: A Unit

Proof of Theorem 3.1: Let S be a finite set of clauses. We may suppose that S does not contain the empty clause, since the empty clause is by definition unsatisfiable. We construct a graph G which has a partition by maximal cliques iff S is 1-in-3 satisfiable. For each occurrence of a literal in S, G will contain a subgraph of the form given in Fig. 4, which we call a *unit*.

Call the the cycle along the top of the diagram the *top boundary* of the unit, and the two cycles around the windows the *window boundary*, both together being the *boundary*. A *weak partition* of a unit is a collection of disjoint maximal cliques (triangles) of the unit which covers all the edges except, possibly, boundary edges. A unit has exactly two weak partitions. One covers the top boundary and the boundary edges in the window boundary marked '+'; call this the *positive* partition, and the edges marked '+', *positive edges*. The other weak partition consists of all the triangles not in the positive partition; call this the *negative* partition. It covers no boundary edges but those around the windows which are not marked with a '+'; call these edges *negative*. Let $p_1, \ldots, p_m$ be the propositional variables occurring in S, and for each $i = 1 \ldots m$, let $p_{i1}, \ldots, p_{in_i}$, be be all the occurrences of p_i in clauses of S. By repeating clauses, if necessary, we may assume that $n_i \geq 2$. G will contain a unit U_{ij} for each p_{ij}. Units are strung together as follows. For $l \leq i \leq m$ and $1 \leq j \leq n_i$, identify the boundary of window 2 of U_{ij} with the boundary of window 1 of $U_{i(j+1)}$ so that positive edges are identified with negative edges, and do the same for the boundary of window 2 of U_{in_i} and that of window 1 of U_{i1}. Call the resulting graph G_i. Because $n_i \geq 2$, G_i is a simple graph, and has no cliques not contained in one of its constituent units. Hence G_i has exactly two weak partitions (covering all edges of G_i except possibly those belonging to the top boundaries of the constituent units), one which induces the positive partition on each U_{ij}, and one which induces the negative partition. Finally, for each clause $C \in S$, if W_1, W_2, W_3 are the units corresponding to the instances of literals contained in C, join the W_i's by identifying their top boundaries. The resulting graph G has no vertex of degree more than 8. It has a partition into triangles, its maximal cliques, iff S is satisfiable. ∎

4 Polynomial algorithms for CP(4) and CC(5)

We will prove Theorems 1.2 and 2.2 by showing that in graphs of the appropriate low degree, configurations which might make it difficult to find a minimum clique partition or clique cover can only occur in a small clique component of G or of a graph derived from G. The problem can then be solved component by component, on large components by an easy algorithm, and on small components by brute force.

In the constructions for these results and for Theorem 3.2, below, we will use the following terminology. If we are constructing a graph G under the constraint that $\Delta(G) \leq k$, we say that a vertex v of G is *full* at a given stage of the construction if it is adjacent to k distinct vertices. The *current degree* of v, $cd(v)$, is the number of distinct vertices adjacent to v at a given stage of the construction (which, of course, changes during the construction).

Since Pullman has already proved the simplest of the results we have stated, Theorem 1.2, we will just sketch it as follows. Let G be given with $\Delta(G) \leq 4$. It suffices to solve the problem for each clique component separately, so we may assume that G is clique connected. By examining various cases, we can show that either $|e(G)| \leq 12$ or else G contains no clique larger than a triangle, and $\Delta(t(G)) \leq 2$. In the former case, solve the problem by brute force. In the latter case $t(G)$ is either a path or a cycle. Start at one end, if a path, and take every other triangle plus the edges left over.

The proofs of Theorems 2.2 and 3.2 are based on the same general idea, the hard part being to confine the troublesome configurations to small subgraphs. Let us now prove Theorem 2.

We want to construct a minimum clique cover successively adding cliques to a partial cover. Here is a definition to allows to express the idea of completing a clique cover.

Definition 4. *Let H be a subgraph of G.*

1. *A G-clique cover of H is a collection of cliques of G which together contain all edges of H. $cc_G(H)$ is the minimum cardinality of a G-clique cover of H.*
2. *An H-eligible clique of G is either*
   (a) a maximal clique of G which contains at least two edges of H, or
   (b) an edge of H which is contained in no clique of the kind described in 2a.
3. *An edge of H is H-exposed if it belongs to only one H-eligible clique. A clique of G is H-exposed if it is H-eligible and contains an H-exposed edge. $\mathcal{E}_H$ is the class of all H-exposed cliques.*
4. *The set of interior edges of H, $int(H)$, consists of those edges of H which belong to no H-exposed clique.*

Of course, a clique cover of G is just a G-clique cover of G, hence $cc(G) = cc_G(G)$. The following lemma is clear.

Lemma 5. *If C is a minimum G-clique-cover of $int(H)$, then $C \cup \mathcal{E}_H$ is a minimum G-clique-cover of H.*

The Lemma shows how to begin an algorithm to find a minimum clique cover C of G.

$C \leftarrow \emptyset$; $H \leftarrow G$;
Repeat
    $C \leftarrow C \cup \mathcal{E}_H$;
    $H \leftarrow int(H)$;
Until $H = int(H)$

Since the loop is executed at most $e(G)$ times, and each execution takes polynomial time, this part of the algorithm takes polynomial time. After this, we are left with a subgraph H of G which has no H-exposed edges. Thus each edge of H belongs to at least two H-eligible cliques, but not to any G-exposed clique, since $H \subseteq int(G)$. As any H-eligible clique which consisting of a single edge would be exposed, every H-eligible clique must be a maximal clique of G which contains at least two edges of H.

By Lemma 5, we can finish the algorithm by finding a mimimum G-clique cover for H. This done in the following way. Let $elig(H)$ be the graph whose vertices are H-eligible cliques, two cliques being adjacent if they share an edge of H.

Since a minimum G-clique cover of H corresponds to a collection of minimum G-clique covers of the $elig(H)$-components of H, a minimum cover may be found $elig(H)$-component by component. Hence we may assume that H is $elig(H)$-connected.

We will show that either H has no more than some fixed number of edges, or else

(4.1) every edge of H is contained in exactly two H-eligible cliques; and

(4.2) no H-eligible clique is adjacent to more than two other H-eligible cliques.

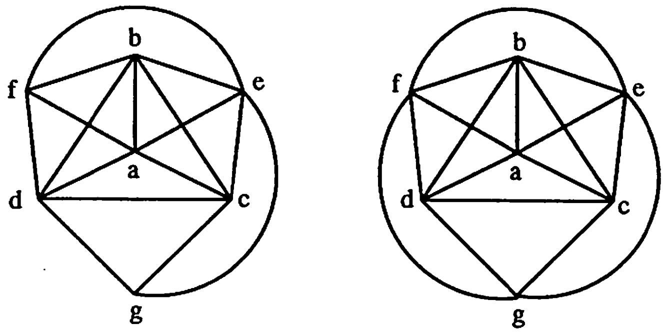


Figure 5: A maximal K_4 in G

By (4.1), a minimum cover of H by eligible cliques is the same as a minimum vertex cover of $elig(H)$. By (4.2), except in small components of $elig(H)$, no vertex of $elig(H)$ has degree more than 2. Hence one can find a minimum vertex cover in polynomial time.

The following sequence of lemmas will establish (4.1) and (4.2), thereby completing the proof of the Theorem.

Lemma 6. *Any K_4 of G which contains an edge of $int(G)$ is contained in a clique component of G which contains at most 15 edges.*

Proof: Since $\Delta(G) \leq 5$, it is easy to see that any K_5 of G either has an exposed edge, or else is contained in a K_6, in which case all the edges are exposed. Hence a K_5 of G contains no edge of $int(G)$.

On the other hand, it follows from looking at cases that any K_4 $abcd$ which is a maximal clique of G and contains an edge of $int(G)$ must be contained in a clique component isomorphic to one of the two forms in Fig. 5, which have 14 and 15 edges respectively. ∎

Lemma 6 implies that, except in small clique components of G, all H-eligible cliques are triangles. It follows that except in such a small clique component of G, edges in two different components of H cannot belong to the same clique of G, hence any $elig(H)$-component of H is connected.

Lemma 7. *Suppose that ab is an edge of $int(G)$ which is contained in at least three maximal cliques of G. Then ab belongs to a clique component of G which has at most twenty edges.*

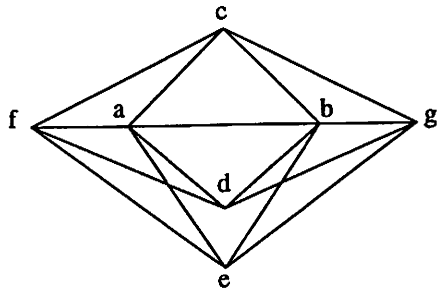


Figure 6: An edge of H contained in three triangles of G

Proof: By the preceding Lemma, we may take the maximal cliques containing ab to be triangles abc, abd and abe. Since $ab \in H$, the triangles are not G-exposed. As we already have four edges incident at a and b, we must have the situation in Fig. 6. $f \neq g$, since otherwise abc would not be a maximal clique. Now every vertex in the diagram has current degree at least 4, so any triangle in the clique component of G of ab contains one of the edges in the diagram. As the diagram has 15 edges and 5 vertices which may each be incident to one more edge, there can be at most 20 edges in the clique component. ∎

Since H has no exposed edges, the Lemma implies that except on small $elig(H)$-components of H, each edge of H belongs to exactly two H-eligible cliques. That is, (4.1) holds.

Lemmas 6 and 7 also imply that, except on small components, any H-eligible clique which is shares an edge with each of three other H-eligible cliques must be a triangle of H. Thus, the following Lemma establishes (4.2).

Lemma 8. *Suppose that no K_4 of G contains an edge of H, and no edge of H is contained in more than two cliques of G. Let abc be an H-eligible triangle which is adjacent to three other H-eligible triangles. Then each vertex of H which is connected to abc has H-distance at most 4 from the nearest vertex of abc.*

Proof: We begin by investigating the structure of G near abc. First, each edge of abc is contained in exactly one other clique of G, since abc is not exposed. These other cliques must be the adjacent H-eligible triangles. Since these triangles are in fact adjacent, i.e. they share an edge of H with abc, each edge ab, bc, ca must be one of those edges, and hence must be in H. Let the adjacent triangles be abe, acd and bcf. d, e, f are distinct since otherwise abc would be contained in a K_4. As these triangles contain edges of H, none of their edges can be exposed.

It follows that G contains one of the configurations of Figure 7. In the sequel, we will assume that we have the first configuration. For the others, the proof is similar, but we have not been able to unify the proofs.

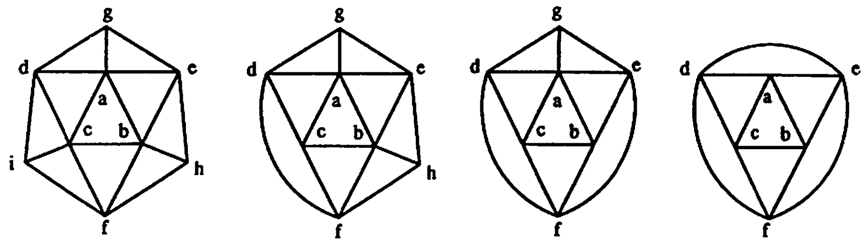


Figure 7: Neighborhood of a triangle of H adjacent to three eligible triangles

Since the vertices d through i are the only vertices of G adjacent to abc, hence the only vertices which may be adjacent in H, it suffices to prove that any vertex which is connected by an H-path to one of the vertices d, e, f, g, h, i is connected by an H-path of length at most two. Suppose, then, that this not the case. Up to isomorphism, we may assume that we have a vertex l which has H-distance exactly three from either g or e. Bear in mind two simple observations, which we will use several times.

1. No H-eligible clique has a G-exposed edge.
2. No H-eligible triangle containing l contains any of the vertices d, e, f, g, h, i, since there would be an H-path of length at most two from l to that vertex.

Case 1: The shortest H-path from l to a vertex d, e, f, g, h, i is $P = (g, j, k, l)$. As the current degree of g is 4, one of the two H-eligible cliques containing gj must be gjd or gje, say gjd. Now, jd must not be G-exposed, hence there must be an edge ij. Now $cd(j) = 4$, so one of the two H-eligible cliques containing jk must be either jki or jkg. Suppose it is jki, as in Fig. 8. As ki must not be G-exposed, there must be a triangle kfi. Now $cd(k) = 4$, so the two eligible triangles containing kl must be jkl and klm. m is a new vertex by Observation 2. But k and all vertices adjacent to it but l and m are full, so km must be G-exposed, impossible since of klm contains an edge of H. On the other hand, if one of the H-eligible cliques containing jk is gjk, then as gk as is not G-exposed, k must be adjacent to e. Again, the two H-eligible cliques containing kl must be klm and jkl, and again km must be exposed, a contradiction.

The subcase where one of the triangles containing jk is gjk, is handled similarly, as is Case 2, where the shortest H-path from l to a vertex $d, \ldots, i$ is $P = (e, j, k, 1)$. ∎

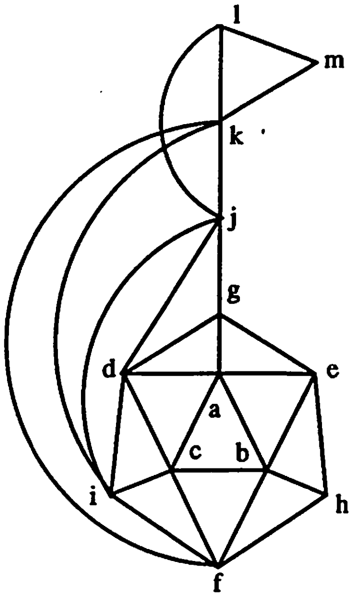


Figure 8: The first subcase of Case 1

5 A polynomial algorithm for PMC(7)

We follow a local reduction plan to attack the problem of partitioning a graph G of degree ≤ 7 into maximal cliques, as we did for CP(4) and CC(5). CC(7), however, is much more complex, and a more interesting problem is left over after the reduction. To carry out our reduction plan we need a more general notion of a partition problem.

A *partition problem* is a pair $(G, \mathcal{L})$ where G is a graph and $\mathcal{L}$ is a collection of maximal cliques of G. $(G, \mathcal{L})$ is *partitionable* iff the edges of G can be partitioned by elements of $\mathcal{L}$. Thus an instance of PMC is a partition problem $(G, \mathcal{M})$, where $\mathcal{M}$ is the set of all maximal cliques of G. We say that $(G, \mathcal{L})$ reduces to $(H, \mathcal{K})$ iff H is a subgraph of G, $\mathcal{K} \subset \mathcal{L}$, and $(G, \mathcal{L})$ is partitionable iff $(H, \mathcal{K})$ is. Often we can find a proper reduction of G by looking only at small subgraphs of G. We say that a subgraph H of G is *small* if its diameter is not more than two. An $\mathcal{L}$-*partition* of H is a covering of the edges of H by edge-disjoint members of $\mathcal{L}$ (the members of $\mathcal{L}$ used may contain edges of G which are not in H). H is said to be *partition independent* in $(G, \mathcal{L})$ if H has an $\mathcal{L}$-*partition* and any $\mathcal{L}$-*partition* of H covers only edges of H.

Here are some ways to reduce $(G, \mathcal{L})$.

(R1) If H has no $\mathcal{L}$-partition, $(G, \mathcal{L})$ is unpartitionable (reduces to a trivially unpartitionable problem).

(R2) If H has only one $\mathcal{L}$-partition $\mathcal{P}$, then $(G, \mathcal{L})$ reduces to $(G', \mathcal{L}')$, where G'

is the subgraph of G induced by those edges not in any element of $\mathcal{P}$, and

$$\mathcal{L}'' = \{C \in \mathcal{L}; C \subseteq G'\}$$

(R4) If H is partition independent in $(G, \mathcal{L})$, then $(G, \mathcal{L})$ reduces to $(G \backslash H, \mathcal{L}')$, where $\mathcal{L}'$ consists of all cliques of $\mathcal{L}$ which contain no edges of H.

Assume that $\Delta(G) \leq 7$.

To decide whether G can be partitioned into maximal cliques, start with $\mathcal{L} = \mathcal{M}$, and repeatedly apply the reductions (R1) through (R3) to $(G, \mathcal{L})$ as H varies over small subgraphs of G. In polynomial time we can decide whether such a reduction exists and, if there is one, apply it, since the number of maximal cliques of G (hence the size of $\mathcal{L}$) and the number of small subgraphs of G are both polynomial in the size of G. Since each reduction reduces the size of either G or $\mathcal{L}$, if we repeatedly reduce this way, in polynomial time we will reduce $(G, \mathcal{L})$ until reductions (R1) through (R4) can no longer be applied with H a small subgraph. When this state is reached, we say that $(G, \mathcal{L})$ is *locally reduced*, or just plain *reduced*. Note that a locally reduced problem has no *$\mathcal{L}$-exposed* cliques, cliques of $\mathcal{L}$ which contain an edge in no other clique of $\mathcal{L}$. In particular, $\mathcal{L}$ contains no maximal cliques consisting of a single edge.

We may assume, then, that $(G, \mathcal{L})$ is locally reduced. Identify $\mathcal{L}$ with the subgraph of the clique graph of G whose vertices are the cliques in $\mathcal{L}$. We assume that $\mathcal{L}$ is connected, since an $\mathcal{L}$-partition of each $\mathcal{L}$-component of G yields a partition of the whole.

Call an edge of G which is contained in three or more members of $\mathcal{L}$ a *triple edge* of $(G, \mathcal{L})$. Suppose that $(G, \mathcal{L})$ has no triple edges. As G has no $\mathcal{L}$-exposed edges, each edge of G belongs to exactly two cliques of $\mathcal{L}$. Pullman, Shank and Wallis [PSW82] observed that in this case one can decide in the following way whether G can be partitioned into maximal cliques: Define an equivalence relation $\equiv$ on $\mathcal{L}$ by letting two members be equivalent iff there is a path of even length in $\mathcal{L}$ from one to the other. $(G, \mathcal{L})$ is partitionable iff $\equiv$ has two equivalence classes (rather than just one), and a partition is formed of all members of either of the two classes. In particular, any clique in $\mathcal{L}$ belongs to a partition.

We will show that if $(G, \mathcal{L})$ is locally reduced and $\mathcal{L}$ is connected, triple edges of $(G, \mathcal{L})$ can occur only in $\mathcal{L}$-components of a special structure for which an $\mathcal{L}$-partition is easily found, so that for the rest of the graph partitionability can be determined by the method of Pullman, Shank and Wallis. The rest of the proof consists of a series of Lemmas showing that triple edges occur only in these special components.

By a subgraph of G *induced by* a set of vertices or edges of G, we mean the subgraph which consists of those vertices or edges and all edges between any pair of the vertices, or of the edges and all vertices to which they are incident.

If H is a subgraph of G, $\ell(H)$, the *locality* of H, is the subgraph of G induced by all edges of G incident to any vertex of H. $\ell(v) = \ell(\{v\})$ for v a vertex of

G. The *neighborhood graph* of v, $nbd(v)$, is the subgraph of G induced by v and all vertices adjacent to it. In carrying out our campaign against triple edges, we will repeatedly use the following facts about $\mathcal{L}$-partitions of $\ell(v)$, v a vertex of G, where $(G, \mathcal{L})$ is a locally reduced partition problem.

1. The only ways to partition $\ell(v)$ are those indicated in Table 1. In particular, the partition cannot consist of a single clique, as that would lead to a reduction by (R2). Note that no edges of G can be maximal cliques, as that would lead to a reduction by either (Rl) or (R2).
2. $\ell(v)$ has at least two $\mathcal{L}$-partitions, because otherwise there would be a reduction by (R2).

Lemma 9. *For each vertex $a \in (G)$, $nbd(a)$ has exactly two $\mathcal{L}$-partitions, unless $nbd(a)$ contained in a complex of the form shown in Fig. 9, where each triangle shown belongs to $\mathcal{L}$, except for possibly cdh and fgi, and a is a vertex of no triangle of $\mathcal{L}$ other than those shown.*

$\delta(v)$	4	5	6	7
possible cliques of an $\mathcal{L}$-partition of $\ell(v)$	2 K_3's	K_3, K_5	3 K_3's 2 K_4's K_5, K_3	K_4, 2 K_3's K_5, K_4 K_6, K_3

Table 1: Partitions of the locality of a vertex in a locally reduced problem $(G, \mathcal{L})$

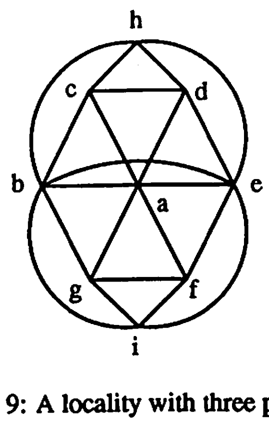


Figure 9: A locality with three partitions

Proof: We will go through all the cases in Table 1 and show that $nbd(a)$ cannot have three or more $\mathcal{L}$-partitions, until the only remaining possibility is that shown in Fig. 9.

We can treat one family of cases generally, those in which there is an $\mathcal{L}$-partition of $\ell(a)$ which consists of a K_3, abc, and one other clique, C. This is depicted in Figure 10. A second partition of $\ell(a)$ must consist of two cliques $\{a,b\} \cup C_1, \{a,c\} \cup C_2$, where C_1, C_2 are disjoint and $C_1 \cup C_2 \cup \{a\} = C$. Now, b cannot be adjacent to any vertex in C_2, nor c to any vertex in C_1, else abc would not be a maximal clique. Hence there can be only these two partitions of $\ell(a)$, hence only two partitions of $nbd(a)$.

This leaves only four cases to consider, two each for $\delta(a) = 6, 7$.

Case 1: $\delta(a) = 7$, and there is a partition of $\ell(a)$ which consists of a K_5 and a K_4, Figure 11 (a).

The second partition can be either: a K_4 and two K_3's, as in Fig. 11(b), a K_5 and a K_4 again, as in (c) or (d). If the second partition is (b), any further adjacencies would violate the maximality of one of the triangles, so there cannot be any other partition.

If we superpose (a) and (c) and consider the edges left over by the partition in (a), the solid lines in (e), we see that extending this partition to $nbd(a)$ would require three distinct cliques to contain the uncovered edges incident at f. This would bring the degree of f to nine, so this is impossible. Similarly superimposing (a) and (d), as in (f), and extending the (a) partition to a partition of $nbd(a)$ would require the degree of e to be at least eight, so that is also impossible.

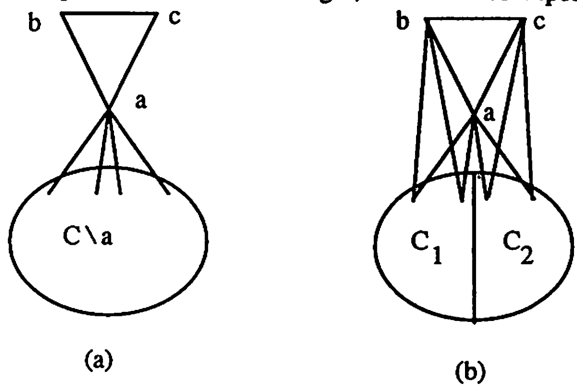


Figure 10: $\ell(a)$ partitioned by a K_3 and one other clique

Case 2: $\delta(a) = 7$, and there is a partition which consists of a K_4 and two K_3's, Figure 12(a). The possibilities for a second partition are a K_5 and a K_4 which we have already discussed, and another way of partitioning into a K_4 and two K_3's. The different ways of doing this (up to isomorphism) are shown in Figure 12(b),(c),(d). They are as follows.

(b) Use the same K_4 and different triangles.
(c) Use a K_4 which contains two of the vertices b, c, d.
(d) Use a K_4 which contains only one of b, c, d (solid lines).

There is only one way to do (b), so to get three partitions, we would have to combine (a) with either (b) and (c), (c) and (d), two ways of doing (c), or two ways doing (d). (b) and (d) are incompatible, as the triangle afg, or one of the others, is not maximal if (d), or a copy of it, is superimposed. Similarly, superimposing (a) and two copies of (d) violates the maximality of one of triangles in the superposition of (a) and (d).

If (a), (b), and (c), are superimposed, consider the edges left not covered by partition (b). These are the solid edges in (e). In a partition of *nbd*(a) which contains afg, and hence all of (b), the solid edges incident at e must be contained in separate cliques, since edges between them which are not already in (b) would violate the assumed maximality of some clique we already have. But putting them in separate cliques would bring the degree of e to at least 8, a contradiction. Therefore $(G, \mathcal{L})$ must be reducible by either (Rl) or (R2).

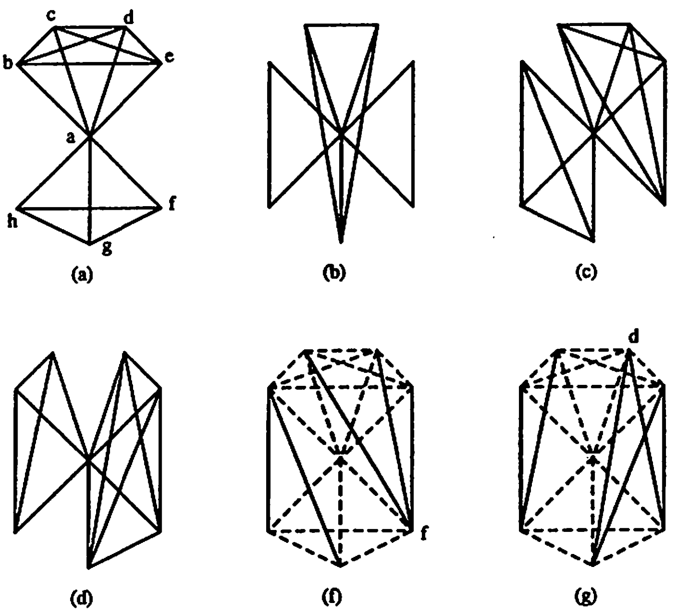


Figure 11: $\ell(a)$ partitioned by a K_5 and a K_4

Consider superposition of (a), (c) and (d). A copy of (c) can be superimposed on diagram (d) by either connecting b and f or d and g, which are isomorphic. The latter is shown in (f). Each of the resulting partitions of $nbd(a)$ requires a different partition of $\ell(c)$. Hence nbd(c) must have three $\mathcal{L}$-partitions. If c has degree 6, then the partition of $\ell(c)$ induced by (d) must consist of two K_4's. This reduces to the next case, so we need consider it no further. If c has degree 7, then each partition of $\ell(c)$ must consist of the K_4 given and two K_3's. By isomorphism with (d), there must be triangles bci, cdj in the extension of (d) to $\ell(c)$, and cgi, cfj in the extension of (a) to $\ell(c)$. Consider a partition extending (a) to $nbd(a)$. dg and fg are so far uncovered, and must be covered by distinct cliques since an edge df would violate the maximality of aef. But this brings the degree of g to 8.

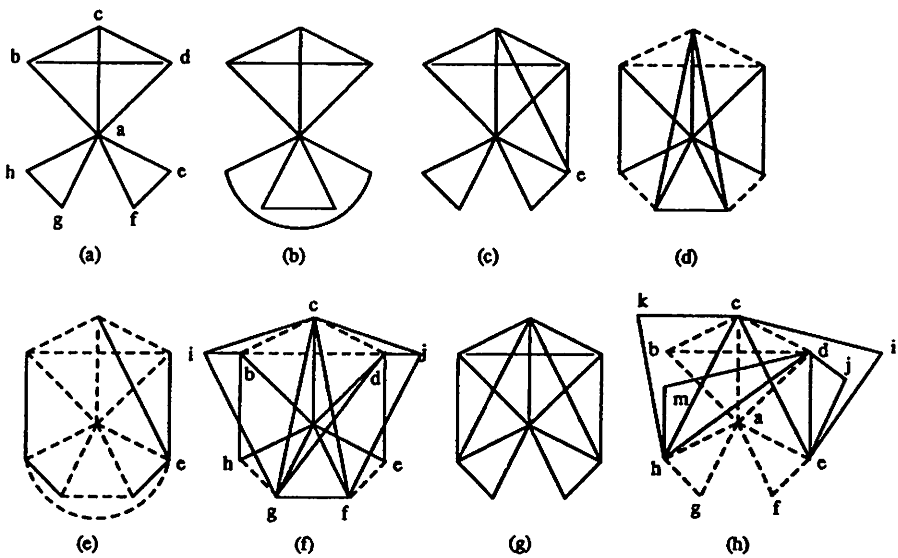


Figure 12: ℓ(a) partitioned by a K_4 and two K_3's

Superimposing (a) and two copies of (c) can be done in two ways. If done as in (g), so that the K_4's from the two copies of (c), $acde$ and $abcx$ have only a and one other vertex in common, we can argue as in the (a), (c), (d) superposition. Suppose, then (a) and two ways of doing (c) are superimposed as in (h), so that the new K_4's $acde$ and $acdh$ share two vertices besides a. Consider first a $\mathcal{L}$-partition of $nbd(a)$ which extends (a). ce, de, ch and dh must be contained in triangles cei, dej, chk, chm. The cliques containing these edges can only be triangles, because this already brings the degrees of c and d to 7. Hence e and h must have degree 6, and must have one $\mathcal{L}$-partition which consists of two K_4s ($acde$ and $efij$, for e), the other of the triangles just introduced. The possibility of a third $\mathcal{L}$-partition is eliminated in the next case, so we assume that these are the only $\mathcal{L}$-partitions of $\ell(e)$ and $\ell(h)$. Hence, if we consider an $\mathcal{L}$-partition of $nbd(a)$ which contains $acde$, it must also contain agh, chk and dhm. In order to cover ab, bc, and bd, b must be adjacent to f, i, and j. But if we consider an $\mathcal{L}$-partition containing $acdh$, the same reasoning shows that b must be adjacent to g, k and m. This brings the degree of b to nine, a contradiction.

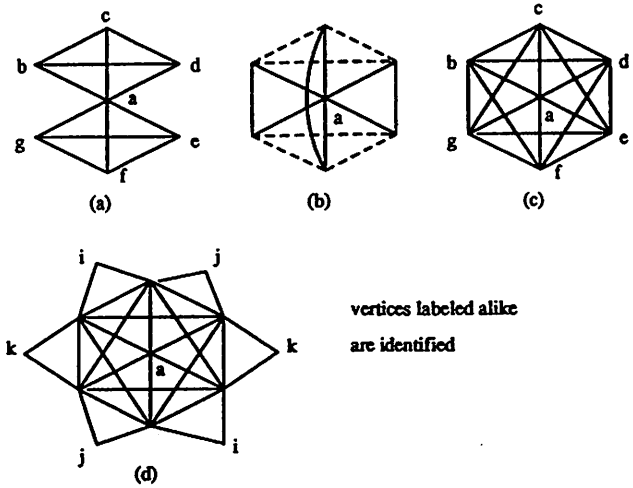


Figure 13: $\ell(a)$ partitioned into a two K_4's

Case 3: δ(a)=6, and $\ell(a)$ has an $\mathcal{L}$-partition which consists of two K_4's, Figure 13. If, as in 13(b), a second partition consists of three K_3's, then any further connections are impossible because the triangles are maximal, so there cannot be a third partition.

The second and third partitions, then, must both consist of two K_4's and this gives us the complex in Fig. 13(c). To extend any three of the four possible partitions of $\ell(a)$ to $nbd(a)$, we must have the complex in 13(c). It has diameter 2 and is partition independent, hence is reducible by (R3).

Case 4: δ(a)=6, and $\ell(a)$ has a $\mathcal{L}$-partition which consists of three K_3's.

Superimposing three ways of partitioning $\ell(a)$ into triangles lead to one of the complexes in Figure 14(a)-(c), where every triangle shown belongs to $\mathcal{L}$ and every

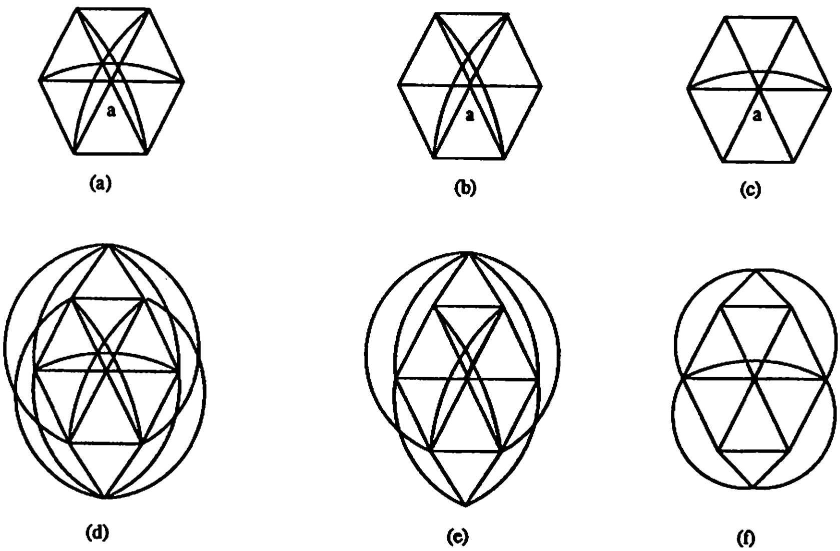


Figure 14: $\ell(a)$ partitioned into three K_3's

triangle of $\mathcal{L}$ which has a as a vertex is shown. By irreducibility, each neighbor of a which is a vertex of three triangles shown, must have three partitions, and hence must be the center of a similar complex. If follows that the complexes (a), (b), (c) are contained in (d), (e), (f), respectively. (d) and (e) are partition independent and of diameter two, hence are reducible by (R3). The remaining complex (f) is the one allowed by the statement of this Lemma. ∎

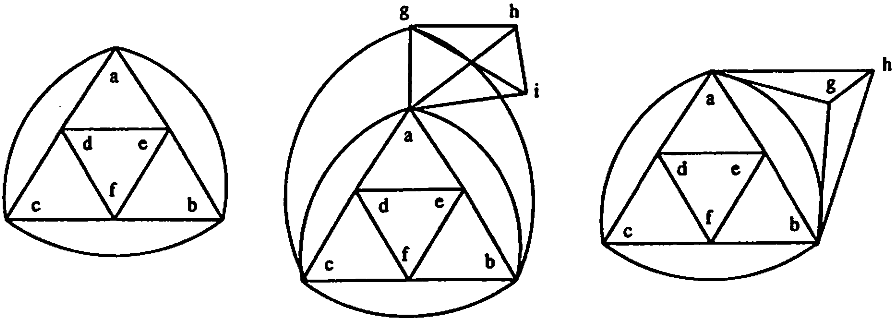
Observe that in a reduced problem $(G, \mathcal{L})$, if an edge ab is contained in three distinct cliques $C_1, C_2, C_3 \in \mathcal{L}$, then each of $nbd(a)$ and $nbd(b)$ has at least three distinct $\mathcal{L}$-partitions, one containing each of the three cliques C_1, C_2, C_3. Hence, our problem will be solved if we can show that each a such that $nbd(a)$ has three or more $\mathcal{L}$-partitions is contained in a clique component for which PMC is easy to solve. To this end, we make the following definition.

A *segment* is a subgraph of G as shown in Figure 15, such that

1. $abe, bcf, acd, ade, bef, cdf \in \mathcal{L}$, and
2. the edges ad, ae, be, bf, cf, cd do not belong to any other cliques of $\mathcal{L}$.

We call the triangles abc and def *end triangles* of the segment, and the edges not contained in these triangles *side edges*.

A *cylinder* is a graph consisting of a sequence of segments S_i, $1 \leq i \leq n$, seg-



A segment. A K4 containing abc. A K4 containing ab only.

Figure 15: A Segment

ment S_i having vertices are denoted $a_i, b_i, c_i, d_i, e_i, f_i$ corresponding to those in Fig. 15, so that for $i < n$, $a_{i+1} = d_i$, $b_{i+1} = e_i$, and $c_{i+1} = f_i$, and no other identifications are made. A *torus* is a similar graph which has also $a_1 = d_n . b_1 = e_n$, and $c_1 = f_n$, or some other one-to-one identification of the end triangles of a cylinder. A torus must have $n \geq 3$, for its constituents to be actually segments. When a torus is a subgraph of G for a reduced problem $(G, \mathcal{L})$, any end triangle of a constituent segment may or may not belong to $\mathcal{L}$. The same goes for cylinders, except the end triangles of the entire cylinder, $a_1 b_1 c_1$ and $d_n e_n f_n$, must be contained in cliques in $\mathcal{L}$, else their edges would be exposed. We see below that the cliques must be the end triangles themselves, or else triangles of another segment extending the cylinder.

The complex in Fig. 14(f), that is the only way a vertex a with three partitions of $\ell(a)$ can occur is a reduced graph, is a cylinder of two segments, fused at the triangle abe. The following Lemma implies that in such a case, $\ell(a)$ is contained in a cylinder or a torus. We continue our assumption that $(G, \mathcal{L})$ is a reduced problem.

Lemma 10. *The $\mathcal{L}$-component of a segment of G is either a cylinder or a torus.*

Proof: It will suffice to show that given a segment S, if any of its edges is contained in another clique, then that edge is an edge of one of the end triangles of S, the clique is a triangle, and it is contained in a segment joined to S in the manner prescribed for a cylinder or a torus.

Suppose that the vertices of S are named as in Fig. 15.

The edge definitely has to be an edge of an end triangle, because side edges are not in any cliques but those in the segment, by definition. We show that the clique cannot be a K_4. The argument also shows that it also cannot be larger. We suppose without loss of generality that the edge in question is ab. There are two cases to consider.

Case 1: the K_4 contains the entire end triangle abc. The K_4 must be $abcg$, where g is a new vertex. Since the subgraph H generated by a, b, c, d, e, f, g has diameter two, and $abe \in \mathcal{L}$, H must have an $\mathcal{L}$-partition $\mathcal{P}_1$ which contains abe, and hence does not contain $abcg$. $\mathcal{P}_1$ must contain a clique containing ag. Since all existing edges containing a are already in a clique that already must be in $\mathcal{P}_1$, there must be a $K_4\, aghi \in \mathcal{P}_1, h, i$ a new vertices. We know it is a K_4 because $\ell(a)$ has a partition containing a $K_4\, abcg$ and a triangle ade, hence must have degree 5 or 7. Now consider an $\mathcal{L}$-partition $\mathcal{P}_2$ of the $\ell(a)$ which contains $abcg$. $\mathcal{P}_2$ must contain ade, since it does not contain abe or acd and ad and ae are not contained in any other cliques. Since there are already 7 edges containing a and all others are already in cliques in $\mathcal{P}_2$, ah and ai cannot be covered, a contradiction.

Case 2: the K_4 contains only the edge ab. Say the K_4 is $abgh, g$ and h new edges. As before, any $\mathcal{L}$-partition must contain a K_4 containing each of a and b. Consider again a partition which contains abe, bcf, and acd. $\mathcal{P}$ must have K_4's containing a and b. They must be $aghi$ and $bghj$, since there is room for only one more edge at each vertex a and b. But these two K_4's have the edge gh in common, a contradiction.

Thus no K_4 contains an edge of S, and any adjacent clique must be a triangle containing just one edge of an end triangle. Consider such a triangle, which we may take to be abg. A partition that contains this triangle must contain ade, cdf, and bef, hence must also contain triangles ach and bci. Looking at a partition which contains abe, and hence acd and bcf, it must also contain agh, bgi and chi. To show that a, b, c, g, h, i induce a segment, it remains to show that each of ag, ah, bg, bi, ch, ci is not contained an any other clique of $\mathcal{L}$. To be specific, consider ag. Since no more edges can be incident at a, such a clique would have to be a triangle agx where x is one of c, d, or e. $x = c$ would form a $K_4\, abcg$, which has been ruled out already, and $x = c$ or $x = d$ would make a third clique of $\mathcal{L}$ containing ac or ad, contradicting assumption 2 of the definition of a segment.
∎

Recall that a *triple edge* is an edge contained in three cliques of $\mathcal{L}$.

Lemma 11. *A reduced problem $(G, \mathcal{L})$ is partitionable iff $(G, \mathcal{L}')$ is partitionable, where*

$$\mathcal{L}' = \mathcal{L} \backslash \{\text{triangles of triple edges of } \mathcal{L}\}.$$

Proof: By the previous results, the $\mathcal{L}$-component of any triple edge is a torus or a cylinder, and triple edge to a triangle of triple edges, which is an end triangle of a segment which is not at the end of a cylinder. Consider a cylinder or torus formed from segments $S_1, \ldots, S_n$, joined as prescribed. If it is a torus, then $\mathcal{P} = \{a_i b_i e_i, b_i c_i f_i, a_i c_i d_i; i = 1, \ldots, n\}$ is a partition. If it is a cylinder then $\mathcal{P} \cup$

$\{d_n e_n f_n\}$ is a partition. These do not contain any of the end triangles of constituent segments that may be triangles of triple edges of $\mathcal{L}$, such such triangles may be deleted without affecting partitionability ∎

This Lemma concludes the proof of Theorem 2 since we have reduced an arbitrary partition problem with $\Delta(G) \leq 7$ to a partition problem $(G, \mathcal{L})$ in which every edge of G is covered by exactly two triangles in $\mathcal{L}$. Let us summarize the algorithm.

Polynomial algorithm for PMC(7). *This produces a partition into maximal cliques if there is one.*

$\mathcal{P}$ denotes the partition being constructed.
Excise_and_list(S), where $S \subseteq \mathcal{L}$, means

$\mathcal{P} \leftarrow \mathcal{P} \cup S$,
$G \leftarrow$ subgraph of G induced by edges not covered by S,
$\mathcal{L} \leftarrow \mathcal{L}|G\ (= \{C \in \mathcal{L}; C \subseteq G\})$

The Algorithm.

$\mathcal{L} \leftarrow \{$maximal cliques of $G\}$,
$\mathcal{P} \leftarrow \emptyset$

(Part 1—Reduce G locally)
While there is a reduction by a small subgraph do

if type (Rl), then stop and reject,
if type (R2) or (R3), then Excise_and_List $\mathcal{L}$-partition of subgraph,

end while

$\mathcal{L} \leftarrow \mathcal{L} \setminus \{$triangles of triple edges$\}$,

(now every edge of G is covered by exactly two cliques of $\mathcal{L}$)

(Part 2—find partition for G when each edge is contained in exactly two edges of $\mathcal{L}$) while G is not empty, do

Excise_and_list($\{C\}$), for any $C \in \mathcal{L}$
while there is an $\mathcal{L}$-exposed or edge not covered by $\mathcal{L}$, do
if uncovered edge, stop and reject,
else Excise_and_list(any clique with an $\mathcal{L}$-exposed edge)
end (inner while)

end (outer while).

Return partition $\mathcal{P}$ and accept
end (of Algorithm).

The last part of the algorithm works because, as observed above, if any edge of G belongs to exactly two cliques of $\mathcal{L}$, and $(G, \mathcal{L})$ is partitionable, then any clique of $\mathcal{L}$ belongs to a partition, so we may excise and list any clique. After that, there will be $\mathcal{L}$-exposed or uncovered cliques, and any exposed cliques must belong to any partition of the remaining problem. When we run out of exposed cliques, if there are no uncovered cliques, either G is empty, and we are done, or $(G, \mathcal{L})$ again has the property that each edge is covered by exactly two cliques of $\mathcal{L}$, and we may repeat the process of excising any clique.

Remark. *In [PSW82] the problem of partitioning a graph into maximal cliques is posed as the problem of finding such a partition which is of minimum cardinality. To find such a partition, modify our algorithm as follows.*

- When reducing by a partition independent subgraph, always choose the smallest partition.
- Once every edge is contained in exactly two edges of $\mathcal{L}$, proceed by component by component, and choose the smaller of the two partitions of each component.

Other steps involve putting cliques into the partition which must be there anyhow, or removing triangles of triple edges from triangles and tori. Since maximal cliques contained in triangles and tori are all triangles, they have only one size of partition anyway.

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