Sums involving multinomial coefficients

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Abstract. Let $g_k(n) = \sum_{\underline{v}\in C_k(n)} \binom{n}{\underline{v}} 2^{v_1v_2+v_2v_3+v_3v_4+\cdots+v_{k-1}v_k}$ where $C_k(n)$ denotes the set of k-compositions of n. We show that

i) $g_k(n+p-1) \equiv g_k(n) \pmod{p}$ for all $k, n \geq 1$, prime p;
ii) $g_k(n)$ is a polynomial in k of degree n for $k \geq n+1$;

and, moreover, that these properties hold for wider classes of functions which are sums involving multinomial coefficients.

1 Introduction

Let N denote the set of non-negative integers and let k, n be positive integers. If $\underline{v} = (v_1, v_2, \ldots, v_k) \in N^k$ satisfies the condition $\sum_{i=1}^{k} v_i = n$, then $\underline{v}$ is called a k-composition of n. The set of all such k-compositions of n will be denoted by $C_k(n)$. For $\underline{v} \in C_k(n)$, the multinomial coefficient $\binom{n}{\underline{v}}$ is defined to be $\frac{n!}{v_1!v_2!\ldots v_k!}$. The function $g_k\colon N - \{0\} \to N$ defined by

$$g_k(n) = \sum_{\underline{v}\in C_k(n)} \binom{n}{\underline{v}} 2^{v_1v_2+v_2v_3+v_3v_4+\cdots+v_{k-1}v_k} \tag{1}$$

occurs in the enumeration of graded graphs [5,6]. The present authors were extending the table of computed values of $g_k(n)$ to $1 \leq k, n \leq 9$ (see Table 1) when they noticed that the resulting table appeared to have some interesting number-theoretic properties. The most startling pattern appears in the final digits of the numbers $g_k(5)$ and $g_k(9)$. In fact, the authors initially assumed that their computer program was in error. Closer inspection proved the observation valid (it follows from feature 3 below) and produced the following list of features.

feature 1: $g_2(n+1) = 2g_3(n)$
feature 2: $g_k(p) \equiv k \pmod{p}$ for $p = 2, 3, 5, 7$
feature 3: $g_k(m+1) \equiv g_k(m) \pmod{2}$
$g_k(m+2) \equiv g_k(m) \pmod{3}$
$g_k(m+4) \equiv g_k(m) \pmod{5}$
$g_k(m+6) \equiv g_k(m) \pmod{7}$

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feature 4: This concerned the apparent growth rate of the various columns. It is clear that $g_k(1) = k$ for all $k \geq 1$ so that the first column grows in a linear fashion. Using the standard techniques of differences it is easy to show that

$g_k(2) = k^2 + 2k - 2,\ k \geq 1$
$g_k(3) = k^3 + 6k^2 + 6k - 18,\ k \geq 2$
$g_k(4) = k^4 + 12k^3 + 48k^2 + 50k - 266,\ k \geq 3$

The question arises: Are these features true in general? We were able quickly to dispose of the validity of features 1 and 2 and include the proofs at the end of this section.

	$g_k(n)$								
$k \backslash n$	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	2	6	26	162	1442	18306	330626	8488962	309465602
3	3	13	81	721	9153	165313	4244481	154732801	8005686273
4	4	22	166	1726	24814	494902	13729846	531077086	28697950174
5	5	33	287	3309	50975	1058493	29885567	1156711869	61815727295
6	6	46	450	5650	91866	1957066	55363650	2109599650	109773407466
7	7	61	661	8953	152917	3334921	94354981	3528929353	177999003157
8	8	78	926	13446	240758	5381118	152654846	5615217126	274588808678
9	9	97	1251	19381	363339	8337037	238002291	8643818581	410796186939

Table 1

The main purpose of this paper is to prove the conjectures suggested in features 3 and 4 — namely

Conjecture 1. $g_k(n+p-1) \equiv g_k(n) \pmod{p}$ *for all* $k, n \geq 1$, *prime* p.

Conjecture 2. $g_k(n)$ *is a polynomial in* k *of degree* n *for* $k \geq n+1$.

It will be seen that there is a more general setting in which to state and prove the theorems which have as corollaries the proofs of the conjectures and the Dickson-Glaisher results on multinomial sums [1,2,3]. Read [9] and Wright [10] considered the polynomial character of a related function

$$m_k(n) = \sum_{\underline{v} \in C_k(n)} \binom{n}{\underline{v}} 2^s \qquad \text{where} \quad s = \sum_{i<j} v_i v_j$$

Their result also follows as a special case of our Theorem 2. Moreover it will be seen that $m_k(n)$ satisfies the congruence $m_k(n+p-1) \equiv m_k(n) \pmod{p}$.

We now return to the disposition of features 1 and 2.

Proposition 1. *For all* $n \geq 1, g_2(n+1) = 2g_3(n)$.

Proof:

$$\begin{aligned}
g_3(n) &= \sum_{0 \leq i, j \leq n} \binom{n}{i, j, n-i-j} 2^{ij + j(n-i-j)} \\
&= \sum_{0 \leq i, j \leq n} \binom{n}{i, j, n-i-j} 2^{j(n-j)} \\
&= \sum_{j=0}^{n} \binom{n}{j} 2^{j(n-j)} \left\{ \sum_{i=0}^{n-j} \binom{n-j}{i} \right\} \\
&= \sum_{j=0}^{n} \binom{n}{j} 2^{j(n-j)} 2^{n-j} \\
&= \sum_{j=0}^{n} \binom{n}{j} 2^{(j+1)(n-j)} \\
&= \sum_{j=0}^{n} \binom{n}{j} 2^{j(n-j+1)} \qquad \text{by replacing } j \text{ with } n-j.
\end{aligned}$$

Now

$$\begin{aligned}
g_2(n+1) &= \sum_{i=0}^{n+1} \binom{n+1}{i} 2^{i(n+1-i)} \\
&= \sum_{i=0}^{n} \binom{n}{i} 2^{i(n+1-i)} + \sum_{i=1}^{n+1} \binom{n}{i-1} 2^{i(n+1-i)} \\
&= \sum_{i=0}^{n} \binom{n}{i} 2^{i(n+1-i)} + \sum_{i=0}^{n} \binom{n}{i} 2^{(i+1)(n-i)} \\
&= 2g_3(n).
\end{aligned}$$

∎

Proposition 2. *For all primes* p *and all integers* $k \geq 1, g_k(p) \equiv k \pmod p$.

Proof: Note that $p \mid \binom{p}{\underline{v}}$ unless one of the v_i's equals p (and therefore the rest equal 0) since otherwise p divides the numerator but not the denominator. Thus

$$g_k(p) = \sum_{\text{all } v_i < p} \binom{p}{\underline{v}} 2^{v_1 v_2 + \cdots + v_{k-1} v_k} + k \equiv k \pmod{p}.$$

∎

2 Conjecture 1

Let p be a fixed prime ≥ 2. In order to obtain the proper setting for our theorem we make the following

Definition 1. *Let* $\underline{v}, \underline{w} \in N^k$. *We write* $\underline{v} \equiv \underline{w} \pmod{p-1}$ *if and only if* $v_i \equiv w_i \pmod{p-1}$ *for* $1 \leq i \leq k$.

It is easy to see that $\equiv$ mod $p-1$ is an equivalence relation.

Definition 2. *Let* $f\colon N^k \to Z$. f *is said to have property* $P(k)$ *if* $\underline{v} \equiv \underline{w} \pmod{p-1} \Rightarrow f(\underline{v}) \equiv f(\underline{w}) \pmod{p}$.

It should be evident that f has property $P(k) \iff \exists \bar{f}\colon N^k/\equiv \to Z_p$ such that the following diagram commutes.

$$\begin{array}{ccc} N^k & \xrightarrow{f} & Z \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ N^k/\equiv & \xrightarrow[\bar{f}]{} & Z_p \end{array}$$

A large class of functions satisfy property $P(k)$. If g is a polynomial in k variables and a is a positive integer, then $f(\underline{v}) = a^{g(\underline{v})}$ has property $P(k)$ for any prime $p \neq a$. To see this, note that $\underline{v} \equiv \underline{w} \pmod{p-1}$ implies that $g(\underline{v}) \equiv g(\underline{w}) \pmod{p-1}$. Hence using Fermat's theorem [4], $a^{p-1} \equiv 1 \pmod{p}$, we obtain $f(\underline{v}) \equiv f(\underline{w}) \pmod{p}$. The particular choices $g_1(\underline{v}) = \sum_i v_i v_{i+1}$ and $g_2(\underline{v}) = \sum_{i<j} v_i v_j$ give rise to the functions $g_k(n)$ and $m_k(n)$ respectively.

We begin with two well known lemmas [7,2] that form the basis of our main results. (We believe our proof of Lemma 2 is rather clearer than Glaisher's.)

Lemma 1. *(Lucas)[7]* $\binom{p-1}{j} \equiv (-1)^j \pmod{p}$ *for* p *prime,* $0 \leq j \leq p-1$.

Proof: If $j = 0$, the result is trivial.
So we suppose $j \geq 1$. $(p-1)(p-2)\ldots(p-j) \equiv (-1)^j j! \pmod{p}$. Since $j!$ is a factor of the left side of the equation and since $\gcd(p, j!) = 1$ we can divide both sides by $j!$ to obtain the required result. ∎

Lemma 2. *(Glaisher)[3] Let* $0 \leq w \leq p-2$. *If* $m \equiv n \pmod{p-1}$ *then*

$$\sum_{\substack{i \equiv w \pmod{p-1} \\ 0 \leq i \leq m}} \binom{m}{i} \equiv \sum_{\substack{i \equiv w \pmod{p-1} \\ 0 \leq i \leq n}} \binom{n}{i} \pmod{p}.$$

Proof: It is sufficient to prove the result when $m = n + p - 1$. If $p = 2$ then the left side of the equation equals 2^{n+1} and the right side equals 2^n so the result is established in this case.

Suppose then that $p > 2$.

$$\sum_{\substack{i \equiv w \pmod{p-1} \\ 0 \le i \le n+p-1}} \binom{n+p-1}{i} = \sum_{\substack{i \equiv w \pmod{p-1} \\ 0 \le i \le n+p-1}} \sum_{j=0}^{p-1} \binom{p-1}{j} \binom{n}{i-j}$$

By Lemma 1, the right side is congruent $\pmod p$ to

$$\sum_{\substack{i \equiv w \pmod{p-1} \\ 0 \le i \le n+p-1}} \sum_{j=0}^{p-1} (-1)^j \binom{n}{i-j}$$

Note that $\binom{n}{i}$ is equal to 0 for $i > n$. Considering $j = 0$ and $j > 0$ separately we obtain that this sum equals

$$\sum_{\substack{i \equiv w \pmod{p-1} \\ 0 \le i \le n}} \binom{n}{i} + \sum_{\substack{i \equiv w \pmod{p-1} \\ 0 \le i \le n+p-1}} \sum_{j=1}^{p-1} (-1)^j \binom{n}{i-j}$$

Every integer in interval $[0, n]$ occurs exactly once as $i - j$ in the double summation. Thus, setting $u = i - j$, the second term is equal to $(-1)^w \sum_{u=0}^{n} (-1)^u \binom{n}{u} = 0$, as required. ∎

Lemma 3. *Suppose* $H = \{0, 1, \ldots, p-2\}^k$ *and let* $\underline{w} \in H$. *If* $m \equiv n \pmod{p-1}$ *then*

$$\sum_{\substack{\underline{v} \equiv \underline{w} \pmod{p-1} \\ \underline{v} \in C_k(m)}} \binom{m}{\underline{v}} \equiv \sum_{\substack{\underline{v} \equiv \underline{w} \pmod{p-1} \\ \underline{v} \in C_k(n)}} \binom{n}{\underline{v}} \pmod p$$

Proof: Again, it is sufficient to prove the result for $m = n + p - 1$. If $p = 2$ then the left side equals k^{n+1} whereas the right side equals k^n, and clearly $k^{n+1} \equiv k^n \pmod 2$.

So we suppose that $p > 2$ in what follows. We will prove the result by induction on k. If $\underline{v} = (v_1, \ldots, v_{k+1})$ we let $\underline{v}' = (v_2, \ldots, v_{k+1})$.

The case $k = 2$ is Lemma 2 and serves as the basis of the induction. Assume the result for a $k \ge 2$. Then

$$\sum_{\substack{\underline{v} \equiv \underline{w} \pmod{p-1} \\ \underline{v} \in C_{k+1}(n+p-1)}} \binom{n+p-1}{\underline{v}}$$

$$= \sum_{v_1 \equiv w_1 \pmod{p-1}} \binom{n+p-1}{v_1} \left\{ \sum_{\substack{\underline{v}' \equiv \underline{w}' \pmod{p-1} \\ \underline{v}' \in C_k(n+p-1-v_1)}} \binom{n+p-1-\underline{v}_1}{\underline{v}'} \right\}$$

Now by the inductive hypothesis the inner term is constant $\pmod{p}$ for each $v_1 \equiv w_1$ (using $n - v_1$ instead of n). Thus the right side is congruent $\pmod{p}$ to

$$\sum_{v_1 \equiv w_1 \pmod{p-1}} \binom{n+p-1}{v_1} \sum_{\substack{\underline{v}' \equiv \underline{w}' \pmod{p-1} \\ \underline{v}' \in C_k(n-v_1)}} \binom{n-v_1}{\underline{v}'}$$

By Lemma 2 we obtain that this expression equals

$$\sum_{v_1 \equiv w_1 \pmod{p-1}} \binom{n}{v_1} \sum_{\substack{\underline{v}' \equiv \underline{w}' \pmod{p-1} \\ \underline{v}' \in C_k(n-v_1)}} \binom{n-v_1}{\underline{v}'} \pmod{p}$$

which equals $\displaystyle\sum_{\substack{\underline{v} \equiv \underline{w} \pmod{p-1} \\ \underline{v} \in C_{k+1}(n)}} \binom{n}{\underline{v}}$, as required. ∎

Theorem 1. *Suppose $f: N^k \to Z$ has property $P(k)$ for a fixed prime $p \geq 2$. Let $F_k(n) = \sum_{\underline{v} \in C_k(n)} \binom{n}{\underline{v}} f(\underline{v})$. Then F_k has property $P(k)$; that is, $F_k(n+p-1) \equiv F_k(n) \pmod{p}$ for all $n \geq 1$.*

Proof: As before, let $H = \{0, 1, \ldots, p-2\}^k$. Then

$$F_k(n+p-1) = \sum_{\underline{v} \in C_k(n+p-1)} \binom{n+p-1}{\underline{v}} f(\underline{v})$$

$$= \sum_{\underline{w} \in H} \sum_{\substack{\underline{v} \equiv \underline{w} \\ \underline{v} \in C_k(n+p-1)}} \binom{n+p-1}{\underline{v}} f(\underline{v})$$

By property $P(k)$, $f(\underline{v}) \equiv f(\underline{w}) \pmod{p}$ so the last sum is

$$\equiv \sum_{\underline{w} \in H} f(\underline{w}) \sum_{\underline{v} \in C_k(n+p-1)} \binom{n+p-1}{\underline{v}} \pmod{p}$$

$$\equiv \sum_{\underline{w} \in H} f(\underline{w}) \sum_{\underline{v} \in C_k(n)} \binom{n}{\underline{v}} \pmod{p}$$

$$\equiv F_k(n) \pmod{p}.$$

∎

Corollary 1. *$g_k(n+p-1) \equiv g_k(n) \pmod{p}$ for all $k, n \geq 1$, p prime.*

Proof: If p is a prime $\neq 2$, then Theorem 1 can be applied directly since we have already remarked that the summand $2^{v_1 v_2 + v_2 v_3 + \cdots + v_{k-1} v_k}$ has property $P(k)$.

So we can suppose $p = 2$. Let $q(\underline{v}) = v_1 v_2 + v_2 v_3 + \cdots + v_{k-1} v_k$. Since $2^{q(\underline{v})} \equiv 0 \pmod 2$ if $q(\underline{v}) > 0$, it is sufficient to show that

$$\sum_{\substack{\underline{v}\in C_k(n)\\ q(\underline{v})=0}} \binom{n}{\underline{v}} \equiv k \pmod 2 \quad n \geq 1. \tag{2}$$

The proof proceeds by induction. The cases $k = 2$ and $k = 3$ are easily proved.

Inductive step: If $\underline{v} \in C_k(n)$ and $q(\underline{v}) = 0$ we can express $\underline{v}$ either in the form $\underline{v} = (v_1, 0, \underline{v}'')$; $v_1 \geq 1, \underline{v}'' \in C_{k-2}(n - v_1)$ or $\underline{v} = (0, \underline{v}')$; $\underline{v}' \in C_{k-1}(n)$ and we have $q(\underline{v}'') = 0, q(\underline{v}') = 0$. Then

$$\sum_{\substack{\underline{v}\in C_k(n)\\ q(\underline{v})=0}} \binom{n}{\underline{v}} = \sum_{v_1=1}^{n} \binom{n}{v_1} \sum_{\substack{\underline{v}''\in C_{k-2}(n-v_1)\\ q(\underline{v}'')=0}} \binom{n}{\underline{v}''} + \sum_{\substack{\underline{v}'\in C_{k-1}(n)\\ q(\underline{v}')=0}} \binom{n}{\underline{v}'}$$

By the inductive hypothesis, we obtain

$$\begin{aligned}\sum_{\substack{\underline{v}\in C_k(n)\\ q(\underline{v})=0}} \binom{n}{\underline{v}} &\equiv \sum_{v_1=1}^{n-1} \binom{n}{v_1} (k-2) + 1 + (k-1) \pmod 2\\ &\equiv (2^n - 2)(k-2) + k\\ &\equiv k \pmod 2\end{aligned}$$

and the inductive step is verified. ∎

Corollary 2. *(Dickson-Glaisher)[1,2,3]: Let* $m \equiv n \pmod{p-1}$, $1 \leq m \leq p-1$, *and let* $\underline{w} \in \{0, 1, \ldots, p-2\}^k$.

$$\sum_{\substack{\underline{v}\in C_k(n)\\ \underline{v}\equiv\underline{w} \pmod{p-1}}} \binom{n}{\underline{v}} \equiv \begin{cases} \binom{m}{\underline{w}} & \pmod p \ \textit{if} \sum w_i = m\\ 0 & \pmod p \ \textit{if} \sum w_i \neq m \end{cases}$$

Proof: Let $f(\underline{v}) = \begin{cases} 1 & \text{if } \underline{v} \equiv \underline{w} \pmod{p-1};\\ 0 & \text{otherwise.}\end{cases}$

From Theorem 1 we obtain $F_k(n) \equiv F_k(m) \pmod p$, but

$$F_k(m) = \sum_{\substack{\underline{v}\in C_k(m)\\ \underline{v}\equiv\underline{w} \pmod{p-1}}} \binom{m}{\underline{v}} = \begin{cases} \binom{m}{\underline{w}} & \text{if } \underline{w} \in C_k(m),\\ 0 & \text{otherwise;}\end{cases}$$

which gives the result. ∎

3 Conjecture 2

In order for us to settle the conjecture it seems necessary to introduce the following definitions and to prove some introductory results.

Definition 3. *Let $C(n) = \cup_{k\geq 1} C_k(n)$ be the set of all compositions of n. For $\underline{v} \in C(n)$ we define $\ell(\underline{v})$ to be the number of components of $\underline{v}$ and $m(\underline{v})$ the number of zero components of $\underline{v}$. Also we define $\rho: C(n) \to C(n)$, a compression map as follows: if $\underline{v} \in C(n)$, then $\rho(\underline{v})$ is the vector obtained by compressing consecutive zeroes in $\underline{v}$.*

For example if $\underline{v} = (0,0,2,0,3,0,0,0,4)$ then $\rho(\underline{v}) = (0,2,0,3,0,4)$.

Proposition 3. *Let $\mathcal{A}(n) = \{\underline{a} \in C(n): \rho(\underline{a}) = \underline{a}\}$. Further define $\sim$ on $C(n)$ by $\underline{v} \sim \underline{w} \Leftrightarrow \rho(\underline{v}) = \rho(\underline{w})$. Then*

i) *$\mathcal{A}$ is finite: in fact $|\mathcal{A}(n)| = 4 \cdot 3^{n-1}$;*
ii) *$\sim$ is an equivalence relation on $C(n)$;*
iii) *$|C(n)/\sim| = |\mathcal{A}(n)|$.*

Proof:

i) Let $\underline{a} \in \mathcal{A}(n)$. Since $\rho(\underline{a}) = \underline{a}$, $\underline{a}$ has no consecutive zeroes. Thus $\underline{a}$ corresponds to a non-empty composition of n into $t(= \ell(\underline{a}) - m(\underline{a}))$ parts together with the addition of some single 0's between the non-zero components, which can be placed in 2^{t+1} ways. Thus

$$|\mathcal{A}(n)| = \sum_{t=1}^{n} \binom{n-1}{t-1} 2^{t+1} = 2^2 \sum_{t=1}^{n} \binom{n-1}{t-1} 2^{t-1} = 4 \cdot 3^{n-1}.$$

ii) It is obvious that $\sim$ is an equivalence relation on $C(n)$.
iii) Since $\rho \circ \rho = \rho$ we see that $\rho: C(n) \to \mathcal{A}(n)$ is surjective and $\sim$ is the relation $\ker \rho$ so the result follows [8]. ∎

Proposition 4. *Let $a \in \mathcal{A}(n)$; then*

a) *If $m(\underline{a}) = 0$ then $\rho^{-1}(\underline{a}) \cap C_k(n) = \begin{cases} \{\underline{a}\} & \text{if } k = \ell(\underline{a}) \\ 0 & \text{otherwise} \end{cases}$*
b) *If $m(\underline{a}) \geq 1$ then $\left|\rho^{-1}(a) \cap C_k(n)\right| = \binom{k-\ell(\underline{a})+m(\underline{a})-1}{m(\underline{a})-1}$*

Proof:

a) is obvious.
b) $\underline{v} \in \rho^{-1}(\underline{a}) \cap C_k(n) \Leftrightarrow \rho(\underline{v}) = \underline{a}, \underline{v} \in C_k(n)$. Thus $\left|\rho^{-1}(\underline{a}) \cap C_k(n)\right|$ is the number of ways we can "pad" $\underline{a}$ with $k - \ell(\underline{a})$ zeroes at places where there is already a single zero. This is the number equivalent to the number of ways of distributing r identical objects in s identical boxes which is $\binom{r+s-1}{s-1}$ [8]. ∎

Definition 4. *Let $h: C(n) \to Z$. h is said to have property Q provided that $\underline{v} \sim \underline{w} \Rightarrow h(\underline{v}) = h(\underline{w})$*

We see that h has property $Q \Longleftrightarrow \exists \bar{h}: C(n)/\sim \longrightarrow Z$ such that the following diagram commutes.

$$\begin{array}{ccc} C(n) & \xrightarrow{h} & Z \\ \pi\big\downarrow & \nearrow \bar{h} & \\ C(n)/\sim & & \end{array}$$

Examples of such functions arise from Section 1. If $h_1: C(n) \to Z$ is obtained by setting $h_1(\underline{v}) = \binom{n}{\underline{v}}$ then h_1 has property Q. Similarly $h_2: C(n) \to Z$ obtained from $h_2(\underline{v}) = 2^{v_1 v_2 + \cdots + v_{k-1} v_k}$, and $h_3: C(n) \to Z$, where $h_3(\underline{v}) = a^{s(\underline{v})}$ in which s is any symmetric polynomial, also have property Q. Moreover if h_1, h_2 have property Q so does $h_1 h_2$. We thus see that the proof of Conjecture 2 will follow immediately from:

Theorum 2. *Let $h: C(n) \to Z$ have property Q and let $H_n(k) = \sum_{\underline{v} \in C_k(n)} h(\underline{v})$. Then $H_n(k)$ is a polynomial of degree n in k for $k \geq n+1$.*

Proof: Since h has property Q we can write

$$H_n(k) = \sum_{\underline{v} \in C_k(n)} h(\underline{v}) = \sum_{\underline{a} \in A(n)} \sum_{\substack{\rho(\underline{v}) = \underline{a} \\ \underline{v} \in C_k(n)}} h(\underline{v}) = \sum_{\underline{a} \in A(n)} h(\underline{a}) \left| \rho^{-1}(a) \cap C_k(n) \right|$$

Using Proposition 4

$$= \sum_{\substack{\underline{a} \in A(n) \\ m(\underline{a}) \geq 1}} h(\underline{a}) \binom{k - \ell(\underline{a}) + m(\underline{a}) - 1}{m(\underline{a}) - 1} + \sum_{\substack{\underline{a} \in A(n) \\ m(\underline{a}) = 0}} h(\underline{a}).$$

The first term is a finite (by prop. 3) weighted sum of binomial coefficients which are polynomials of degree at most $m(\underline{a}) - 1 \leq n$. In fact it is a polynomial of degree exactly n, since $\underline{a} = (0, 1, 0, 1, \ldots, 1, 0)$ will have $m(\underline{a}) - 1 = n$. The second term vanishes when $k \geq n+1$ since $m(\underline{a}) \geq 1$ in that case. ∎

Close examination of the data presented suggests that, in fact, $H_n(k)$ is a polynomial of degree n in k for $k \geq n-1$. We have been unable to settle this outstanding conjecture.

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