A Note On The Size Ramsey Number For Stars[1]

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Abstract. In this paper, we prove that the size ramsey number

$$\hat{r}(a_1K_{1,n_1}, a_2K_{1,n_2}, \dots a_\ell K_{1,n_\ell}) = \left[\sum_{i=1}^{\ell}(a_i-1)+1\right]\left[\sum_{i=1}^{\ell}(n_i-1)+1\right].$$

For the most part our notation will conform with that used in [1]. All graphs considered will be finite, undirected and without loops and multiple edges. Let $G, G_1, G_2, \dots, G_\ell$ be graphs. The edges of G are colored in any fashion with ℓ colors. If for some i, $1 \le i \le \ell$ the ith colored subgraph $G[E_i]$ contains G_i as a subgraph. i.e. $G[E] \supseteq G_i$, then G will be said to arrow $(G_1, G_2, \dots, G_\ell)$ and will be written $G \to (G_1, G_2, \dots, G_\ell)$. Let $\mathcal{D} = \{G | G \to (G_1, G_2, \dots, G_\ell)\}$. The size ramsey number of $\{G_1, G_2, \dots, G_\ell\}$, denoted $\hat{r}(G_1, G_2, \dots, G_\ell)$, is defined as $\hat{r}(G_1, G_2, \dots, G_\ell) = \min_{G \in \mathcal{D}} |E(G)|$.

In 1967, Kalbfleisch [4] first raised the concept of the size ramsey number, and then it was considered [2] and [3]. There is an open problem in [2]: 'Show that $\hat{r}(mK_{1,k}, nK_{1,\ell}) = (k+\ell-1)(m+n-1)$ for $k \neq \ell$'. This problem is a corollary of the main result of this paper.

Theorem. $\hat{r}(a_1K_{1,n_1}, a_2K_{1,n_2}, \dots, a_\ell K_{1,n_\ell}) = [\sum_{i=1}^{\ell}(a_i-1)+1] \cdot [\sum_{i=1}^{\ell}(n_i-1)+1]$, *where* a_i, n_i $(1 \le i \le \ell)$ *are positive integers.*

Proof: Since $[\sum_{i=1}^{\ell}(a_i-1)+1]K_{1,\sum_{i=1}^{\ell}(n_i-1)+1} \to (a_1K_{1,n_1}, a_2K_{1,n_2}, \dots, a_\ell K_{1,n_\ell})$, $\hat{r}(a_1K_{1,n_1}, a_2K_{1,n_2}, \dots, a_\ell K_{1,n_\ell}) \le |E([\sum_{i=1}^{\ell}(a_i-1)+1] K_{1,\sum_{i=1}^{\ell}(n_i-1)+1}| = [\sum_{i=1}^{\ell}(a_i-1)+1]\ [\sum_{i=1}^{\ell}(n_i-1)+1]$. So we only need to prove that:

$$\hat{r}(a_1K_{1,n_1}, a_2K_{1,n_2}, \dots, a_\ell K_{1,n_\ell}) \ge \left[\sum_{i=1}^{\ell}(a_i-1)+1\right]\left[\sum_{i=1}^{\ell}(n_i-1)+1\right] \quad \text{(A)}$$

[1] Project supported by NSFC

Now, using induction on $a = \sum_{i=1}^{\ell}(a_i - 1)$ and $n = \sum_{i=1}^{\ell}(n_i - 1)$, we have:

(1) When $a = 0$, i.e. $a_1 = a_2 = \cdots = a_\ell = 1$, for any G with $|E(G)| \le n$, there is an ℓ-edge coloring of G such that $|E(G[E_i])| \le n_i - 1$, $i = 1, 2, \ldots, \ell$, where $G[E_i]$ is the ith colored subgraph in G. Then $G[E_i] \not\supseteq K_{1,n_i}$, $i = 1, 2, \ldots, \ell$. Hence $G \not\to (K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_\ell})$. i.e. (A) is true in this case.

In the following, we assume that when $0 \le a < a_0 = \sum_{i=1}^{\ell}(a_i^0 - 1)$, (A) is true. We will prove that when $a = a_0$, (A) is also true. Since $a = a_0 > 0$, without loss of generality, $a_1^0 > 1$.

(2) When $n = 0$, i.e. $n_1 = n_2 = \cdots = n_\ell = 1$, for any G with $|E(G)| \le a_0$, there is an ℓ-edge coloring such that $|E(G[E_i])| \le a_i^0 - 1$, $i = 1, 2, \ldots, \ell$. Then $G[E_i] \not\supseteq a_i^0 K_{1,1}$ $i = 1, 2, \ldots, \ell$. Hence $G \not\to (a_1^0 K_{1,1}, a_2^0 K_{1,1}, \ldots, a_\ell^0 K_{1,1})$. So (A) is true in this case.

In the following, we assume that when $0 \le n < n_0$, (A) is true.

(3) When $n = n_0 = \sum_{i=1}^{\ell}(n_i - 1)$, without loss of generality, we can assume that $n_1^0 > 1$. For any G with $|E(G)| \le (a_0 + 1)(n_0 + 1) - 1$, we divide the problem into three cases to prove (A).

Case 1. $\Delta(G) \ge n_0 + 1$.

There is $v_0 \in V(G)$ such that $d(v_0) = \Delta(G) \ge n_0 + 1$. Since $|E(G - v_0)| = |E(G)| - d(v_0) < (a_0 + 1)(n_0 + 1) - (n_0 + 1) = a_0(n_0 + 1) = \hat{r}((a_1^0 - 1)K_{1,n_1^0}, a_2^0 K_{1,n_2^0}, \ldots, a_\ell^0 K_{1,n_\ell^0})$ by the induction hypothesis, there is an ℓ-edge coloring of $G - v_0$ such that the 1st colored subgraph does not contain $(a_1^0 - 1)K_{1,n_1^0}$ and the ith, $2 \le i \le \ell$, colored subgraph does not contain $a_i^0 K_{1,n_i^0}$. Then the edges which are incident with v_0 are colored in the 1st color. Hence we get an ℓ-edge coloring of G such that $G[E_i] \not\supseteq a_i^0 K_{1,n_i^0}$, $i = 1, 2, \ldots, \ell$. So $G \not\to (a_1^0 K_{1,n_1^0}, a_2^0 K_{1,n_2^0}, \ldots, a_\ell^0 K_{1,n_\ell^0})$;

Case 2. $\Delta(G) \le n_0$ *and there is a component, say* C_1, *with* $|E(C_1)| \ge a_0(n_0 + 1)$.

a) When $C_1 \supseteq M_1 = (a_0+1)K_{1,1}$, then $|E(C_1 - (a_0+1)K_{1,1})| = |E(C_1)| - (a_0 + 1) \le |E(G)| - (a_0 + 1) < (a_0 + 1)(n_0 + 1) - (a_0 + 1) = (a_0 + 1)n_0 = [\sum_{i=1}^{\ell}(a_i^0 - 1) + 1][n_1^0 - 2 + \sum_{i=2}^{\ell}(n_i^0 - 1) + 1] = \hat{r}(a_1^0 K_{1,n_1^0-1}, a_2^0 K_{1,n_2^0}, \ldots, a_\ell^0 K_{1,n_\ell^0})$ by induction hypothesis. Hence there is an ℓ-edge coloring of $C_1 - (a_0+1)K_{1,1}$ such that the 1st colored subgraph does not contain $a_1^0 K_{1,n_1^0-1}$ and the ith, $2 \le i \le \ell$, colored subgraph does not contain $a_i^0 K_{1,n_i^0}$. Then the edges of M_1 are colored in 1st color. Thus we get an ℓ-edge coloring of C_1 such that $C_1[E_i] \not\supseteq a_i^0 K_{1,n_i^0}$, $1 \le i \le \ell$, where $C_1[E_i]$ is the ith colored subgraph in C_1.

b) When $C_1 \not\supseteq (a_0 + 1)K_{1,1}$, then we have a maximum matching of C_1 $M = \{u_i v_i, i = 1, 2, \ldots, \alpha'\}$, $\alpha' \le a_0$. Let $U = \{u_i, i = 1, 2, \ldots, \alpha'\}$,

$V = \{v_i, i = 1, 2, \ldots, \alpha'\}$, $W = V(C_1)\backslash\{U \cup V\}$ and $W_0 = \{v | d_{c_1}(v) = \Delta(C_1), v \in W\}$. Thus $E(C_1[W]) = \phi$. Now, let $C_0 = C_1 \cap [W \vee (U \cup V)]$, where the join $\vee$ is defined in page 58 of [1]. Thus for any $S \subseteq W_0 \subseteq W$, we have $|S|\Delta(C_1) = \sum_{V \in S} d_{c_1}(v) = \sum_{V \in S} d_{c_0}(v) = \sum_{V \in N_{c_0}(S)} d_{c_0}(v) \leq |N_{c_0}(S)|\Delta(C_0) \leq |N_{c_0}(S)|\Delta(C_1)$. Hence $|N_{c_0}(S)| \geq |S|$. By the Hall Theorem ([1] Theorem 5.2), C_0 contains a matching M_0 that saturates every vertex in W_0. i.e. $W_0 \subseteq V(C_0[M_0]) = V(C_1[M_0])$. Thus $V(G_1[M \cup M_0]) = W_0 \cup U \cup V \supseteq \{v | d_{c_1}(v) = \Delta(C_1), v \in V(C_1)\}$. Hence we have:

$$\chi'(C_1 - M \cup M_0) \leq 1 + \Delta(C_1 - M \cup M_0) \leq \Delta(C_1) \leq n_0.$$

So there is an n_0-edge coloring C of $C_1 - M \cup M_0$. Now the edges of $C_1 - M \cup M_0$ are recolored as follows. For $j = 1, 2, \ldots, \ell$, recolor all of the $\sum_{i=1}^{j-1}(n_i^0 - 1) + 1$th, $\sum_{i=1}^{j-1}(n_i^0 - 1)$ +2th,..., $\sum_{i=1}^{j}(n_i^0 - 1)$th color edges in C with jth color. Thus $(C_1 - M \cup M_0)[E_i] \not\supseteq K_{1,n_i^0}, i = 1, 2, \ldots, \ell$.

Note that $C_1[M \cup M_0]$ has $\alpha'(\leq a_0)$ components which are ismorphic to $K_{1,1}$ or $K_{1,2}$. Hence there is an ℓ-edge coloring of $C_1[M \cup M_0]$ such that the edges of each component are colored the same color and there are at most $a_i^0 - 1$ ith color components $i = 1, 2, \ldots, \ell$. Thus we have an ℓ-edge coloring of C_1 with $C_1[E_i] = (C_1 - M \cup M_0)[E_i] \cup (C_1[M \cup M_0])[E_i] \not\supseteq a_i K_{1,n_i^0}, i = 1, 2, \ldots, \ell$.

Combining a) and b), we have

$$C_1 \not\rightarrow (a_1^0 K_{1,n_1^0}, a_2^0 K_{1,n_2^0}, \ldots, a_\ell^0 K_{1,n_\ell^0}).$$

On the other hand, $|E(G - C_1)| = |E(G)| - |E(C_1)| < (a_0 + 1)(n_0 + 1) - a_0(n_0 + 1) = n_0 + 1 = \hat{r}(K_{1,n_1^0}, K_{2,n_2^0}, \ldots, K_{1,n_\ell^0})$ by the induction hypothesis. So

$$G - C_1 \not\rightarrow (K_{1,n_1^0}, K_{1,n_2^0}, \ldots, K_{1,n_\ell^0}).$$

Hence $G \not\rightarrow (a_1^0 K_{1,n_1^0}, a_2^0 K_{1,n_2^0}, \ldots, a_\ell^0 K_{1,n_\ell^0})$.

Case 3. *For every component C_j of G, $|E(C_j)| \leq a_0(n_0 + 1) - 1$, $j = 1, 2, \ldots, \omega$.*

In this case, for any j, $1 \leq j \leq \omega$, there is k_j with $0 \leq k_j \leq a_0 - 1$ such that

$$k_j(n_0 + 1) \leq |E(C_j)| < (k_j + 1)(n_0 + 1) \qquad \text{(B)}$$

By the induction hypothesis, for any $\{a_{1j}, a_{2j}, \ldots, a_{\ell j}\}$ with $\sum_{v=1}^{\ell}(a_{ij} - 1) = k_j$, we have:

$$\hat{r}(a_{1j}K_{1,n_1^0}, a_{2j}K_{1,n_2^0}, \ldots, a_{\ell j}K_{1,n_\ell^0}) = (k_j + 1)(n_0 + 1).$$

Thus by (B), we have

$$C_j \not\to (a_{1j}K_{1,n_1^0}, a_{2j}K_{1,n_2^0}, \ldots, a_{\ell j}K_{1,n_\ell^0}).$$

Hence $G = \cup_{j=1}^{\omega} C_j \not\to ([\sum_{j=1}^{\omega}(a_{1j}-1)+1]K_{1,n_1^0}, [\sum_{j=1}^{\omega}(a_{2j}-1)+1]K_{1,n_2^0}, \ldots, [\sum_{j=1}^{\omega}(a_{\ell j}-1)+1]K_{1,n_\ell^0})$.

On the other hand, by (B), we have $\sum_{i=1}^{\ell}[\sum_{j=1}^{\omega}(a_{ij}-1)+1] = \sum_{j=1}^{\omega}[\sum_{j=1}^{\ell}(a_{ij}-1)]+\ell = \sum_{j=1}^{\omega} k_j+\ell \leq [\sum_{j=1}^{\omega}|E(C_j)|/(n_0+1)]+\ell = [|E(G)|/(n_0+1)]+\ell \leq a_0+\ell = \sum_{i=1}^{\ell}(a_i^0-1)+\ell = \sum_{i=1}^{\ell} a_i^0$.

So we can choose positive integers a_{ij}, $1 \leq i \leq \ell$ and $1 \leq j \leq \omega$, such that $\sum_{i=1}^{\ell}(a_{ij}-1) = k_j$ and $\sum_{j=1}^{\omega}(a_{ij}-1)+1 = a_i^0$. Hence

$$G \not\to (a_1^0 K_{1,n_1^0}, a_2^0 K_{1,n_2^0}, \ldots, a_\ell^0 K_{1,n_\ell^0}).$$

Combining cases 1,2 and 3, (A) is true. The proof is complete.

References

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