On The Third Ramsey Numbers Of Graphs With Five Edges

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Abstract. With the help of a computer, the third Ramsey number is determined for each of the 25 graphs with five edges, five or more vertices and no trivial components.

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. For a graph G with vertex-set $V(G)$ and edge-set $E(G)$, we write $p(G) = |V(G)|$ and $q(G) = |E(G)|$. The mth Ramsey number of G is the least $n \in \mathcal{N}$ such that every assignment of m colours to $E(K_n)$ results in a monochromatic subgraph isomorphic to G. The second Ramsey numbers for graphs with at most seven edges and no isolated vertices were given in [2,8]. In [11] we gave the third Ramsey numbers for all graphs with at most four edges and no isolated vertices. Here we give the corresponding Ramsey numbers for all but one of the graphs with five edges. The exception arises when the graph in question has only four vertices: in this case we know only that $28 \leq r_3(K_4 - e) \leq 32$, these bounds having been established recently by Exoo [6].

There are 26 graphs D with $q(D) = 5$ and no isolated vertices: we label them $D_1, \ldots, D_{26}$ as shown in Fig.1, with $D_1 = K_4 - e$. For $1 \leq k \leq 26$, let $F_k(n)$ be the set of n-vertex graphs which contain no D_k, and let $t_k(n) = \max\{q(G): G \in F_k(n)\}$. If K_n is the edge-disjoint union of m graphs in $F_k(n)$ then $\binom{n}{2} = q(K_n) \leq mt_k(n)$ and so we have:

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Lemma 1.1. *If* $mt_k(n) < \binom{n}{2}$ *then* $r_m(D_k) \leq n$.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$t_1(n)$	1	3	4	6	9	12	16	20	25	30	36	42	49	56	64	72	81	90	100	110	121	132
$t_2(n)$	1	3	6	7	9	12	16	20	25	30	36	42	49	56	64	72	81	90	100	110	121	132
$t_3(n)$	1	3	6	6	7	9	12	13	16	18	21	24	27	30	33	36	39	42	46	50	52	56
$t_4(n)$	1	3	6	7	9	12	16	20	25	30	36	42	49	56	64	72	81	90	100	110	121	132
$t_5(n)$	1	3	6	7	9	12	16	20	25	30	36	42	49	56	64	72	81	90	100	110	121	132
$t_6(n)$	1	3	6	7	9	12	16	20	25	30	36	42	49	56	64	72	81	90	100	110	121	132
$t_7(n)$	1	3	6	10	10	11	13	16	20	20	21	23	26	30	30	31	33	36	40	40	41	43
$t_8(n)$	1	3	6	10	10	11	13	16	20	20	21	23	26	30	30	31	33	36	40	40	41	43
$t_9(n)$	1	3	6	10	10	11	13	16	20	20	21	23	26	30	30	31	33	36	40	40	41	43
$t_{10}(n)$	1	3	6	10	10	11	13	16	20	20	21	23	26	30	30	31	33	36	40	40	41	43
$t_{11}(n)$	1	3	6	10	10	11	13	16	20	20	21	23	26	30	30	31	33	36	40	40	41	43
$t_{12}(n)$	1	3	6	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46
$t_{13}(n)$	1	3	6	10	10	11	13	15	17	19	21	24	27	30	33	36	39	42	46	50	52	56
$t_{14}(n)$	1	3	6	10	10	12	16	20	25	30	36	42	49	56	64	72	81	90	100	110	121	132
$t_{15}(n)$	1	3	6	10	10	12	16	20	25	30	36	42	49	56	64	72	81	90	100	110	121	132
$t_{16}(n)$	1	3	6	10	15	15	15	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43
$t_{17}(n)$	1	3	6	10	15	15	15	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43
$t_{18}(n)$	1	3	6	10	15	15	16	16	16	16	18	19	21	22	24	25	27	28	30	31	33	34
$t_{19}(n)$	1	3	6	10	15	15	16	16	17	19	21	23	25	27	29	31	33	35	37	39	41	43
$t_{20}(n)$	1	3	6	10	15	15	16	16	17	17	18	18	19	21	22	24	25	27	28	30	31	33
$t_{21}(n)$	1	3	6	10	15	15	18	21	25	30	36	42	49	56	64	72	81	90	100	110	121	132
$t_{22}(n)$	1	3	6	10	15	21	21	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63
$t_{23}(n)$	1	3	6	10	15	21	21	21	21	21	21	23	25	27	29	31	33	35	37	39	41	43
$t_{24}(n)$	1	3	6	10	15	21	21	21	21	21	21	23	25	27	29	31	33	35	37	39	41	43
$t_{25}(n)$	1	3	6	10	15	21	28	28	28	28	30	33	36	39	42	45	48	51	54	57	60	63
$t_{26}(n)$	1	3	6	10	15	21	28	36	36	36	38	42	46	50	54	58	62	66	70	74	78	82

Table 1: The values of $t_k(n)$

Clearly $t_k(n) = \max\{q(G): G \in C_k(n)\}$ where $C_k(n)$ denotes the set of n-vertex graphs which are critical in our context: thus $G \in C_k(n)$ if and only if $G \in F_k(n)$ and the addition of any edge results in a subgraph isomorphic to D_k. In [9] we reported on a computer search for critical graphs having no 4-cycle. Here we use the same algorithm for determining $t_k(n)$ and $C_k(n)$ for $1 \leq k \leq 26$ and $2 \leq n \leq 23$. In practice it suffices to find only those graphs in $C_k(n)$ with sufficiently many edges, as explained in §2. The values of $t_k(n)$ are shown in Table 1. From this table and Lemma 1.1 we obtain upper bounds for $r_3(D_k)$ for all $k \in \{1, \ldots, 26\}$ except $k = 1, 2, 4, 5, 6, 14, 15, 21$. These are shown in Table 2. It was already known that $r_3(D_{12}) = 14$, from general results of Burr and Roberts [3] on Ramsey numbers of stars. The numbers $r_3(D_k)$ $(2 \leq k \leq 26)$ are determined in the next section and the results are given in Table 3. The notation is that of [[1], Chapter 1].

k	3	7	8	9	10	11	12	13	16
$r_3(D_k) \leq$	11	12	12	12	12	12	14	12	12
k	17	18	19	20	22	23	24	25	26
$r_3(D_k) \leq$	12	11	12	11	17	12	12	17	23

Table 2: Some upper bounds for $r_3(D_k)$.

k	1	2	3	4	5	6	7	8	9	10	11	12	13
$r_3(D_k)$	28-32	17	11	21	21	21	10	12	10	10	12	14	11
k	14	15	16	17	18	19	20	21	22	23	24	25	26
$r_3(D_k)$	17	17	11	11	11	11	10	17	14	12	12	15	18

Table 3: The Ramsey numbers $r_3(D_k)$ $(1 \leq k \leq 26)$.

2. The Ramsey numbers $r_3(D_k)$

For a given graph D_k, an m-colouring of K_n will be taken to mean an assignment of m colours to the edges of K_n in such a way that K_n has no monochromatic subgraph isomorphic to D_k. An m-colouring is specified by m monochromatic subgraphs $SC_1, \ldots, SC_m$ (with colours $1, \ldots, m$), whose edge-disjoint union is K_n. Such an m-colouring is *extremal* if (i) $q(SC_1) \leq \cdots \leq q(SC_m)$, and (ii) whenever $1 \leq i < j \leq m$, $SC_j + e \notin F_k(n)$ for all $e \in E(SC_i)$. In this situation, $q(SC_m) \geq \lceil \binom{n}{2}/m \rceil$ and $SC_m \in C_k(n)$. As noted in [[10], Lemma 3.1] K_n has an extremal m-colouring if and only if K_n has an m-colouring. Thus if K_n has no extremal m-colouring then $r_m(D_k) \leq n$. Of course, if K_n has an m-colouring then $r_m(D_k) \geq n+1$. We denote the values for $r_3(D_k)$ in table 3 as $\beta_k (2 \leq k \leq 26)$. First, for lower bounds on $r_3(D_k)$ we have:

Lemma 2.1. $r_3(D_k) \geq \beta_k (2 \leq k \leq 26)$.

Proof: The graphs D_3 and and D_{13} contain a 4-cycle, and so from [4] we have $r_3(D_k) \geq 11 = \beta_k$ $(k = 3, 13)$. The graphs D_{14}, D_{15} and D_{21} contain a triangle, and so from [7] we have $r_3(D_k) \geq 17 = \beta_k$ $(k = 14, 15, 21)$. Since $D_{26} = 5K_2$, we have from [5] that $r_3(D_{26}) \geq 5.4 - 2 = \beta_{26}$. Figs. 2–5 show a 3-colouring of $K_{\beta_k - 1}$ with no D_k for $k = 7, 8, 9, 10, 11, 12, 20$; so we have $r_3(D_k) \geq \beta_k$ for these values of k. For the remaining values of k the 3-colourings of $K_{\beta_k - 1}$ with no D_k may be determined by means of Table 4. Hence we have $r_3(D_k) \geq \beta_k$ $(k = 1, 2, 4, 5, 6, 16, 17, 18, 19, 22, 23, 24, 25)$. ∎

k	β_k	SC_3	SC_2	SC_1
2	17	$K_{8,8}$	$2K_{4,4}$	$4K_4$
4,5,6	21	$C_5[\overline{K_4}]$	$C_5[\overline{K_4}]$	$5K_4$
16,17	11	K_2+8K_1	$2K_1\cup(K_2+6K_1)$	$4K_1\cup K_6$
18	11	$K_4\cup K_{3,3}$	$K_4\cup K_{3,3}$	$K_4\cup K_{3,3}$
19	11	K_2+8K_1	$2K_1\cup K_2\cup K_6$	$2K_1\cup K_{2,6}$
22	14	K_3+10K_1	$3K_1\cup(K_3+7K_1)$	$6K_1\cup K_7$
23,24	12	K_2+9K_1	$2K_1\cup(K_2+7K_1)$	$4K_1\cup K_7$
25	15	K_3+11K_1	$3K_1\cup(K_3+8K_1)$	$6K_1\cup K_8$

Table 4: 3-colourings of K_{β_k-1} with no D_k.

For upper bounds on $r_3(D_k)$ we have first from Table 2:

Lemma 2.2. $r_3(D_k)\le\beta_k$ $(k=3,8,11,12,18,23,24)$. ∎

For the graphs D_k which do not contain a triangle we have:

Lemma 2.3. $r_3(D_k)\le\beta_k$ $(k=2,7,9,10,13,16,17,19,20,22,25,26)$.

Proof: Suppose by way of contradiction that there is an extremal 3-colouring $SC_3\cup SC_2\cup SC_1$ of K_{β_k} with no D_k. Then $SC_3\in C_k(\beta_k)$ and $q(SC_3)\ge\lceil\binom{\beta_k}{2}/3\rceil$; moreover $\overline{SC_3}$ has a 2-colouring and hence $K_{\alpha_k}\not\subseteq\overline{SC_3}$ where $\alpha_k=r_2(D_k)$. Let $S_k=\{G\colon G\in C_k(\beta_k),q(G)>\lceil\binom{\beta_k}{2}/3\rceil$ and $K_{\alpha_k}\not\subseteq\overline{G}\}$.

By using an algorithm similar to that for determining $t_k(n)$ and $C_k(n)$ (see [9]), we can construct all the graphs in S_k. Table 5 shows the values of $|S_k|$, and where the zeros occur we can deduce that $r_3(D_k)\le\beta_k$ $(k=13,16,17,19,22,25)$. By using the algorithm described in [10], we can establish that no graph G with $\overline{G}$ in S_k has an extremal 2-colouring with no D_k. Hence we have $r_3(D_k)\le\beta_k$ $(k=2,7,9,10,20,26)$. ∎

k	2	7	9	10	13	16	17	19	20	22	25	26
$\|S_k\|$	52	2	2	4	0	0	0	0	1	0	0	1

Table 5: The values of $|S_k|$ for Lemma 2.3.

The remaining lemmas deal with the graphs D_k which contain a triangle.

Lemma 2.4. $r_3(D_4)\le 21$.

Proof: Suppose that there is a 3-colouring $SC_3\cup SC_2\cup SC_1$ of K_{21} with no D_4. Since $r_3(K_3)=17$ [7], we may suppose that SC_3 (say) contains a triangle. If there is a vertex u of the triangle with degree 2 in SC_3, we take $w = u$. Otherwise we take w to be a third vertex adjacent to u. Then w has degree at most 3 in SC_3 and at least 17 in $\overline{SC_3}$. Without loss of generality, w has degree at least 9 in SC_2. Let $v_i (1 \leq i \leq d, d \leq 9)$ be the vertices adjacent to w in SC_2, and $u_j (1 \leq j \leq 20 - d)$ be the vertices not adjacent to w in SC_2. We show that $\overline{SC_2}$ contains K_9. This is immediate if the v_i are independent in SC_2; otherwise, since SC_2 has no D_4, there is precisely one pair of the v_i which are adjacent, say $v_1 \sim v_2$. Thus if $d \geq 11$ then again $\overline{SC_2}$ contains K_9. Accordingly we may suppose that $20 - d \geq 10$. In this case we may assume that u_1 and u_2 are non-adjacent because the u_j cannot induce a complete subgraph. Now, since SC_2 has no D_4, the vertices $u_1, u_2, v_3, v_4, \ldots, v_9$ are independent. Thus always $\overline{SC_2}$ contains K_9; but this contradicts the fact that $r_2(D_4) = 9$ [2]. Hence $r_3(D_4) \leq 21$. ∎

Lemma 2.5. $r_3(D_5) \leq 21$.

Proof: Suppose that there is a 3-colouring $SC_3 \cup SC_2 \cup SC_1$ of K_{21} with no D_5. As before we may suppose that SC_3 contains a triangle. Since SC_3 has no D_5, some vertex u of this triangle has degree ≤ 3 in SC_3. Hence without loss in generality u has degree at least 9 in SC_2. Let $v_i (1 \leq i \leq d, d \geq 9)$ be the vertices adjacent to u in SC_2. Since SC_2 has no D_5, each v_i, is adjacent to at most one $v_j (1 \leq i \neq j \leq d)$. We can now show that $\overline{SC_2}$ contains $K_9 - E(3K_2)$. This is clear unless the graph induced by $v_1, \ldots, v_d$ has four independent edges. In this case we may suppose that $v_1 \sim v_2$, $v_3 \sim v_4$, $v_5 \sim v_6$ and $v_7 \sim v_8$. Of the 12 vertices different from u and $v_1, v_2, \ldots, v_8$ there must be two which are non-adjacent, say u_1 and u_2. Since SC_2 contains no D_5 neither u_1 nor u_2 is adjacent to any of $v_1, v_2, \ldots, v_8$. It follows that the subgraph of SC_2 induced by $u_1, u_2, v_1, v_2, \ldots, v_7$ is $3K_2 \cup 3K_1$. Thus always $\overline{SC_2}$ contains $K_9 - E(3K_2)$. But by using the algorithm described in [10] we know that $K_9 - E(3K_2)$ has no extremal 2-colouring with no D_5. It follows that $r_3(D_5) \leq 21$. ∎

Lemma 2.6. $r_3(D_6) \leq 21$.

Proof: Suppose that there is a 3-colouring $SC_3 \cup SC_2 \cup SC_1$ of K_{21} with no D_6. Again we may suppose that SC_3 contains a triangle. The vertices of this triangle have at most degree 3 in SC_3 because SC_3 has no D_6. Such vertices have degree at least 17 in $\overline{SC_3}$ and so without loss of generality, SC_2 has a vertex v of degree at least 9. Since SC_2 has no D_6, the vertices adjacent to v are pairwise non-adjacent in SC_2. Thus $\overline{SC_2}$ contains K_9, a contradiction since $r_2(D_6) = 9$ [2]. ∎

Lemma 2.7. $r_3(D_{14}) \leq 17$.

Proof: Suppose that there is a 3-colouring $SC_3 \cup SC_2 \cup SC_1$ of K_{17} with no D_{14}. In the notation of [11], $D_{14} = H_{10} \cup K_2$ where $H_{10} = K_4 - E(P_3)$, with third Ramsey number equal to 13. Accordingly we may suppose that SC_3 contains

H_{10}. The remaining 13 vertices are independent in SC_3 and so $\overline{SC_3}$ contains K_{13}, a contradiction since $r_2(D_{14}) = 8$ [2]. ∎

Lemma 2.8. $r_3(D_{15}) \leq 17$.

Proof: Suppose that there is a 3-colouring $SC_3 \cup SC_2 \cup SC_1$ of K_{17} with no D_{15}. As before we may suppose that SC_3 contains a triangle; moreover $\overline{SC_3}$ contains no K_8 because $r_2(D_{15}) = 8$ [2]. In addition SC_3 contains no D_{15} and the only graph to satisfy all these conditions is $K_5 \cup 6K_2$. Now $\overline{SC_3}$ contains K_6 and $r_2(K_3) = 6$. Accordingly we may assume that SC_2 has a triangle, and so similarly $SC_2 = K_5 \cup 6K_2$. From Table 1, $q(SC_1) \leq 72$ and so $q(SC_1) + q(SC_2) + q(SC_3) \leq 104$, a contradiction. Hence $r_3(D_{15}) \leq 17$. ∎

Lemma 2.9. $r_3(D_{21}) \leq 17$.

Proof: Suppose that there is a 3-colouring $SC_3 \cup SC_2 \cup SC_1$ of K_{17} with no D_{21}. Now D_{21} contains $K_3 \cup K_2$ and we know from [11] that $r_3(K_3 \cup K_2) = 17$. Accordingly we may assume that SC_3 contains $K_3 \cup K_2$. The remaining 12 vertices are independent in SC_3, and so $\overline{SC_3}$ contains K_{12}. This is a contradiction because $r_2(D_{21}) = 10$ [2]. Hence $r_3(D_{21}) \leq 17$. ∎

Lemmas 2.1 to 2.9. now serve to verify that the Ramsey numbers $r_3(D_k)$ $(2 \leq k \leq 26)$ are as given in Table 3.

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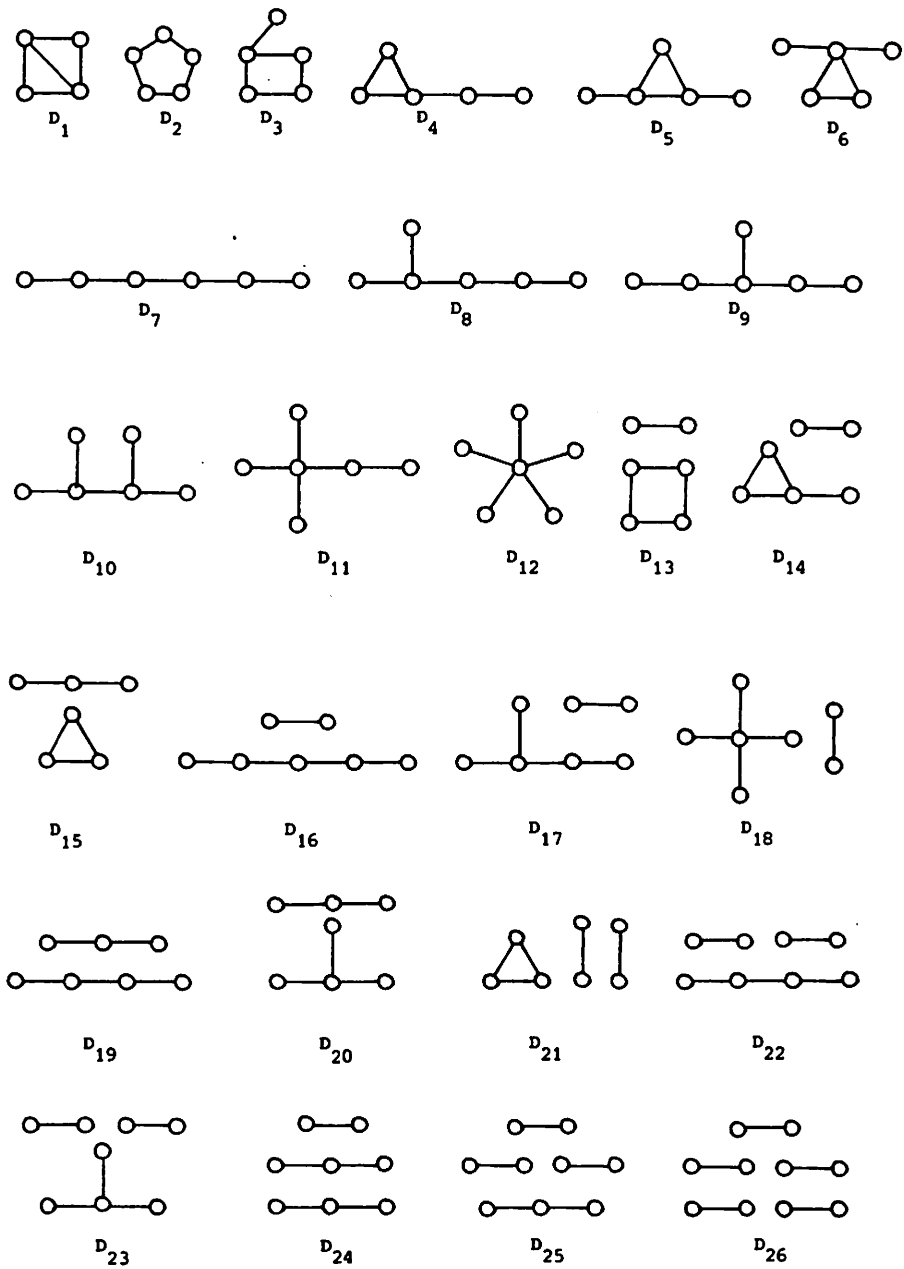

Fig. 1: The graphs with five edges and no isolated vertices.

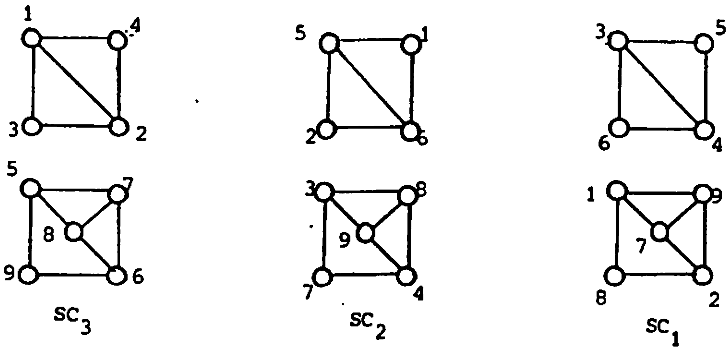

Fig. 2: A 3-colouring of K_9 with no D_7, D_9 or D_{10}.

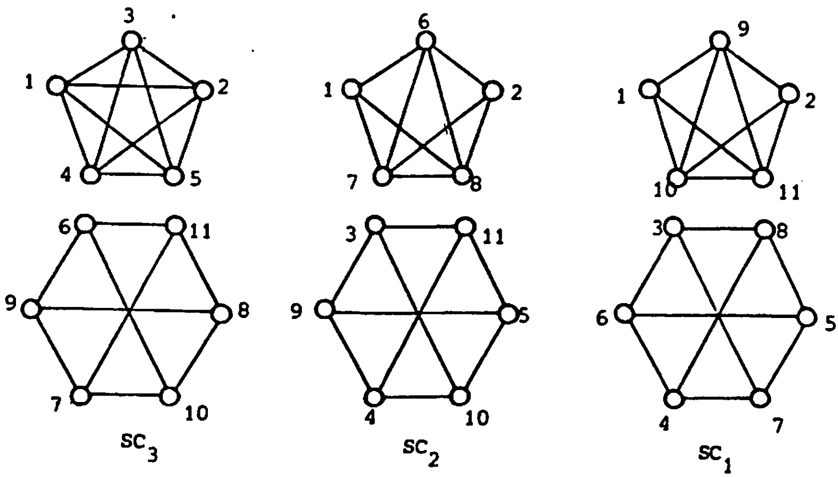

Fig. 3: A 3-colouring of K_{11} with no D_8, or D_{11}.

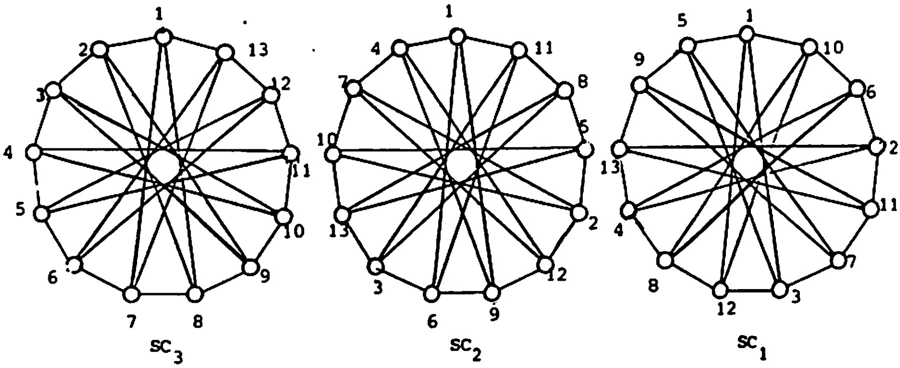

Fig. 4: A 3-colouring of K_{13} with no D_{12}.

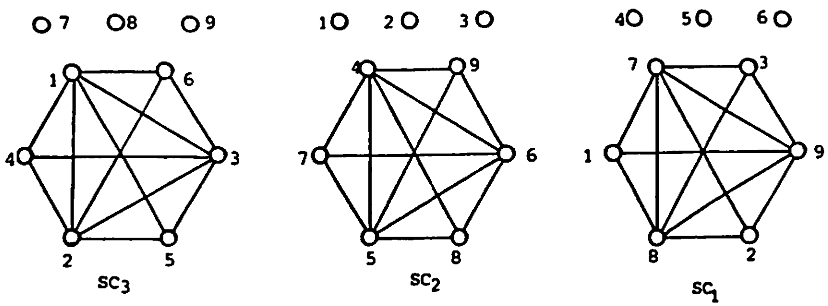

Fig. 5: A 3-colouring of K_9 with no D_{20}.